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On the properties of δ-interior and δ-closure in generalized topological spaces

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ABSTRACT

Some family of generalized topologies using the closure and interior operators of the generalized topology of δ-open sets are defined in [7]. We discuss the relation between their interior and closure operators with the other interior and closure operators and characterize some well known generalized open sets.

Keywords and Phrases: *μ-closed and μ-open sets; δ-open and δ-closed sets, generalized topology.*

AMS subject Classification (2000): *Primary 54 A 05, 54 A 10.*

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1. INTRODUCTION

The paper [1] of Prof. Á. Császár, is a base to study generalized topology and its properties. A generalized topology or simply GT μ [2] on a nonempty set X is a collection of subsets of X such that $\varphi\in\mu$ and μ is closed under arbitrary union. Elements of μ are called μ-open sets. A subset A of X is said to be μ-closed if X-A is μ-open. The pair (X, μ) is called a generalized topological space (GTS) or simply, a generalized space. If A is a subset of a space (X, μ), then $c_\mu(A)$ is the smallest μ -closed set containing A and $i_\mu(A)$ is the largest μ-open set contained in A. If $\gamma : \rho(X) \to \rho(X)$ be a monotonic function defined on a nonempty set X and $\mu = \{A|\ A\subset \gamma(A)\}$, the family of all γ-open sets is also a generalized topology [1], $i_\mu = i_\gamma$, and $c_\mu = c_\gamma$. By a space (X, μ), we will always mean a generalized topological space (X, μ). A subset A of a space (X, μ) is said to be α-open [3] (resp., semiopen [3], preopen [3], b-open [9], β-open [3]) if $A \subset i_\mu c_\mu i_\mu(A)$ (resp., $A \subset c_\mu i_\mu(A)$, $A \subset i_\mu c_\mu(A)$, $A \subset i_\mu c_\mu(A)\cup c_\mu i_\mu(A)$, $A \subset c_\mu i_\mu c_\mu(A)$). We will denote the family of all α-open sets by α, the family of all semiopen sets by σ, the family of all preopen sets by π, the family of all *b*-open sets by *b* and the family of all β-open sets by β. If (X, μ) is a generalized topological space, then we say that a subset $A\in\delta\subset\rho(X)$ [5] if for every $x\in A$, there exists a μ-closed Q such that $x\in i_\mu(Q) \subset A$. Then (X, δ) is a generalized topological space [5, Proposition 2.1] such that $\delta \subset \mu$ [5, Theorem1]. Elements of δ are called the δ-open sets of (X, δ). For $A \subset X$, $i_\delta(A)$ and $c_\delta(A)$ are the interior and closure of A in (X, δ). In [5], using the interior and closure operators of the generalized topologies δ and μ on X, we introduce the following family of generalized open sets, namely, the family of μ_δ-α-open sets, denoted by *ν*, the family of μ_δ-semiopen sets, denoted by ξ, the family of μ_δ-preopen sets, denoted by η, the family of μ_δ-*b*-open sets, denoted by ε, the family of μ_δ-β-open sets, denoted by ψ, and study their characterizations and properties. Also, we prove that ν (resp. ξ , η, ε, ψ) is nothing but the family of all α-open (resp. semiopen, preopen, *b*-open, β-open) sets of the generalized topological spaces (X, δ) and (X, μ). Let (X, μ) be a space. A subset A of X is said to be μ_δ-α -open (resp. μ_δ-semiopen, μ_δ-preopen, μ_δ-*b*-open, μ_δ- β-open) if $A \subset i_\mu c_\mu i_\delta(A)$ (resp. $A \subset c_\mu i_\delta(A)$, $A \subset i_\mu c_\delta(A)$, $A \subset c_\mu i_\delta(A)\cup i_\mu c_\delta(A)$, $A \subset c_\mu i_\mu c_\delta(A)$). We will denote by ν (resp. ξ, π, ε, ψ), the family of all μ_δ-α-open (resp. μ_δ-semiopen, μ_δ-preopen, μ_δ -b-open, μ_δ-β-open) sets.

If $\kappa\in\{\mu, \alpha, \sigma, \pi, b, \beta, \delta, \gamma, \xi, \eta, \varepsilon, \psi\}$ and A is a subset of a space (X, κ), then $c_\kappa(A)$ is the smallest κ-closed set containing A and $i_\kappa(A)$ is the largest *κ*-open set contained in A. Note that the operator c_κ is monotonic, increasing and idempotent and the operator i_κ is monotonic, decreasing and idempotent. Clearly, A is κ-open if and only if $A = i_\kappa(A)$ and A is κ-closed if and only if $A = c_\kappa(A)$. Also, for every subset A of a space (X, κ), $X\text{-}i_\kappa(A) = c_\kappa(X\text{-}A)$. Let X be a nonempty set. Let $\lambda \subset \rho(X)$ and $\gamma\in\Gamma$. γ is said to be *λ-friendly* [4] if $L\cap\gamma(A)\subset \gamma(L\cap A)$ for every subset A of X and $L\in\lambda$. In [9], it is denoted that $\Gamma_4 = \{\gamma\ |\gamma$ is μ-friendly where μ is the GT of all γ-open sets$\}$ and if $\gamma\in\Gamma_4$, the space (X, γ) (resp. (X, μ) is called a γ-space). By [9, Theorem 2.1], the intersection of two μ-open sets is again a μ-open set and so every γ-space is a quasi-topological space [4]. By [9, Theorem 2.3], it is established that in a γ-space, i_μ and c_μ preserves finite intersection and finite union respectively. Later, in [4], it is established that the above result is also true for quasi-topological spaces. One can easily prove that $\delta \subset \nu \subset \eta \subset \varepsilon \subset \psi$, $\delta \subset \nu \subset \xi \subset \varepsilon \subset \psi$ and $\nu = \xi\cap\eta$. Refer [6] for more such relations.

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The following Lemma 1.1 is essential to proceed further.

Theorem: 1.1 [7, Lemma 1.3] Let (X, μ) be a space and $A \subset X$. Then the following hold.
(a) If A is μ-open, then $c_\mu(A) = c_\delta(A)$.
(b) If A is μ-closed, then $i_\mu(A) = i_\delta(A)$.

Lemma: 1.2 [7, Theorem 2.4] Let (X, μ) be a generalized topological space where μ is the family of all γ -open sets of a $\gamma\in\Gamma_4$. Then the following hold.
(a) The intersection of two δ-open set is a δ-open set.
(b) $i_\delta(A)\cap i_\delta(A) = i_\delta(A\cap B)$ for every subsets A and B of X.
(c) $c_\delta(A) \cup c_\delta(B) = c_\delta(A\cup B)$ for every subsets A and B of X.
(d) $i_\delta\in\Gamma_4$.

Theorem: 1.3 [7, Theorem 2.6] Let (X, μ) be a space. Then the following hold.
(a) $i_\nu(A) = A\cap i_\mu c_\mu i_\delta(A)$.
(b) $c_\nu(A) = A\cup c_\mu i_\mu c_\delta(A)$
(c) $i_\xi(A) = A\cap c_\mu i_\delta(A)$.
(d) $c_\xi(A) = A\cup i_\mu c_\delta(A)$.

Theorem: 1.4 [7, Theorem 2.13] Let (X, μ) be a space where μ is the family of all γ-open sets, $\gamma\in\Gamma_4$ and $A\subset X$. Then the following hold.
(a) $i_\eta(A) = A\cap i_\mu c_\delta(A)$.
(b) $c_\eta(A) = A\cup c_\mu i_\delta(A)$.
(c) $i_\psi(A) = A\cap c_\mu i_\mu c_\delta(A)$.
(d) $c_\psi(A) = A\cup i_\mu c_\mu i_\delta(A)$.
(e) $c_\varepsilon(A) = c_\xi(A)\cap c_\eta(A)$.
(f) $i_\varepsilon(A) = i_\xi(A)\cup i_\eta(A)$.

2. PROPERTIES OF THE INTERIOR AND CLOSURE OPERATOR

In this section, we study the relations between the operators i_δ and c_δ with the other interior and closure operators, namely i_μ, c_μ, i_ξ, c_ξ, i_ν, c_ν, i_η, c_η, i_ε, c_ε, i_ψ and c_ψ. The dual of an identity is obtained by replacing the interior operator by the corresponding closure operator and '⊂' by '⊃'.

Theorem: 2.1 Let (X,μ) be a space and $A \subset X$. Then the following hold.
(a) $i_\delta i_\eta(A) = i_\delta(A)$.
(b) $c_\delta c_\eta(A) = c_\delta(A)$.
(c) $i_\delta c_\eta(A) \subset c_\delta i_\delta(A)$.
(d) $c_\delta i_\delta c_\eta(A) = c_\delta i_\delta(A)$.
(e) $i_\delta c_\eta(A) = i_\delta c_\delta i_\delta(A)$.
(f) $i_\delta i_\xi(A) = i_\delta(A)$.
(g) $c_\delta i_\xi(A) = c_\delta i_\delta(A) = c_\mu i_\delta(A)$.
(h) $i_\delta i_\psi(A) = i_\delta(A)$.
(i) $c_\delta c_\psi(A) = c_\delta(A)$.
(j) $i_\delta c_\psi(A) = i_\delta c_\delta i_\delta(A)$.
(k) $c_\delta i_\psi(A) = c_\delta i_\delta c_\delta(A)$.
(l) $i_\delta i_\varepsilon(A) = i_\delta(A)$.
(m) $c_\delta c_\varepsilon(A) = c_\delta(A)$.
(n) $i_\delta c_\varepsilon(A) = i_\delta c_\delta i_\delta(A)$.
(o) $c_\delta i_\varepsilon(A) = c_\delta i_\delta c_\delta(A)$.
(p) $i_\xi i_\eta(A) = i_\xi(A)\cap i_\eta(A)$
(q) $i_\delta c_\nu(A) = i_\delta c_\delta(A)$.
(r) $c_\delta c_\nu(A) = c_\delta(A)$.

Proof: (a) $i_\delta i_\eta(A) = i_\delta(A\cap i_\mu c_\delta(A)) \supset i_\delta(i_\delta(A)\cap i_\mu(A)) = i_\delta i_\delta(A) = i_\delta(A)$. $i_\delta i_\eta(A) = i_\delta(A\cap i_\mu c_\delta(A)) \subset i_\delta(A)$.

(b) The proof follows from (a) since the statement (b) is the dual of (a).

(c) Let $x\in i_\delta c_\eta(A)$ and $x\notin c_\delta i_\delta(A)$. Then there exists a δ open set U such that $x\in U \subset c_\eta(A)$, $U\cap i_\delta(A) = \varphi$. Since $U\subset c_\eta(A) = A\cup c_\mu i_\delta(A)$ and so $U\subset A$ which implies that $x\in i_\delta(A)$, which is not possible.

Hence, $x\in c_\delta i_\delta(A)$.

Therefore, $i_\delta c_\eta(A) \subset c_\delta i_\delta(A)$.

(d) By (c), $c_\delta i_\delta c_\eta(A) \subset c_\delta i_\delta(A)$. But $c_\delta i_\delta(A) \subset c_\delta i_\delta c_\eta(A)$. Hence, $c_\delta i_\delta c_\eta(A) = c_\delta i_\delta(A)$.

(e) By (c), $i_\delta c_\eta(A) \subset c_\delta\, i_\delta(A)$ which implies that $i_\delta c_\eta(A) \subset i_\delta c_\delta i_\delta(A)$. $i_\delta c_\eta(A) = i_\delta(A\cup c_\mu i_\delta(A)) \supset i_\delta c_\mu i_\delta(A) = i_\delta c_\delta i_\delta(A)$.

Hence, $i_\delta c_\eta(A) = i_\delta c_\delta i_\delta(A)$.

(f) The proof follows from 2.1(7) of [8].

(g) The proof follows from Theorem 2.1(10) of [8].

(h) $i_\delta i_\psi(A) = i_\delta(A\cap c_\mu i_\mu c_\delta(A)) \subset i_\delta(c_\delta(A)\cap i_\mu c_\delta(A)) = i_\delta(A)$. $i_\delta i_\psi(A) = i_\delta(A\cap c_\mu i_\mu c_\delta(A)) \supset i_\delta(i_\delta(A)\cap i_\mu(A))= i_\delta(A)$.

Hence, $i_\delta i_\psi(A) = i_\delta(A)$.

(i) The proof follows from (h).

(j) $i_\delta c_\psi(A) = i_\delta(A\cup i_\mu c_\mu i_\delta(A)) \supset i_\delta i_\mu c_\mu i_\delta(A) = i_\delta i_\delta c_\delta i_\delta(A) = i_\delta c_\delta i_\delta(A)$. $i_\delta c_\psi(A)$ is a subset of $i_\delta c_\eta(A)$ and $i_\delta c_\eta(A) = i_\delta c_\delta i_\delta(A)$, by (e). Hence, $i_\delta c_\psi(A) = i_\delta c_\delta i_\delta(A)$.

Therefore, $i_\delta c_\psi(A) = i_\delta c_\delta i_\delta(A)$.

(k) The proof follows from (j).

(l) $i_\delta i_\varepsilon(A) = i_\delta(i_\xi(A) \cup i_\eta(A)) \supset i_\delta i_\xi(A)\cup i_\delta c_\eta(A) = i_\delta(A)\cup i_\delta(A)$, by (a) and (e) and so $i_\delta i_\varepsilon(A) = i_\delta(A)$.

(m) The proof follows from (l).

(n) $i_\delta c_\varepsilon(A) = i_\delta(c_\xi(A)\cap c_\eta(A)) = i_\delta c_\xi(A)\cap i_\delta c_\eta(A) = i_\delta c_\delta(A)\cap i_\delta c_\delta i_\delta(A)$, by 2.1(10) of [8] and (e) and so $i_\delta c_\varepsilon(A) = i_\delta c_\delta i_\delta(A)$.

(o) The proof follows from (n).

(p) $i_\xi i_\eta(A) = i_\eta(A)\cap c_\mu i_\delta i_\eta(A) = i_\eta(A)\cap c_\mu i_\delta(A)$,by (a), and so $i_\xi i_\eta(A) = (A\cap i_\mu c_\delta(A))\cap c_\mu i_\delta(A) =(A\cap c_\mu i_\delta(A))\cap(A\cap i_\mu c_\delta(A)) = i_\xi(A)\cap i_\eta(A)$.

(q) $i_\delta c_\nu(A) = i_\delta(A\cup c_\mu i_\mu c_\delta(A)) \subset i_\delta(A\cup c_\mu c_\delta(A)) = i_\delta(A\cup c_\delta(A)) = i_\delta c_\delta(A)$. $i_\delta c_\nu(A) = i_\delta(A\cup c_\mu i_\mu c_\delta(A)) \supset i_\delta(A\cup i_\mu c_\delta(A)) = i_\delta(A\cup i_\delta c_\delta(A)) \supset i_\delta c_\delta(A)$.

Hence the proof follows.

(r) $c_\delta c_\nu(A) = c_\delta(A\cup c_\mu i_\mu c_\delta(A))\subset c_\delta(A\cup c_\mu c_\delta(A)) = c_\delta(A)$. Again, $c_\delta c_\nu(A) = c_\delta(A\cup c_\mu i_\mu c_\delta(A)) \supset c_\delta(A)$.

Hence, $c_\delta c_\nu(A) = c_\delta(A)$.

The following Theorem 2.2 gives the properties of the operators i_ν and c_ν.

Theorem: 2.2 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.
(a) $i_\nu i_\xi(A) = i_\nu(A)$.
(b) $i_\nu i_\eta(A) = i_\nu(A)$.
(c) $i_\nu i_\psi(A) = i_\nu(A)$.
(d) $i_\nu c_\xi(A) = i_\delta c_\delta(A)$.
(e) $i_\nu c_\eta(A) = i_\delta c_\delta(A)$.
(f) $i_\nu c_\psi(A) = i_\mu c_\mu i_\delta(A)$.
(g) $c_\nu i_\xi(A) = c_\mu i_\delta(A)$.
(h) $c_\nu i_\eta(A) = c_\mu i_\delta(A)$.
(i) $c_\nu i_\psi(A) = c_\mu i_\mu c_\delta(A)$.
(j) $c_\nu c_\xi(A) = c_\nu(A)$.
(k) $c_\nu c_\eta(A) = c_\nu(A)$.
(l) $c_\nu c_\psi(A) = c_\nu(A)$.

Proof:
(a) $i_\nu i_\xi(A) = i_\xi(A)\cap i_\mu c_\mu i_\delta(i_\xi(A)) = i_\xi(A)\cap i_\mu c_\mu i_\delta(A)$, by Theorem2.1(f), and so $i_\nu i_\xi(A) = (A\cap c_\mu i_\delta(A))\cap i_\mu c_\mu i_\delta(A) =A\cap i_\mu c_\mu i_\delta(A) = i_\nu(A)$.

(b) $i_\nu i_\eta(A) = i_\eta(A)\cap i_\mu c_\mu i_\delta(i_\eta(A)) = i_\eta(A)\cap i_\mu c_\mu i_\delta(A)$, by Theorem 2.1(a) and so $i_\nu i_\eta(A) = (A\cap i_\mu c_\delta(A))\cap i_\mu c_\mu i_\delta(A) = A\cap i_\mu c_\mu i_\delta(A) = i_\nu(A)$.

(c) $i_\nu i_\psi(A) = i_\psi(A)\cap i_\mu c_\mu i_\delta(i_\psi(A)) = i_\psi(A)\cap i_\mu c_\mu i_\delta(A)$, by Theorem 2.1(h) and so $i_\nu i_\psi(A) =(A\cap c_\mu i_\mu c_\delta(A))\cap i_\mu c_\mu i_\delta(A) = A\cap i_\mu c_\mu i_\delta(A) = i_\nu(A)$.

(d) $i_\nu c_\xi(A) = c_\xi(A)\cap i_\mu c_\mu i_\delta(c_\xi(A)) = c_\xi(A)\cap i_\mu c_\mu i_\delta c_\delta(A)$, by Theorem 2.1(10) of [8] and so $i_\nu c_\xi(A) = c_\xi(A)\cap i_\delta c_\delta(A) = (A\cup i_\mu c_\delta(A))\cap i_\delta c_\delta(A) = i_\delta c_\delta(A)$.

(e) $i_\nu c_\eta(A) =c_\eta(A)\cap i_\mu c_\mu i_\delta(c_\eta(A)) = c_\eta(A)\cap i_\mu c_\mu i_\delta c_\delta i_\delta(A)$, by Theorem 2.1(e) and so $i_\nu c_\eta(A) =(A\cup c_\mu i_\delta(A))\cap i_\delta c_\delta i_\delta(A) = i_\delta c_\delta i_\delta(A)$.

(f) $i_\nu c_\psi(A) = c_\psi(A)\cap i_\mu c_\mu i_\delta(c_\psi(A)) = c_\psi(A)\cap i_\mu c_\mu i_\delta c_\delta i_\delta(A)$, by Theorem 2.1(j) and so $i_\nu c_\psi(A) =(A\cup i_\mu c_\mu i_\delta(A))\cap i_\delta c_\delta i_\delta(A) = i_\delta c_\delta i_\delta(A)$.

(g) $c_\nu i_\xi(A) = i_\xi(A)\cup c_\mu i_\mu c_\delta(i_\xi(A)) = i_\xi(A)\cup c_\mu i_\mu c_\delta i_\delta(A)$, by Theorem 2.1(g) and so $c_\nu i_\xi(A) = (A\cap c_\mu i_\delta(A)) \cup c_\delta i_\delta(A) = c_\delta i_\delta(A)$.

(h) The proof follows from (e).

(i) The proof follows from (f).

(j) The proof follows from (a).

(k) The proof follows from (b).

(l) The proof follows from (c).

Theorem: 2.3 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.
(a) $c_\delta c_\nu(A) = c_\nu c_\delta(A) = c_\delta(A)$.
(b) $i_\delta i_\nu(A) = i_\nu i_\delta(A) = i_\delta(A)$.
(c) $c_\nu i_\delta(A) = c_\delta i_\nu(A) = c_\delta i_\delta(A)$.
(d) $i_\nu c_\delta(A) = i_\delta c_\nu(A) = i_\delta c_\delta(A)$.
(e) $i_\psi i_\delta(A) = i_\delta i_\psi(A) = i_\delta(A)$.
(f) $i_\eta i_\xi(A) = i_\nu(A)$.
(g) $c_\eta c_\xi(A) = c_\nu(A)$.
(h) $c_\eta i_\xi(A) = i_\xi(A) \cup c_\mu i_\delta(i_\xi(A)) = c_\eta(A) \cap c_\mu i_\delta(A)$.
(i) $i_\eta c_\xi(A) = c_\xi(A) \cap i_\mu c_\delta(c_\xi(A)) = i_\eta(A) \cup i_\mu c_\delta(A)$.
(j) $i_\xi i_\eta(A) = i_\eta(A) \cap c_\mu i_\delta(A)$.
(k) $c_\xi c_\eta(A) = c_\eta(A) \cup i_\mu c_\delta(A)$.

Proof:
(a) $c_\delta c_\nu(A) = c_\delta(A)$, by Theorem 2.1(r).

Again, $c_\nu c_\delta(A) = c_\delta(A) \cup c_\mu i_\mu c_\delta(c_\delta(A)) = c_\delta(A)$.

(b) The proof follows from (a).

(c) $c_\nu i_\delta(A) = i_\delta(A) \cup c_\mu i_\mu c_\delta(i_\delta(A)) = i_\delta(A) \cup c_\delta i_\delta(A) = c_\delta i_\delta(A)$. Also, $c_\delta i_\nu(A) = c_\delta(A \cap i_\mu c_\mu i_\delta(A)) \subset c_\delta(c_\delta(A) \cap c_\delta i_\delta(A)) = c_\delta i_\delta(A)$.

Again, $c_\delta i_\nu(A) = c_\delta(A \cap i_\mu c_\mu i_\delta(A)) \supset c_\delta(A \cap i_\mu i_\delta(A)) = c_\delta i_\delta(A)$.

(d) The proof follows from (c).

(e) $i_\psi i_\delta(A) = i_\delta(A) \cap c_\mu i_\mu c_\delta(i_\delta(A)) = i_\delta(A) \cap c_\delta i_\delta(A) = i_\delta(A)$.

Again, $i_\delta i_\psi(A) = i_\delta(A \cap c_\mu i_\mu c_\delta(A)) \subset i_\delta(A)$. Also, $i_\delta i_\psi(A) \supset i_\delta(A \cap i_\mu c_\delta(A)) \supset i_\delta(i_\delta(A) \cap i_\mu c_\delta(A)) = i_\delta(A)$.

Hence, $i_\psi i_\delta(A) = i_\delta i_\psi(A) = i_\delta(A)$.

(f) $i_\eta i_\xi(A) = i_\xi(A) \cap i_\mu c_\delta(i_\xi(A)) = i_\xi(A) \cap i_\mu c_\delta i_\delta(A)$,

by Theorem 2.1(g) and so $i_\eta i_\xi(A) = (A \cap c_\mu i_\delta(A)) \cap i_\mu c_\delta i_\delta(A) = A \cap i_\mu c_\mu i_\delta(A) = i_\nu(A)$.

(g) The proof follows from (f).

(h) $c_\eta(i_\xi(A)) = i_\xi(A) \cup c_\mu i_\delta(i_\xi(A)) = i_\xi(A) \cup c_\mu i_\delta(A)$, by Theorem 2.1(f) and so $c_\eta(i_\xi(A)) = (A \cap c_\mu i_\delta(A)) \cup c_\mu i_\delta(A) = (A \cup c_\mu i_\delta(A)) \cap c_\mu i_\delta(A) = c_\eta(A) \cap c_\mu i_\delta(A)$.

(i) The proof follows from (h).

(j) $i_\xi i_\eta(A) = i_\eta(A) \cap c_\mu i_\delta(i_\eta(A)) = i_\eta(A) \cap c_\mu i_\delta(A)$, by Theorem 2.1(a).

(k) The proof follows from (j).

Theorem: 2.4 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.
(a) $i_\nu c_\nu(A) = i_\eta(A) \cup i_\delta c_\delta(A)$.
(b) $c_\nu i_\nu(A) = c_\eta(A) \cap c_\mu i_\delta(A)$.
(c) $A \cup c_\nu i_\nu(A) = c_\eta(A)$.
(d) $A \cap i_\nu c_\nu(A) = c_\nu(A) \cup i_\eta(A) = i_\eta(A)$.
(e) $A \cap c_\nu i_\nu(A) = i_\xi(A) \cap c_\eta(A) = i_\xi(A)$.
(f) $A \cup i_\nu c_\nu(A) = c_\xi(A) \cup i_\eta(A) = c_\xi(A)$.

Proof:
(a) $i_\nu c_\nu(A) = c_\nu(A) \cap i_\mu c_\mu i_\delta(c_\nu(A)) = c_\nu(A) \cap i_\mu c_\mu i_\delta c_\delta(A)$,
by Theorem 2.1(q) and so $i_\nu c_\nu(A) = (A \cup c_\mu i_\mu c_\delta(A)) \cap i_\delta c_\delta(A) = (A \cap i_\delta c_\delta(A)) \cup (c_\mu i_\mu c_\delta(A) \cap i_\delta c_\delta(A))$
$= (A \cap i_\mu c_\delta(A)) \cup i_\delta c_\delta(A) = i_\eta(A) \cup i_\delta c_\delta(A) = i_\eta(A) \cup i_\mu c_\delta(A)$.
(b) The proof follows from (a).

(c) $A \cup c_\nu i_\nu(A) = A \cup (c_\eta(A) \cap c_\mu i_\delta(A))$, by (b) and so $A \cup c_\nu i_\nu(A) = (A \cup c_\eta(A)) \cap (A \cup c_\mu i_\delta(A)) = c_\eta(A) \cap c_\eta(A) = c_\eta(A)$.

(d) The proof follows from (c).

(e) $A \cap c_\nu i_\nu(A) = A \cap (c_\eta(A) \cap c_\mu i_\delta(A))$, by (b) and so $A \cap c_\nu i_\nu(A) = (A \cap c_\mu i_\delta(A)) \cap c_\eta(A) = i_\xi(A) \cap c_\eta(A) = i_\xi(A)$.

(f) The proof follows from (e).

Theorem: 2.5 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.
(a) $i_\xi c_\nu(A) = c_\nu(A) \cap c_\mu i_\mu c_\delta(A)$.
(b) $c_\xi i_\nu(A) = i_\nu(A) \cup i_\mu c_\mu i_\delta(A)$.
(c) $c_\xi c_\nu(A) = c_\nu(A) \cup c_\xi(A)$.
(d) $i_\xi i_\nu(A) = i_\nu(A) \cap i_\xi(A)$.
(e) $i_\xi i_\psi(A) = i_\xi(A)$.
(f) $c_\xi c_\psi(A) = c_\psi(A)$.

Proof:
(a) $i_\xi c_\nu(A) = c_\nu(A) \cap c_\mu i_\delta(c_\nu(A)) = c_\nu(A) \cap c_\mu i_\delta c_\delta(A)$, by Theorem 2.1(q) and so $i_\xi c_\nu(A) = c_\nu(A) \cap c_\mu i_\mu c_\delta(A)$.

(b) The proof follows from (a).

(c) $c_\xi c_\nu(A) = c_\nu(A) \cup i_\mu c_\delta(c_\nu(A)) = c_\nu(A) \cup i_\mu c_\delta(A)$, by Theorem 2.1(r) and so $c_\xi c_\nu(A) = c_\nu(A) \cup (A \cup i_\mu c_\delta(A)) = c_\nu(A) \cup c_\xi(A)$.

(d) The proof follows from (c).

(e) $i_\xi i_\psi(A) = i_\psi(A) \cap c_\mu i_\delta(i_\psi(A)) = i_\psi(A) \cap c_\mu i_\delta(A)$, by Theorem 2.1(h) and so $i_\xi i_\psi(A) = (A \cap c_\mu i_\mu c_\delta(A)) \cap c_\mu i_\delta(A) = A \cap c_\mu i_\delta(A) = i_\xi(A)$.

(f) The proof follows from (e).

Theorem: 2.6 Let (X, μ) be a quasi-topological space and A be a subset of X. Then the following hold.
(a) $i_\eta c_\nu(A) = c_\nu(A) \cap i_\mu c_\delta(A)$.
(b) $c_\eta i_\nu(A) = i_\nu(A) \cup c_\mu i_\delta(A)$.
(c) $i_\eta i_\psi(A) = i_\eta(A)$.
(d) $c_\eta c_\psi(A) = c_\eta(A)$.
(e) $i_\eta i_\nu(A) = i_\nu(A)$.
(f) $c_\eta c_\nu(A) = c_\nu(A)$.
(g) $i_\varepsilon i_\nu(A) = i_\nu(A)$.
(h) $c_\varepsilon c_\nu(A) = c_\nu(A)$.

Proof:
(a) $i_\eta c_\nu(A) = c_\nu(A) \cap i_\mu c_\delta(c_\nu(A)) = c_\nu(A) \cap i_\mu c_\delta(A)$, by Theorem 2.1(r).

(b) The proof follows from (a).

(c) $i_\eta i_\psi(A) = i_\psi(A) \cap i_\mu c_\delta(i_\psi(A)) = (A \cap c_\mu i_\mu c_\delta(A)) \cap i_\mu c_\delta i_\delta c_\delta(A)$, by Theorem 2.1(k) and so $i_\eta i_\psi(A) = (A \cap c_\mu i_\mu c_\delta(A)) \cap i_\delta c_\delta(A) = A \cap i_\delta c_\delta(A) = i_\eta(A)$.

(d) The proof follows from (c).

(e) $i_\eta i_\nu(A) = i_\nu(A) \cap i_\mu c_\delta i_\nu(A) = i_\nu(A) \cap i_\mu c_\delta i_\delta(A)$, by Theorem 2.3(c) and so $i_\eta i_\nu(A) = (A \cap i_\mu c_\mu i_\delta(A)) \cap i_\mu c_\mu i_\delta(A) = A \cap i_\mu c_\mu i_\delta(A) = i_\nu(A)$.

(f) The proof follows from (e).

(g) $i_\varepsilon i_\nu(A) = i_\xi i_\nu(A) \cup i_\eta i_\nu(A) = (i_\nu(A) \cap c_\mu i_\delta i_\nu(A)) \cup i_\nu(A)$, by (e) and so $i_\varepsilon i_\nu(A) = i_\nu(A)$.

(h) The proof follows from (g).

Theorem: 2.7 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.
(a) $i_\psi c_\nu(A) = c_\nu(A) \cap c_\mu i_\mu c_\delta(A) = c_\mu i_\mu c_\delta(A)$.
(b) $c_\psi i_\nu(A) = i_\mu c_\mu i_\delta(A)$.

Proof: (a) $i_\psi c_\nu(A) = c_\nu(A) \cap c_\mu i_\mu c_\delta(c_\nu(A)) = c_\nu(A) \cap c_\mu i_\mu c_\delta(A)$, by Theorem 2.1(r) and so $i_\psi c_\nu(A) = (A \cup c_\mu i_\mu c_\delta(A)) \cap c_\mu i_\mu c_\delta(A) = c_\mu i_\mu c_\delta(A)$.

(b) The proof follows from (a).

Theorem: 2.8 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.
(a) $i_\delta i_\varepsilon(A) = i_\varepsilon i_\delta(A) = i_\delta(A)$.
(b) $c_\delta c_\varepsilon(A) = c_\varepsilon c_\delta(A) = c_\delta(\mathrm{A})$.

(c) $i_\xi i_\varepsilon(A) = i_\varepsilon i_\xi(A) = i_\xi(A)$.
(d) $c_\xi c_\varepsilon(A) = c_\varepsilon c_\xi(A) = c_\xi(A)$.
(e) $i_\nu i_\varepsilon(A) = i_\varepsilon i_\nu(A) = i_\nu(A)$.
(f) $c_\nu c_\varepsilon(A) = c_\varepsilon c_\nu(A) = c_\nu(A)$.
(g) $i_\varepsilon i_\psi(A) = i_\psi i_\varepsilon(A) = i_\varepsilon(A)$.
(h) $c_\varepsilon c_\psi(A) = c_\psi c_\varepsilon(A) = c_\varepsilon(A)$.
(i) $i_\varepsilon c_\nu(A) = c_\mu i_\mu c_\delta(A)$.
(j) $c_\varepsilon i_\nu(A) = i_\mu c_\mu i_\delta(A)$.
(k) $c_\nu i_\varepsilon(A) = i_\varepsilon(A)\cup c_\delta i_\delta(A)$.
(l) $i_\nu c_\varepsilon(A) = c_\varepsilon(A)\cap i_\delta c_\delta(A)$.
(m) $c_\nu c_\varepsilon(A) = c_\varepsilon(A)\cup c_\mu i_\mu c_\delta(A)$.
(n) $i_\nu i_\varepsilon(A) = i_\varepsilon(A)\cap i_\mu c_\mu i_\delta(A)$.
(o) $c_\xi c_\varepsilon(A) = c_\varepsilon(A)\cup i_\mu c_\delta(A)$.
(p) $i_\xi i_\varepsilon(A) = i_\varepsilon(A)\cap c_\mu i_\delta(A)$.
(q) $c_\eta c_\varepsilon(A) = c_\varepsilon(A)\cup c_\delta i_\delta(A)$.
(r) $i_\eta i_\varepsilon(A) = i_\varepsilon(A)\cap i_\delta c_\delta(A)$.
(s) $c_\psi c_\varepsilon(A) = c_\varepsilon(A)\cup i_\mu c_\delta i_\delta(A)$.
(t) $i_\psi i_\varepsilon(A) = i_\varepsilon(A)\cap c_\mu i_\delta c_\delta(A)$.
(u) $i_\eta i_\varepsilon(A) = i_\varepsilon i_\eta(A) = i_\eta(A)$.
(v) $c_\eta c_\varepsilon(A) = c_\varepsilon c_\eta(A) = c_\eta(A)$.

Proof:

(a) $i_\delta i_\varepsilon(A) = i_\delta(A)$, by Theorem 2.1(l). Again, $i_\varepsilon i_\delta(A) = i_\xi i_\delta(A)\cup i_\eta i_\delta(A) = (i_\delta(A)\cap c_\mu i_\delta(i_\delta(A)))\cup(i_\delta(A)\cap i_\mu c_\delta(i_\delta(A))) = i_\delta(A)\cup i_\delta(A) = i_\delta(A)$.

Hence, $i_\delta i_\varepsilon(A) = i_\varepsilon i_\delta(A) = i_\delta(A)$.

(b) The proof follows from (a).

(c) $i_\xi i_\varepsilon(A) = i_\xi(i_\xi(A) \cup i_\eta(A)) \supset i_\xi(i_\xi(A))\cup i_\xi(i_\eta(A)) = i_\xi(A)\cup(i_\xi(A)\cap i_\eta(A)) = i_\xi(A))$.

Clearly, $i_\xi(i_\varepsilon(A))) \subset i_\xi(A))$. Hence, $i_\xi i_\varepsilon(A)) = i_\xi(A))$.

Again, $i_\varepsilon i_\xi(A) = i_\xi(i_\xi(A)))\cup i_\eta(i_\xi(A))) = i_\xi(A))\cup i_\nu(A))$, by Theorem 2.3(f) and so $i_\varepsilon i_\xi(A) = i_\xi(A))$. Hence, the proof follows.

(d) The proof follows from (c).

(e) $i_\nu i_\varepsilon(A))=i_\varepsilon(A))\cap i_\mu c_\mu i_\delta(i_\varepsilon(A))) = i_\varepsilon(A))\cap i_\mu c_\mu i_\delta(A))$, by Theorem 2.1(l) and so $i_\nu i_\varepsilon(A)) = (i_\xi(A)) \cup i_\eta(A))) \cap i_\mu c_\mu i_\delta(A)) = ((A\cap c_\mu i_\delta(A)) \cup(A\cap i_\mu c_\delta(A)))\cap i_\mu c_\mu i_\delta(A) = A\cap i_\mu c_\mu i_\delta(A) = i_\nu(A)$.

Again, $i_\varepsilon i_\nu(A) = i_\xi i_\nu(A)\cup i_\eta i_\nu(A) =(i_\nu(A)\cap i_\xi(A))\cup i_\nu(A)$, by Theorems 2.5(d) and 2.6(e) and so $i_\varepsilon i_\nu(A) = i_\nu(A)\cup i_\nu(A) =i_\nu(A)$.

(f) The proof follows from (e).

(g) $i_\varepsilon i_\psi(A) = i_\xi i_\psi(A)\cup i_\eta i_\psi(A) = i_\xi(A)\cup i_\eta(A) = i_\varepsilon(A)$, by Theorem 2.5(e) and Theorem 2.6(c). Also, $i_\psi i_\varepsilon(A) = i_\psi(i_\xi(A) \cup i_\eta(A)) \supset i_\psi(i_\xi(A))\cup i_\psi(i_\eta(A)) = i_\xi(A)\cup i_\eta(A)$, and so $i_\psi i_\varepsilon(A) = i_\varepsilon(A)$.

(h) The proof follows from (g).

(i) $i_\varepsilon c_\nu(A) = i_\xi c_\nu(A) \cup i_\eta c_\nu(A) = (c_\nu(A)\cap c_\mu i_\delta(c_\nu(A)) \cup(c_\nu(A)\cap i_\mu c_\delta(c_\nu(A))) = c_\nu(A)\cap(c_\mu i_\delta c_\nu(A) \cup i_\mu c_\delta(c_\nu(A)) = c_\nu(A)\cap(c_\mu i_\delta c_\delta(A) \cup i_\mu c_\delta(A))$, by Theorem 2.1(q) and (r) and so $i_\varepsilon c_\nu(A) = c_\nu(A)\cap c_\mu i_\mu c_\delta(A) = (A\cup c_\mu i_\mu c_\delta(A))\cap c_\mu i_\mu c_\delta(A) = c_\mu i_\mu c_\delta(A)$.

(j) The proof follows from (i).

(k) $c_\nu i_\varepsilon(A) = i_\varepsilon(A)\cup c_\mu i_\mu c_\delta(i_\varepsilon(A)) = i_\varepsilon(A)\cup c_\mu i_\mu(c_\delta i_\delta c_\delta(A))$, by Theorem 2.1(o) and so $c_\nu i_\varepsilon(A) = i_\varepsilon(A)\cup c_\delta i_\delta c_\delta(A)$.

(l) The proof follows from (k).

(m) $c_\nu c_\varepsilon(A) = c_\varepsilon(A)\cup c_\mu i_\mu c_\delta(c_\varepsilon(A)) = c_\varepsilon(A)\cup c_\mu i_\mu c_\delta(A)$, by Theorem 2.1(m).

(n) The proof follows from (m).

(o) $c_\xi c_\varepsilon(A) = c_\varepsilon(A)\cup i_\mu c_\delta(c_\varepsilon(A)) = c_\varepsilon(A)\cup i_\mu c_\delta(A)$, by Theorem 2.1(m).

(p) The proof follows from (o).
(q) $c_\eta c_\varepsilon(A) = c_\varepsilon(A)\cup c_\mu i_\delta c_\varepsilon(A) = c_\varepsilon(A)\cup c_\mu i_\delta c_\delta i_\delta(A)$, by Theorem 2.1(n) and so $c_\eta c_\varepsilon(A) = c_\varepsilon(A)\cup c_\delta i_\delta(A)$.

(r) The proof follows from (q).

(s) $c_\psi c_\varepsilon(A) = c_\varepsilon(A)\cup i_\mu c_\mu i_\delta c_\varepsilon(A) = c_\varepsilon(A)\cup i_\mu c_\mu i_\delta c_\delta i_\delta(A)$, by Theorem 2.1(n) and so $c_\psi c_\varepsilon(A) = c_\varepsilon(A)\cup i_\mu c_\delta i_\delta(A)$.

(t) The proof follows from (s).

(u) $i_\eta i_\varepsilon(A) = i_\eta(i_\xi(A) \cup i_\eta(A)) \supset i_\eta(i_\xi(A)) \cup i_\eta(i_\eta(A)) = i_\nu(A) \cup i_\eta(A)$, by Theorem 2.3(f) and so $i_\eta i_\varepsilon(A) \supset i_\eta(A)$. Clearly, $i_\eta i_\varepsilon(A) \subset i_\eta(A)$.

Hence $i_\eta i_\varepsilon(A) = i_\eta(A)$.

(v) The proof follows from (u).

Theorem: 2.9 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.
(a) $c_\xi i_\xi(A) = (A \cap c_\mu i_\delta(A)) \cup i_\mu c_\delta i_\delta(A)$.
(b) $i_\xi c_\xi(A) = (A \cup i_\mu c_\delta(A)) \cap c_\mu i_\delta c_\delta(A)$.

Proof:
(a) $c_\xi i_\xi(A) = i_\xi(A) \cup i_\mu c_\delta(i_\xi(A)) = (A \cap c_\mu i_\delta(A)) \cup i_\mu c_\delta i_\delta(A)$, by Theorem 2.1(g).
(b) The proof follows from (a).

Theorem: 2.10 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.
(a) $c_\psi i_\psi(A) = i_\psi c_\psi(A) = (A \cup i_\delta c_\delta i_\delta(A)) \cap c_\delta i_\delta c_\delta(A)$.
(b) $c_\eta i_\eta(A) = i_\eta(A) \cup c_\mu i_\delta(A) = c_\eta(A) \cap (i_\mu c_\delta(A)) \cup c_\mu i_\delta(A))$.
(c) $i_\eta c_\eta(A) = c_\eta(A) \cap i_\mu c_\delta(A) = (A \cup c_\mu i_\delta(A)) \cap i_\mu c_\delta(A)$.
(d) $A \cup c_\eta i_\eta(A) = c_\eta(A)$.
(e) $A \cap c_\eta i_\eta(A) = i_\eta(A) \cup i_\xi(A)$.
(f) $A \cup i_\eta c_\eta(A) = c_\eta(A) \cap c_\xi(A)$.
(g) $A \cap i_\eta c_\eta(A) = i_\eta(A)$.
(h) $i_\eta c_\eta(A) \subset c_\eta i_\eta(A)$ and so $i_\eta c_\eta(A) \cup c_\eta i_\eta(A) = c_\eta i_\eta(A)$.
(i) $c_\eta i_\eta c_\eta(A) = c_\eta i_\eta(A)$.
(j) $i_\eta c_\eta i_\eta(A) = i_\eta c_\eta(A)$.
(k) $i_\eta c_\eta(A) \subset i_\xi c_\xi(A)$.
(l) $i_\xi c_\xi(A) = c_\xi(A) \cap c_\mu i_\mu c_\delta(A)$.
(m) $c_\xi i_\xi(A) \subset c_\eta i_\eta(A)$.
(n) $c_\eta(i_\delta(A)) = c_\mu i_\delta(A)$.
(o) $c_\psi i_\delta(A) = i_\delta c_\psi(A) = i_\delta c_\delta i_\delta(A) = i_\mu c_\mu i_\delta(A)$.
(p) $c_\xi i_\delta(A) = i_\mu c_\mu i_\delta(A)$.
(q) $i_\xi c_\delta(A) = c_\mu i_\mu c_\delta(A)$.
(r) $i_\psi(c_\delta(A)) = c_\mu i_\mu c_\delta(A)$.
(s) $c_\xi(i_\xi(A)) \subset i_\psi(c_\psi(A)) \subset i_\xi c_\xi(A)$.

Proof:
(a) $i_\psi c_\psi(A) = c_\psi(A) \cap c_\mu i_\mu c_\delta(c_\psi(A)) = (A \cup i_\mu c_\mu i_\delta(A)) \cap c_\mu i_\mu c_\delta(A)$, by Theorem 2.1(i) and so $i_\psi c_\psi(A) = (A \cap c_\mu i_\mu c_\delta(A)) \cup i_\mu c_\mu i_\delta(A) = (A \cap c_\mu i_\mu c_\delta(A)) \cup i_\mu c_\mu i_\delta i_\psi(A)$, by Theorem 2.1(h) and so $i_\psi c_\psi(A) = i_\psi(A) \cup i_\mu c_\mu i_\delta(i_\psi(A)) = c_\psi(i_\psi(A))$.

(b) $c_\eta i_\eta(A) = i_\eta(A) \cup c_\mu i_\delta(i_\eta(A)) = i_\eta(A) \cup c_\mu i_\delta(A)$, by Theorem 2.1(a). Again,
$(c_\eta(A) \cap (i_\mu c_\delta(A) \cup c_\mu i_\delta(A)) = (A \cup c_\mu i_\delta(A)) \cap (i_\mu c_\delta(A) \cup c_\mu i_\delta(A)) = c_\mu i_\delta(A) \cup (A \cap i_\mu c_\delta(A)) = i_\eta(A) \cup c_\mu i_\delta(A)$.

(c) $i_\eta c_\eta(A) = c_\eta(A) \cap i_\mu c_\delta c_\eta(A) = c_\eta(A) \cap i_\mu c_\delta(A)$, by Theorem 2.1(b) and so $i_\eta c_\eta(A) = (A \cup c_\mu i_\delta(A)) \cap i_\mu c_\delta(A)$.

(d) $A \cup c_\eta i_\eta(A) = A \cup (i_\eta(A) \cup c_\mu i_\delta(i_\eta(A)) = A \cup (i_\eta(A) \cup c_\mu i_\delta(A))$, by Theorem2.1(a) and so $A \cup c_\eta i_\eta(A) = (A \cup c_\mu i_\delta(A)) \cup i_\eta(A) = c_\eta(A) \cup i_\eta(A) = c_\eta(A)$.

(e) $A \cap c_\eta i_\eta(A) = A \cap (i_\eta(A) \cup c_\mu i_\delta(i_\eta(A)) = (A \cap i_\eta(A)) \cup c_\mu i_\delta(A)$, by Theorem 2.1(a) and so $A \cap c_\eta i_\eta(A) = i_\eta(A) \cup (A \cap c_\mu i_\delta(A) = i_\eta(A) \cup i_\xi(A)$.

(f) The proof follows from (e).

(g) The proof follows from (d).
(h) By (c), $i_\eta c_\eta(A) = c_\eta(A) \cap i_\mu c_\delta(A) = (A \cup c_\mu i_\delta(A)) \cap i_\mu c_\delta(A) = (A \cap i_\mu c_\delta(A)) \cup (c_\mu i_\delta(A) \cap i_\mu c_\delta(A)) \subset i_\eta(A) \cup c_\mu i_\delta(A) = c_\eta(i_\eta(A))$, by (b).

Hence, $i_\eta c_\eta(A) \cup c_\eta i_\eta(A) = c_\eta\, i_\eta(A)$.

(i) $c_\eta i_\eta(c_\eta(A)) \subset c_\eta(c_\eta(i_\eta(A)))$, by (h) and so $c_\eta i_\eta(c_\eta(A)) \subset c_\eta(i_\eta(A))$. Clearly, $c_\eta(i_\eta(A)) \subset c_\eta i_\eta(c_\eta(A))$.

Hence, the proof follows.

(j) The proof follows from (i).

(k) $i_\eta(c_\eta(A)) = (A \cap i_\mu c_\delta(A)) \cup (c_\mu i_\delta(A) \cap i_\mu c_\delta(A)) \subset (A \cap c_\mu i_\mu c_\delta(A)) \cup i_\mu c_\delta(A) = (A \cup i_\mu c_\delta(A)) \cap c_\mu i_\mu c_\delta(A) = i_\xi c_\xi(A)$, by Theorem 2.9 (b).

(l) $i_\xi c_\xi(A) = c_\xi(A) \cap c_\mu i_\delta(c_\xi(A)) = c_\xi(A) \cap c_\mu i_\delta c_\delta(A)$, by 2.1(10) of [8].

(m) By (b), $c_\eta(i_\eta(A) = c_\eta(A)\cap(i_\mu c_\delta(A)\cup c_\mu i_\delta(A))=(A\cup c_\mu i_\delta(A))\cap(i_\mu c_\delta(A)\cup c_\mu i_\delta(A)) \supset (A\cup i_\mu c_\mu i_\delta(A)) \cap c_\mu i_\delta(A) =(A\cap c_\mu i_\delta(A)) \cup i_\mu c_\mu i_\delta(A) = c_\xi i_\xi(A)$, by Theorem 2.9(a).

(n) $c_\eta(i_\delta(A)) = i_\delta(A)\cup c_\mu i_\delta(i_\delta(A)) = i_\delta(A)\cup c_\mu i_\delta(A) = c_\mu i_\delta(A)$.

(o) $c_\psi i_\delta(A) = i_\delta(A)\cup i_\mu c_\mu i_\delta(i_\delta(A)) = i_\mu c_\mu i_\delta(A)= i_\delta c_\delta i_\delta(A)$. Again, $i_\delta(c_\psi(A)) = i_\delta(A\cup i_\mu c_\mu i_\delta(A)) \supset i_\delta(i_\mu c_\mu i_\delta(A)) = i_\delta(i_\delta c_\delta i_\delta(A)) \supset i_\delta c_\delta i_\delta(A)$.

Again, $i_\delta(c_\psi(A))\subset i_\delta(c_\eta(A)) \subset c_\delta i_\delta(A)$, by Theorem 2.1(c) and so $i_\delta(c_\psi(A)) \subset i_\delta c_\delta i_\delta(A)$.

Hence, $c_\psi i_\delta(A) = i_\delta c_\psi(A) = i_\delta c_\delta i_\delta(A) = i_\delta c_\mu(i_\delta(A)$.

(r) The proof follows from (p).

(s) $i_\psi c_\delta(A) = c_\delta(A)\cap c_\mu i_\mu c_\delta(c_\delta(A))= c_\delta(A)\cap c_\mu i_\mu c_\delta(A) = c_\mu i_\mu c_\delta(A)$.

(t) $c_\xi i_\xi(A)= i_\xi(A)\cup i_\mu c_\delta(i_\xi(A)) = i_\xi(A)\cup i_\mu c_\delta i_\delta(A)$, by Theorem 2.1(g) and so $c_\xi i_\xi(A) =(A\cap c_\mu i_\delta(A))\cup i_\mu c_\delta i_\delta(A) = (A\cup i_\mu c_\mu i_\delta(A))\cap c_\mu i_\delta(A) \subset (A\cup i_\mu c_\mu i_\delta(A))\cap c_\mu i_\mu c_\delta(A) = (A\cup i_\mu c_\mu i_\delta(A))\cap c_\mu i_\mu c_\delta c_\psi(A)$, by Theorem 2.1(i) and so $c_\xi i_\xi(A) \subset i_\psi c_\psi(A)$.

Again, $i_\psi c_\psi(A) = c_\psi(A)\cap c_\mu i_\mu c_\delta(c_\psi(A)) = (A\cup i_\mu c_\mu i_\delta(A))\cap c_\mu i_\mu c_\delta(A)$, by Theorem 2.1(i) and so $i_\psi c_\psi(A)\subset A\cup i_\mu c_\delta(A)) \cap c_\mu i_\delta c_\delta(A) = c_\xi(A)\cap c_\mu i_\delta(c_\xi(A))$, by Theorem 2.1(10) of [8] and so $i_\psi c_\psi(A) = i_\xi(c_\xi(A))$.

Hence, the proof follows.

Theorem: 2.11 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.

(a) $A \in \eta(\delta)$ if and only if $c_\xi(A) = i_\mu c_\delta(A)$.
(b) A is η-closed if and only if $i_\xi(A) = c_\mu i_\delta(A)$.
(c) $A \in \xi(\delta)$ if and only if $c_\eta(A) = c_\mu i_\delta(A)$.
(d) A is ξ-closed if and only if $i_\eta(A) = i_\mu c_\delta(A)$.
(e) $A \in \nu(\delta)$ if and only if $c_\psi(A) = i_\mu c_\mu i_\delta(A)$.
(f) A is ν-closed if and only if $i_\psi(A) = c_\mu i_\mu c_\delta(A)$.
(g) $A \in \psi(\delta)$ if and only if $c_\nu(A) = c_\mu i_\mu c_\delta(A)$.
(h) A is $\psi(\delta)$-closed if and only if $i_\nu(A) = i_\mu c_\mu i_\delta(A)$.

Proof:
(a) $A \in \eta(\delta)$ if and only if $A \subset i_\mu c_\delta(A)$ if and only if $c_\xi(A) = A\cup i_\mu c_\delta(A) = i_\mu c_\delta(A)$.
(b) The proof follows from (a).
(c) $A\in \xi(\delta)$ if and only if $A \subset c_\mu i_\delta(A)$ if and only if $c_\eta(A) =A\cup c_\mu i_\delta(A) = c_\mu i_\delta(A)$.
(d) The proof follows from (c).
(e) $A\in \nu(\delta)$ if and only if $A \subset i_\mu c_\mu i_\delta(A)$ if and only if $c_\psi(A) = A\cup i_\mu c_\mu i_\delta(A) = i_\mu c_\mu i_\delta(A)$.
(f) The proof follows from (e).
(g) $A\in \psi(\delta)$ if and only if $A \subset c_\mu i_\mu c_\delta(A)$ if and only if $c_\nu(A) = A\cup c_\mu i_\mu c_\delta(A) = c_\mu i_\mu c_\delta(A)$.
(h) The proof follows from (g).

Theorem: 2.12 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.

(a) $c_\psi i_\delta(A) = i_\mu c_\mu i_\delta(A) = i_\delta c_\delta i_\delta(A)$.
(b) $i_\psi c_\delta(A) = c_\mu i_\mu c_\delta(A) = c_\delta i_\delta c_\delta(A)$.
(c) $c_\delta c_\eta i_\delta(A) = c_\mu i_\mu(A)$.
(d) $i_\delta i_\eta c_\delta(A)= i_\mu c_\mu(A)$.
(e) $i_\eta c_\varepsilon(A) = c_\eta(A)\cap i_\eta c_\xi(A)$.
(f) $c_\eta i_\varepsilon(A)= i_\eta(A)\cup c_\eta i_\xi(A)$.
(g) $i_\delta c_\varepsilon(A) = i_\delta c_\delta i_\delta(A)$.
(h) $c_\delta i_\varepsilon(A) = c_\delta i_\delta c_\delta(A)$.
(i) $c_\varepsilon i_\delta(A) = i_\mu c_\mu i_\delta(A)$.
(j) $i_\varepsilon c_\delta(A) = c_\mu i_\mu c_\delta(A)$.

Proof:
(a) $c_\psi i_\delta(A) = i_\delta(A)\cup i_\mu c_\mu i_\delta(i_\delta(A)) = i_\delta(A)\cup i_\mu c_\mu i_\delta(A) = i_\mu c_\mu i_\delta(A) = i_\delta c_\delta i_\delta(A)$.

(b) The proof follows from (a).

(c) $c_\delta c_\eta i_\delta(A) = c_\delta(i_\delta(A)\cup c_\mu i_\delta(i_\delta(A)) \supset c_\delta i_\delta(A)$. Again, $c_\delta c_\eta i_\delta(A) = c_\delta(i_\delta(A)\cup c_\mu i_\delta(i_\delta(A)) \subset c_\delta(i_\delta(A)\cup c_\delta i_\delta(A)) = c_\delta i_\delta(A)$. This proves (c).

(d) The proof follows from (c).

(e) $i_\eta(c_\varepsilon(A)) = c_\varepsilon(A)\cap i_\mu c_\delta(c_\varepsilon(A)) = (c_\xi(A)\cap c_\eta(A))\cap i_\mu c_\delta(c_\varepsilon(A)) = c_\eta(A)\cap(c_\xi(A)\cap i_\mu c_\delta(c_\varepsilon(A)))$. Again, $c_\xi(A)\cap i_\mu c_\delta(c_\varepsilon(A)) = c_\xi(A)\cap i_\mu c_\delta(c_\xi(A)\cap c_\eta(A))= c_\xi(A)\cap i_\mu c_\delta c_\eta(A) = c_\xi(A)\cap i_\mu c_\delta(A)$, by Theorem 2.1(b). But $i_\eta c_\xi(A) = c_\xi(A)\cap i_\mu c_\delta(c_\xi(A))= c_\xi(A)\cap i_\mu c_\delta(A)$, by 2.1(7) of [8] and so $c_\xi(A)\cap i_\mu c_\delta(c_\varepsilon(A)) = i_\eta c_\xi(A)$. Hence, $i_\eta(c_\varepsilon(A)) = c_\eta(A)\cap i_\eta(c_\xi(A))$.

(f) The proof follows from (e).

(g) $i_\delta(c_\varepsilon(A)) = i_\delta(c_\xi(A)\cap c_\eta(A)) = i_\delta(c_\xi(A))\cap i_\delta(c_\eta(A)) = i_\delta c_\mu(A)\cap i_\delta c_\delta i_\delta(A)$, by Theorem 2.1(10) of [8] and 3.1(e) and so $i_\delta(c_\varepsilon(A)) = i_\delta c_\delta i_\delta(A)$.

(h) The proof follows from (g).

(i) $c_\varepsilon(i_\delta(A)) = c_\xi(i_\delta(A))\cap c_\eta(i_\delta(A)) = i_\mu c_\delta i_\delta(A)\cap c_\delta i_\delta(A)$, by Theorem 2.10(p) and (n), and so $c_\varepsilon(i_\delta(A)) = i_\mu c_\delta i_\delta(A)$.

(j) The proof follows from (i).

Theorem: 2.13 Let (X,μ) be a quasi-topological space and A be a subset of X. Then the following hold.

(a) $i_\delta(c_\varepsilon(A))= c_\varepsilon(i_\delta(A)) = i_\delta c_\delta i_\delta(A)$.
(b) $c_\delta(i_\varepsilon(A)) = i_\varepsilon(c_\delta(A)) = c_\delta i_\delta c_\delta(A)$.
(c) $i_\varepsilon(c_\xi(A)) = i_\xi(c_\xi(A))$.
(d) $c_\varepsilon(i_\xi(A)) = c_\xi(i_\xi(A))$.
(e) $i_\xi(c_\varepsilon(A)) = c_\xi(A)\cap c_\delta i_\delta(A)$.
(f) $c_\xi(i_\varepsilon(A)) = i_\xi(A)\cup i_\delta c_\delta(A)$.
(g) $i_\eta(c_\varepsilon(A)) = c_\varepsilon(i_\eta(A)) = i_\eta(c_\eta(A))$.
(h) $c_\eta(i_\varepsilon(A)) = i_\varepsilon(c_\eta(A)) = c_\eta(i_\eta(A))$.
(i) $i_\psi(c_\psi(A)) = c_\varepsilon(i_\eta(A)) = i_\eta(c_\eta(A))$.
(j) $c_\psi(i_\psi(A)) = i_\varepsilon(c_\eta(A)) = c_\eta(i_\eta(A))$.
(k) $i_\varepsilon(c_\varepsilon(A)) = c_\varepsilon(i_\varepsilon(A))$. (l) $c_\varepsilon(i_\varepsilon(c_\varepsilon(A)) = c_\varepsilon(i_\varepsilon(A))$.
(m) $i_\varepsilon(c_\varepsilon(A)) = i_\varepsilon(c_\varepsilon(i_\varepsilon(A)))$.

Proof:

(a) $i_\delta(c_\varepsilon(A)) = i_\delta(c_\xi(A)\cap c_\eta(A)) = i_\delta(c_\xi(A))\cap i_\delta c_\eta(A) = i_\delta c_\delta(A)\cap i_\delta c_\delta i_\delta(A)$, by Theorem 2.1(10) and 2.1(e) and so $i_\delta(c_\varepsilon(A)) = i_\delta c_\delta i_\delta(A)$.Also, $c_\varepsilon(i_\delta(A)) = c_\xi(i_\delta(A))\cap c_\eta(i_\delta(A)) = (i_\delta(A) \cup i_\mu c_\delta i_\delta(A))\cap(i_\delta(A)\cup c_\mu i_\delta i_\delta(A)) = (i_\delta(A)\cup i_\mu c_\delta i_\delta(A))\cap c_\mu i_\delta(A) = i_\delta(A)\cup i_\mu c_\delta i_\delta(A)$.

(b) The proof follows from (a).

(c) $i_\varepsilon(c_\xi(A)) = i_\xi(c_\xi(A))\cup i_\eta(c_\xi(A))$. Now $i_\eta(c_\xi(A)) = c_\xi(A)\cap i_\mu c_\delta(c_\xi(A)) = c_\xi(A)\cap i_\delta c_\delta(A)$, by 2.1(7) of [8] and so $i_\eta(c_\xi(A)) \subset c_\xi(A)\cap c_\mu i_\mu c_\delta(A) = i_\xi(c_\xi(A))$, by Theorem 2.10(l). Clearly $i_\xi(c_\xi(A)) \subset i_\varepsilon(c_\xi(A))$. Hence, $i_\varepsilon(c_\xi(A)) = i_\xi(c_\xi(A))$.

(d) The proof follows from (c).

(e) $i_\xi(c_\varepsilon(A)) = c_\varepsilon(A)\cap c_\mu i_\delta(c_\varepsilon(A)) = c_\varepsilon(A)\cap c_\mu i_\delta c_\delta i_\delta(A)$, by (a) and so $i_\xi(c_\varepsilon(A)) =(c_\xi(A))\cap c_\eta(A))\cap c_\delta i_\delta(A)= c_\xi(A)\cap c_\delta i_\delta(A)$.

(f) The proof follows from (e).

(g) $i_\eta(c_\varepsilon(A)) = c_\varepsilon(A)\cap i_\mu c_\delta(c_\varepsilon(A)) = (c_\xi(A)\cap c_\eta(A))\cap i_\mu c_\delta(c_\varepsilon(A))$. Now, $c_\xi(A)\cap i_\mu c_\delta(c_\varepsilon(A)) \supset c_\xi(A)\cap i_\mu c_\delta(A) = i_\mu c_\delta(A)$. Again, $c_\xi(A)\cap i_\mu c_\delta(c_\varepsilon(A)) \subset c_\xi(A)\cap i_\mu c_\delta(c_\eta(A)) = c_\xi(A)\cap i_\mu c_\delta(A)$, by Theorem 2.1(b) and so $c_\xi(A)\cap i_\mu c_\delta(c_\varepsilon(A)) \subset i_\mu c_\delta(A)$.

Hence, $i_\eta(c_\varepsilon(A)) = c_\eta(A)\cap i_\mu c_\delta(A) = i_\eta(c_\eta(A))$, by Theorem 2.10(c). To prove the next equality, $c_\varepsilon(i_\eta(A)) = c_\xi (i_\eta(A)) \cap c_\eta (i_\eta(A)) = i_\delta\ c_\delta(A)\cap(c_\eta(A)\cap(i_\delta c_\delta(A)\cup c_\delta i_\delta(A))$ by Theorem 2.10(b) and so $c_\varepsilon(i_\eta(A)) = i_\delta c_\delta(A)\cap c_\eta(A) = i_\eta(c_\eta(A))$, by Theorem 2.10(c).

(h) The proof follows from (g).

(i) $i_\psi(c_\varepsilon(A)) = c_\varepsilon(A)\cap c_\mu i_\mu c_\delta(c_\varepsilon(A)) = (c_\eta(A)\cap c_\xi(A))\cap c_\mu i_\mu c_\delta(A)$, since $c_\delta(c_\varepsilon(A) = c_\delta(A)$, by Theorem 2.1(m).
Therefore, $i_\psi(c_\varepsilon(A)) = c_\eta(A)\cap(c_\xi(A)\cap c_\mu i_\delta(c_\xi(A))$, by 2.1(10) of [8] and so $i_\psi(c_\varepsilon(A)) = c_\eta(A)\cap i_\xi(c_\xi(A)) = i_\eta c_\eta(A)$, by Theorem 2.10(c).

(j) The proof follows from (i).

(k) $i_\varepsilon(c_\varepsilon(A)) = i_\xi(c_\varepsilon(A))\cup i_\eta(c_\varepsilon(A)) = (c_\xi(A)\cap c_\delta i_\delta(A))\cup i_\eta(c_\eta(A))$, by (e) and (g) and so, $i_\varepsilon(c_\varepsilon(A)) = c_\xi(A) \cap c_\delta i_\delta(A)) \cup (A \cup c_\delta i_\delta (A)) \cap i_\delta c_\delta(A)$, by Theorem 2.10(c). Therefore, $i_\varepsilon(c_\varepsilon(A)) = ((A\cup i_\mu c_\delta(A))\cap c_\mu i_\delta(A)) \cup((A\cup c_\mu i_\delta(A))\cap i_\mu c_\delta(A)) = ((A\cap c_\mu i_\delta(A))\cup(i_\mu c_\delta(A)\cap c_\mu i_\delta(A))\cup((A\cap i_\mu c_\delta(A))\cup(c_\mu i_\delta(A)\cap i_\mu c_\delta(A))=(i_\xi(A)\cup(i_\mu c_\delta(A)\cap c_\mu i_\delta(A)))\cup(i_\eta(A)\cup(i_\mu c_\delta(A)\cap c_\mu i_\delta(A))) =(i_\xi(A)\cup i_\eta(A))\cup(i_\mu c_\delta(A)\cap c_\mu i_\delta(A))$. Again, $c_\varepsilon i_\varepsilon(A)= c_\xi(i_\varepsilon(A))\cap c_\eta(i_\varepsilon(A)) = (i_\xi(A)\cup i_\mu c_\delta(A))\cap c_\eta i_\eta(A)$, by (f) and (h) and so $c_\varepsilon i_\varepsilon(A) = (i_\xi(A)\cup i_\mu c_\delta(A))\cap(i_\eta(A)\cup c_\mu i_\delta(A))$, by Theorem 2.10 (b) and so $c_\varepsilon i_\varepsilon(A) = (i_\xi(A)\cup i_\eta(A))\cup(c_\mu i_\delta(A)\cap i_\mu c_\delta(A))$.

Hence, $i_\varepsilon(c_\varepsilon(A)) = c_\varepsilon i_\varepsilon(A)$.

(l) $c_\varepsilon(i_\varepsilon(c_\varepsilon(A)) = c_\varepsilon(c_\varepsilon(i_\varepsilon(A)))$, by (k) and so $c_\varepsilon(i_\varepsilon(c_\varepsilon(A)) = c_\varepsilon(i_\varepsilon(A))$.

(m) By (k), $c_\varepsilon(i_\varepsilon(A)) = i_\varepsilon(c_\varepsilon(A))$. Hence, $i_\varepsilon c_\varepsilon(i_\varepsilon(A))= i_\varepsilon(c_\varepsilon(A))$.

Theorem: 2.14 Let (X,μ) be a quasi-topological space and A be a subset of X, then the following statements are equivalent.

(a) A is ε-open.

(b) $A = i_\eta(A) \cup i_\xi(A)$.
(c) $A \subset c_\eta(i_\eta(A))$.

Proof:
(a) ⇒ (b). If A is ε-open, then $A \subset i_\mu c_\delta(A) \cup c_\mu i_\delta(A)$. Now, $A = i_\eta(A) \cup i_\xi(A) = (A \cap i_\mu c_\delta(A)) \cup (A \cap c_\mu i_\delta(A)) = A \cap (i_\mu c_\delta(A) \cup c_\mu i_\delta(A)) = A$. Hence, $A = i_\eta(A) \cup i_\xi(A)$.

(b) ⇒ (c). If $A = i_\eta(A) \cup i_\xi(A)$, then $A = i_\eta(A) \cup (A \cap c_\mu i_\delta(A)) \subset i_\eta(A) \cup c_\mu i_\delta(A) = c_\eta i_\eta(A)$, by Theorem 2.10.(b).

Hence $A \subset c_\eta i_\eta(A)$.

(c) ⇒ (a). $A \subset c_\eta i_\eta(A)$ implies that $A \subset i_\eta(A) \cup c_\mu i_\delta(i_\eta(A)) = i_\eta(A) \cup c_\mu i_\delta(A)$, by Theorem 2.1(a) and so $A \subset (A \cap i_\mu c_\delta(A)) \cup c_\mu i_\delta(A) \subset i_\mu c_\delta(A) \cup c_\mu i_\delta(A)$ and so A is ε-open.

Corollary: 2.15 Let (X,μ) be a quasi-topological space, then $\xi(\eta(\delta)) = \varepsilon(\delta)$.

Proof: The Proof follows from the Theorem 2.8(c).

Theorem: 2.16 Let (X,μ) be a quasi-topological space and A be a subset of X, then the following hold.
(a) $c_\varepsilon(A) = c_\xi(A) \cap c_\eta(A)$.
(b) $i_\varepsilon(A) = i_\xi(A) \cup i_\eta(A)$.

Proof:
(a) Since $c_\varepsilon(A) \subset c_\xi(A)$ and $c_\varepsilon(A) \subset c_\eta(A)$, we have $c_\varepsilon(A) \subset c_\xi(A) \cap c_\eta(A)$.
Again, $c_\xi(A) \cap c_\eta(A) = (A \cup i_\mu c_\delta(A)) \cap (A \cup c_\mu i_\delta(A)) = A \cup (i_\mu c_\delta(A) \cap c_\mu i_\delta(A)) \subset A \cup (i_\mu c_\delta(c_\varepsilon(A)) \cap c_\mu i_\delta(c_\varepsilon(A))) \subset A \cup c_\varepsilon(A) = c_\varepsilon(A)$.
Hence $c_\varepsilon(A) = c_\xi(A) \cap c_\eta(A)$.

(b) The proof follows from (a).

Theorem: 2.17 Let (X,μ) be a quasi-topological space and A be a subset of X, then the following statements are equivalent.
(a) $A \in \psi(\delta)$.
(b) $A \subset i_\psi(c_\psi(A))$.
(c) $A \subset i_\xi(c_\xi(A))$.

Proof:
(a) ⇒ (b). If $A \in \psi(\delta)$, then $A = i_\psi(A) \subset i_\psi(c_\psi(A))$.
(b) ⇒ (c). The proof follows from Theorem 2.10(s).
(c) ⇒ (a). $A \subset i_\xi(c_\xi(A))$ implies that $A \subset c_\xi(A) \cap c_\mu i_\delta(c_\xi(A)) = c_\xi(A) \cap c_\mu i_\mu c_\delta(A)$, by Theorem 2.1(10) of [8] and so $A \subset c_\mu i_\mu c_\delta(A)$. Hence $A \in \psi(\delta)$.

Theorem: 2.18 Let (X,μ) be a quasi-topological space and A be a subset of X, then the following hold.
(a) $\xi(\eta(\delta)) = \varepsilon(\delta)$.

(b) $\xi(\eta(\delta)) = \psi(\eta(\delta)) = \varepsilon(\delta)$.

Proof:
(a) Suppose $A \in \xi(\eta(\delta))$. Then, $A \subset c_\eta(i_\eta(A))$ which implies that $A \subset (A \cap i_\mu\ c_\delta(A)) \cup c_\mu i_\delta(A)$, by Theorem 2.1(a) which in turn implies that $A \subset i_\mu c_\delta(A) \cup c_\mu\ i_\delta(A)$ and so $A \in \varepsilon(\delta)$. Hence, $\xi(\eta(\delta)) \subset \varepsilon(\delta)$.

Conversely suppose, $A \in \varepsilon(\delta)$. $A \in \varepsilon(\delta)$ if and only if $A \subset i_\mu c_\delta(A) \cup c_\mu i_\delta(A)$ if and only if $A = A \cap (i_\mu c_\delta(A) \cup c_\mu i_\delta(A)) = (A \cap i_\mu c_\delta(A)) \cup (A \cap c_\mu i_\delta(A)) \subset (A \cap i_\mu c_\delta(A)) \cup c_\mu i_\delta(A) = c_\eta(i_\eta(A))$, by Theorem 2.10(b) and so $A \in \xi(\eta(\delta)$ which implies that $\xi(\eta(\delta) \supset \varepsilon(\delta)$. This proves (a).

(b) $\xi(\eta(\delta)) = \psi(\eta(\delta))$, by Theorem 2.10(i) and each is equal to $\varepsilon(\delta)$, by (a).

Theorem: 2.19 Let V be a subset of a space (X,μ). Then the following hold.
(a) V is η-open if and only if $V \subset i_\eta(c_\eta(V)$.

(b) V is ε-open if and only if $V \subset c_\eta(i_\eta(V)$.

Proof:

(a) Let V be η-open. Then $i_\eta(V) = V$ and so $V \subset i_\eta(c_\eta(V))$. Also, $V \subset i_\eta(c_\eta(V)) \subset i_\eta(c_\delta(V)) = c_\delta(V) \cap i_\mu c_\delta(c_\delta(V)) = c_\delta(V) \cap i_\mu c_\delta(V) = i_\mu c_\delta(V)$ and so V is η- open.

(b) Let V be ε-open. Then $V \subset c_\mu i_\delta(V) \cup i_\mu c_\delta(V)$ and so $V = (c_\mu i_\delta(V) \cup i_\mu c_\delta(V)) \cap V = (c_\mu i_\delta(V) \cap V) \cup (i_\mu c_\delta(V) \cap V) \subset i_\eta(V) \cup c_\mu i_\delta(V) = i_\eta(V) \cup c_\mu i_\delta(i_\eta(V)$, by Theorem 2.1(a) and so $V \subset c_\eta(i_\eta(V))$.

Conversely, suppose $V \subset c_\eta(i_\eta(V)) = i_\eta(V) \cup c_\mu(i_\delta(i_\eta(V))) = (V \cap i_\mu c_\delta(V)) \cup c_\mu(i_\delta(V))$, by Theorem 2.1(a) and so $V \subset i_\mu c_\delta(V) \cup c_\mu i_\delta(V)$. Hence, V is ε- open.

We define the following new families of generalized topologies.
$\varepsilon(\nu(\delta)) = \{A \mid A \subset i_\nu(c_\nu(A)) \cup c_\nu(i_\nu(A))\}$,
$\varepsilon(\xi(\delta)) = \{ A \mid A \subset i_\xi(c_\xi(A)) \cup c_\xi(i_\xi(A))\}$,
$\varepsilon(\eta(\delta)) = \{ A \mid A \subset i_\eta(c_\eta(A)) \cup c_\eta(i_\eta(A))\}$,
$\varepsilon(\psi(\delta)) = \{ A \mid A \subset i_\psi(c_\psi(A)) \cup c_\psi(i_\psi(A))\}$,
$\varepsilon(\varepsilon(\delta)) = \{A \mid A \subset i_\varepsilon(c_\varepsilon(A)) \cup c_\varepsilon(i_\varepsilon(A))\}$,
$\nu(\varepsilon(\delta)) = \{A \mid A \subset i_\varepsilon(c_\varepsilon(i_\varepsilon(A)))\}$,
$\xi(\varepsilon(\delta)) = \{A \mid A \subset c_\varepsilon(i_\varepsilon(A))\}$,
$\eta(\varepsilon(\delta)) = \{A \mid A \subset i_\varepsilon(c_\varepsilon(A))\}$ and
$\psi(\varepsilon(\delta)) = \{A \mid A \subset c_\varepsilon(i_\varepsilon(c_\varepsilon(A)))\}$.

The following Theorem 2.20 gives the relations between the above generalized topologies. Theorem 2.20(a) shows that Theorem 3.6.5 of [4] is true for the generalized topology of all δ-open sets, if $\gamma \in \Gamma_4$. Theorem 2.20(b) shows that Lemma 3.8 of [4] is true for the generalized topology of all δ-open sets, if (X,μ) is a quasi-topological space.

Theorem: 2.20 If (X,μ) is a quasi-topological space, then the following hold.
(a) $\xi(\eta(\delta)) = \psi(\eta(\delta)) = \varepsilon(\eta(\delta)) = \varepsilon(\delta)$.
(b) $\nu(\varepsilon(\delta)) = \xi(\varepsilon(\delta)) = \eta(\varepsilon(\delta)) = \varepsilon(\delta)$.

Proof: (a) $\varepsilon(\eta(\delta)) = \varepsilon(\delta)$, by Theorem 2.10(b) and (h), and so (a) follows from Theorem 2.18(b).

(b) $\xi(\varepsilon(\delta)) = \eta(\varepsilon(\delta)) = \varepsilon(\varepsilon(\delta))$by Theorem 2.13(k). $\xi(\varepsilon(\delta)) = \psi(\varepsilon(\delta))$, by Theorem 2.13(l). $\nu(\varepsilon(\delta)) = \eta(\varepsilon(\delta))$, by Theorem 2.13(m). Now, $\nu(\varepsilon(\delta)) = \nu(\xi(\eta(\delta)))$, by Theorem 2.18(a). By Theorem 2.3 of [4], $\nu(\xi(\eta(\delta))) = \xi(\eta(\delta)) = \varepsilon(\delta)$, and so $\nu(\varepsilon(\delta)) = \varepsilon(\delta)$. Hence (b) follows.

3. CHARACTERIZATIONS OF SOME GENERALIZED OPEN SETS

In this section, we characterize some of the family of generalized open sets mentioned above by the interior and closure operators.

Theorem: 3.1 If (X,μ) is a quasi-topological space and A be a subset of X, then the following are equivalent.
(a) A is ν-open.
(b) $i_\nu i_\xi(A) = A$.
(c) $i_\nu i_\eta(A) = A$.
(d) $i_\nu i_\psi(A) = A$.
(e) $A \subset i_\nu c_\eta(A)$.
(f) $A \subset i_\nu c_\psi(A)$.
(g) $c_\xi i_\nu(A) = i_\delta c_\delta i_\delta(A)$.
(h) $A \subset c_\psi i_\nu(A)$. (i) $i_\nu i_\varepsilon(A) = A$.

Proof: (a) and (b) are equivalent by Theorem 2.2(a).
(a) and (c) are equivalent by Theorem 2.2(b).
(a) and (d) are equivalent by Theorem 2.2(c).
(a) and (e) are equivalent by Theorem 2.2(e).
(a) and (f) are equivalent by Theorem 2.2(f).
(a) and (g) are equivalent by Theorem 2.5(b).
(a) and (h) are equivalent by Theorem 2.7(b).
(a) and (i) are equivalent by Theorem 2.8(e).

Theorem: 3.2 If (X,μ) is a quasi-topological space and A be a subset of X, then the following are equivalent.
(a) A is ξ-open.(b) $A \subset c_\nu i_\xi(A)$. (c) $A \subset c_\nu i_\delta(A)$. (d) $A \subset c_\delta i_\nu(A)$. (e) $i_\xi i_\varepsilon(A) = A$.

Proof:
(a) and (b) are equivalent by Theorem 2.2(g).
(a) and (c) are equivalent by Theorem 2.3(c).
(a) and (d) are equivalent by Theorem 2.3(c).
(a) and (e) are equivalent by Theorem 2.8(c).

Theorem: 3.3 If (X,μ) is a quasi-topological space and A be a subset of X, then the following are equivalent.

(a) A is η-open.
(b) $A \subset i_\nu c_\xi(A)$.
(c) $A \subset i_\nu c_\delta(A)$.
(d) $A \subset i_\delta c_\nu(A)$.
(e) $A \subset i_\eta c_\nu(A)$.
(f) $i_\eta i_\psi(A) = A$.
(g) $i_\eta i_\varepsilon(A) = A$.

Proof:
(a) and (b) are equivalent by Theorem 2.2(d).
(a) and (c) are equivalent by Theorem 2.3(d).
(a) and (d) are equivalent by Theorem 2.3(d).
(a) and (e) are equivalent by Theorem 2.6(a).
(a) and (f) are equivalent by Theorem 2.6(c).
(a) and (g) are equivalent by Theorem 2.8(u).

Theorem: 3.4 If (X,μ) is a quasi-topological space and A be a subset of X, then the following are equivalent.
(a) A is ε-open. (b) $i_\varepsilon i_\psi(A) = A$. ‘

Proof: (a) and (b) are equivalent by Theorem 2.8(g).

Theorem: 3.5 If (X,μ) is a quasi-topological space and A be a subset of X, then the following are equivalent.

(a) A is ψ-open.
(b) $A \subset c_\nu i_\psi(A)$.
(c) $A \subset i_\xi c_\nu(A)$.
(d) $A \subset i_\psi c_\nu(A)$.
(e) $A \subset i_\varepsilon c_\nu(A)$.
(f) $i_\psi i_\varepsilon(A) = A$.

Proof:
(a) and (b) are equivalent by Theorem 2.2(i).
(a) and (c) are equivalent by Theorem 2.5(a).
(a) and (d) are equivalent by Theorem 2.7(a).
(a) and (e) are equivalent by Theorem 2.8(i).
(a) and (f) are equivalent by Theorem 2.8(t).

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