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NONSINGULAR PQ-INJECTIVE MODULES

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ABSTRACT

Let M be a right R-module. A right R-module N is called nonsingular principally M-injective (briefly, nonsingular PM-injective) if, for each $m \in M \setminus Z(M)$, any R-homomorphism from mR to N can be extended to an R-homomorphism from M to N. M is called nonsingular principally quasi –injective (briefly, nonsingular PQ-injective) if, it is nonsingular PM-injective. In this paper, we give some characterizations and properties of nonsingular PQ-injective modules.

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1. INTRODUCTION

Let R be a ring. A right R-module M is called *principally injective* (or $P-$*injective*), if every $R-$homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$ where l and r are left and right annihilators, respectively. This notion was introduced by Camillo [2] for commutative rings. In [8], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. Nicholson, Park, and Yousif [9] extended this notion of principally injective rings to the one for modules. In [5], W. Junchao introduced the definition of Jcp-injective rings, a ring R is called right Jcp-injective if for each $a \in R \setminus Z_r$, any R-homomorphism from aR to R can be extended to an R-homomorphism from R to R. A right R-module M is called *almost mininjective* [11] if, for any simple right ideal kR of R, there exists an S-submodule X_k of M such that $l_M(r_R(m)) = Mk \oplus X_k$ as left S-modules. A ring R is called *right almost* mininjective if R_R is almost mininjective. In this note we introduce the definition of nonsingular PQ-injective modules and give some characterizations and properties.

Throughout this paper, R will be an associative ring with identity and all modules are unitary right $R-$modules. For right $R-$modules M and N, $Hom_R(M,N)$ denotes the set of all $R-$homomorphisms from M to N and $S = End_R(M)$ denotes the endomorphism ring of M. If X is a subset of M the right (resp. left) annihilator of X in R (resp. S) is denoted by $r_R(X)$ (resp. $l_S(X)$). By notation $N \subset^{\oplus} M$ $(N \subset^{e} M)$ we mean that N is a direct summand (an essential submodule) of M. We denote the singular submodule of M by $Z(M)$.

2. NONSINGULAR PM-INJECTIVE MODULES

Recall that a submodule K of a right $R-$module M is *essential* (or *large*) in M if, every nonzero submodule L of M, we have $K \cap L \neq 0$. An element $m \in M$ is called *singular* if $r_R(m) \subset^{e} R$. M is called *nonsingular* if it contains no nontrivial singular element.

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Definition 2.1: Let M be a right R-module. A right R-module N is called *nonsingular principally* M*-injective* (briefly, *nonsingular* PM*-injective*) if, for each $m \in M \setminus Z(M)$, any R-homomorphism from mR to N can be extended to an R-homomorphism from M to N.

Lemma 2.2: Let M and N be right $R-$modules. Then N is nonsingular PM-injective if and only if for each $m \in M \setminus Z(M)$,

$$Hom_R(M,N)m = l_N r_R(m).$$

Proof: Clearly, $Hom_R(M,N)m \subset l_N r_R(m)$.

Let $x \in l_N r_R(m)$. Define $\varphi : mR \to xR$ by $\varphi(mr) = xr$ for every $r \in R$. Since $r_R(m) \subset r_R(x)$, φ is well-defined. It is clear that φ is an $R-$homomorphism. Since N is nonsingular PM-injective, there exists an $R-$homomorphism $\hat{\varphi} : M \to N$ such that $\hat{\varphi}\iota_1 = \iota_2\varphi$, where $\iota_1 : mR \to M$ and $\iota_2 : xR \to N$ are the inclusion maps. Hence $x = \varphi(m) = \hat{\varphi}(m) \in Hom_R(M,N)m$.

Conversely, let $m \in M \setminus Z(M)$, and $\varphi : mR \to N$ be an $R-$homomorphism. Then $\varphi(m) \in l_N r_R(m)$ so by assumption, we have $\varphi(m) = \hat{\varphi}(m)$ for some $\hat{\varphi} \in Hom_R(M,N)$. This shows that N is nonsingular PM-injective.

Example 2.3: Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field.

(1) If $M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and $N_R = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, then N is not nonsingular PM-injective.

(2) If $M_R = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, then N is nonsingular PM-injective.

Proof: (1) It is clear that only $\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ are nonzero nonessential principal right ideals of R. Let $m = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in M$ with $x \neq 0$ or $y \neq 0$. Then $m \in M \setminus Z(M)$ and that nonzero submodules mR of M may be $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ or M. It is clear that $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. For any $R-$homomorphism

$\varphi : \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$ with $\varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$ for some $x \in F$,

$\varphi\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for every $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, hence $\varphi = 0$. Then N is not nonsingular PM-injective.

(2) For $M_R = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, let $m = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \in M$ where $x \neq 0$ or $y \neq 0$. Then $r_R(m) = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ is a nonessential right ideal of R and mR may be $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$ or M.

Let $\alpha : \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ be an R-homomorphism.

Then there exists $x_1, x_2 \in F$ such that $\alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$.

Hence

$$\alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix}.$$

It follows that $x_1 = 0$.

Define $\hat{\alpha}: M \to N$ by $\hat{\alpha}\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & ax_2 \\ 0 & 0 \end{pmatrix}$ for every $a, b \in F$.

It is clear that $\hat{\alpha}$ is an $R-$homomorphism and

$\hat{\alpha}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \hat{\alpha}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix}$. This shows that $\hat{\alpha}$ is an extension of α.

Then N is nonsingular PM-injective.

Lemma 2.4:

(1) N is nonsingular PM-injective if and only if N is nonsingular PX-injective for any submodule X of M.

(2) $\oplus_{i=1}^{n} N_i$ is nonsingular PM-injective if and only if N_i is nonsingular PM- injective for all i.

(3) If $m \in M \setminus Z(M)$ and mR is nonsingular PM-injective, then $mR \subset^{\oplus} M$.

Proof:

(1) The sufficiency is trivial. For the necessity, let $x \in X \setminus Z(X)$, and $\varphi: xR \to N$ be an $R-$homomorphism. Since $x \in M \setminus Z(M)$, there exists an $R-$homomorphism $\hat{\varphi}: M \to N$ such that $\varphi = \hat{\varphi}\iota_2\iota_1$ where $\iota_1: xR \to X$ and $\iota_2: X \to M$ are the inclusion maps. Then $\hat{\varphi}\iota_2$ extends φ.

(2) The necessity is trivial. For the sufficiency, let $m \in M \setminus Z(M)$, and $\varphi: mR \to \oplus_{i=1}^{n} N_i$ be an $R-$homomorphism. Then for each i, there exists $R-$homomorphisms $\varphi_i: M \to N_i$ such that $\varphi_i\iota = \pi_i\varphi$ where $\pi_i: \oplus_{i=1}^{n} N_i \to N_i$ is the projection map, and $\iota: mR \to M$ is the inclusion map. Put $\hat{\varphi} = \iota_1\varphi_1 + \ldots + \iota_n\varphi_n : M \to \oplus_{i=1}^{n} N_i$. Then it is clear that $\hat{\varphi}$ extends φ.

(3) Since mR is nonsingular PM-injective, there exists an $R-$homomorphism $\varphi: M \to mR$ such that $\varphi\iota = 1_{mR}$ where $\iota: mR \to M$ is the inclusion map. Then by[1, Lemma 5.1], ι is a split monomorphism, therefore $mR \subset^{\oplus} M$.

Theorem 2.5: The following conditions are equivalent for a projective module M.

(1) Every $m \in M \setminus Z(M)$, mR is projective.

(2) Every factor module of a nonsingular PM-injective module is *nonsingular* PM- injective.

(3) Every factor module of an injective $R-$module is nonsingular PM-injective.

Proof:

$(1) \Rightarrow (2)$: Let N be a nonsingular PM-injective module, X a submodule of N, $m \in M \setminus Z(M)$, and $\varphi: mR \to N/X$ be an $R-$homomorphism. Then by (1), there exists an $R-$homomorphism $\hat{\varphi}: mR \to N$ such that $\varphi = \eta\hat{\varphi}$ where $\eta: N \to N/X$ is the natural $R-$epimorphism. Since N is nonsingular PM-injective, there exists an $R-$homomorphism $\beta: M \to N$ which is an extension of $\hat{\varphi}$ to M. Then $\eta\beta$ is an extension of φ to M.

$(2) \Rightarrow (3):$ is clear.

$(3) \Rightarrow (1):$ Let $m \in M \setminus Z(M)$, $h: A \to B$ an $R-$epimorphism, and let $\alpha: mR \to B$ be an $R-$homomorphism. Embed A in an injective module E [1, 18.6]. Let $\sigma: B \to A/Ker(h)$ be an $R-$isomorphism. Since $E/Ker(h)$ is nonsingular PM-injective, there exists an $R-$homomorphism $\hat{\alpha}: M \to E/Ker(h)$ such that $\iota_1\sigma\alpha = \hat{\alpha}\iota_2$ where $\iota_1: A/Ker(h) \to E/Ker(h)$ and $\iota_2: mR \to M$ are the inclusion maps. Since M is projective, $\hat{\alpha}$ can be lifted to $\beta: M \to E.$ Let $x \in mR.$ Then $\sigma\alpha(x) = a + Ker(h)$ for some $a \in A,$ so $\beta(x) + Ker(h) = \eta\beta(x) = \hat{\alpha}(x) = \sigma\alpha(x) = a + Ker(h)$ where $\eta: E \to E/Ker(h)$ is the natural $R-$epimorphism. Hence $\beta(x) - a \in Ker(h) \subset A$ so $\beta(x) \in A.$ This shows that $\beta(mR) \subset A.$ Therefore we have lifted $\alpha.$

3. NONSINGULAR PQ-INJECTIVE MODULES

A right R-module M is called *nonsingular principally quasi –injective* (briefly, *nonsingular* PQ*-injective*) if, it is nonsingular PM-injective.

Lemma 3.1: Let M be a right $R-$module and $S = End_R(M).$ Then the following conditions are equivalent.

(1) M is nonsingular PQ-injective.

(2) $l_M r_R(m) = Sm$ for each $m \in M \setminus Z(M).$

(3) $r_R(m) \subset r_R(n),$ where $m, n \in M$ with $m \in M \setminus Z(M),$ implies that $Sn \subset Sm.$

(4) $l_M(r_R(m) \cap aR) = l_M(a) + Sm$ for all $a \in R$ and $m \in M$ with $ma \in M \setminus Z(M).$

(5) If $\alpha: mR \to M$ is an $R-$homomorphism, $m \in M \setminus Z(M),$ then $\alpha(m) \in Sm.$

Proof:

$(1) \Leftrightarrow (2):$ by Lemma 2.2

$(2) \Rightarrow (3):$ If $r_R(m) \subset r_R(n),$ where $m, n \in M$ with $m \in M \setminus Z(M),$ then $l_M r_R(n) \subset l_M r_R(m).$ Then $Sn \subset l_M r_R(n) \subset l_M r_R(m) = Sm$ by (2).

$(3) \Rightarrow (4):$ Let $a \in R$ and $m \in M$ with $ma \in M \setminus Z(M)$ and let $x \in l_M(r_R(m) \cap aR).$ Then $r_R(ma) \subset r_R(xa),$ and hence by (3), $Sxa \subset Sma.$ Thus $xa = \varphi(ma),$ $\varphi \in S$ and so $(x - \varphi(m)) \in l_M(a).$ It follows that $x \in l_M(a) + Sm.$ The other hand is clear.

$(4) \Rightarrow (5):$ Put $a = 1_R$ in (4), then $\alpha(m) \in l_M r_R(m) = l_R(r_R(m) \cap 1R) = l_M(1_R) + Sm = Sm$ because $m1 \in M \setminus Z(M).$

$(5) \Rightarrow (1):$ Let $m \in M$ with $m \in M \setminus Z(M)$ and let $\varphi: mR \to M$ be an $R-$homomorphism. Then by (5), $\varphi(m) \in Sm$ so there exists an $R-$homomorphism $\hat{\varphi} \in S$ is an extension of φ to $M.$

Theorem 3.2: Let M be a nonsingular PQ-injective module and $m, n \in M$ with $m \in M \setminus Z(M).$

(1) If mR embeds into $nR,$ then Sm is an image of $Sn.$

(2) If nR is an image of $mR,$ then Sn can be embedded into $Sm.$

(3) If $mR \simeq nR,$ then $Sm \simeq Sn.$

Proof:

(1) Let $\sigma: mR \to nR$ be an $R-$monomorphism and let $\iota_1: mR \to M$ and $\iota_2: nR \to M$ be the inclusion maps. Since M is nonsingular PQ-injective, there exists an $R-$homomorphism $\hat{\sigma}: M \to M$ such that $\hat{\sigma}\iota_1 = \iota_2\sigma.$ Let $\varphi: Sn \to Sm$ defined by $\varphi(\alpha(n)) = \alpha\hat{\sigma}(m)$ for every $\alpha \in S$. Since $\varphi(\alpha(n)) = \alpha(\hat{\sigma}(m)) = \alpha(\sigma(m)) \in \alpha(nR),$ φ is well-defined.It is clear that φ is an S-homomorphism.

Since σ is monic, $r_R(\sigma(m)) \subset r_R(m)$ and $\sigma(m) \in M \setminus Z(M)$ and hence by Lemma 3.1, $Sm \subset S\sigma(m).$ Then $m \in S\sigma(m) \subset \varphi(Sn).$

(2) By the same notations as in (1), let $\sigma : mR \to nR$ be an $R-$epimorphism. Write $\sigma(ms) = n,\ s \in R.$ Since M is nonsingular PQ-injective, σ can be extended to $\hat{\sigma} : M \to M$ such that $\hat{\sigma}\iota_1 = \iota_2\sigma.$ Define $\varphi : Sn \to Sm$ by $\varphi(\alpha(n)) = \alpha\hat{\sigma}(ms)$ for every $\alpha \in S$. It is clear that φ is an $S-$homomorphism. If $\alpha(n) \in Ker(\varphi)$, then $0 = \varphi(\alpha(n)) = \alpha\hat{\sigma}(ms) = \alpha(n).$ This shows that φ is an $S-$monomorphism.

(3) Follows from (1) and (2).

Recall that a right R-module M is called $C2$ [6] if, every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of $M.$ M is called $C3$ if, whenever N and K are direct summands of M with $N \cap K = 0$ then $N \oplus K$ also a direct summand of $M.$

Theorem 3.3: Let $M = mR$ be a principle, nonsingular PQ-injective module.

(1) If $X \simeq e(mR),\ e^2 = e \in S$ and $e(m) \in M \setminus Z(M),$ then $X = g(mR),$ for some $g^2 = g \in S.$

(2) If $e(mR) \cap f(mR) = 0,\ e^2 = e \in S,\ f^2 = f \in S$ and $f(m) \in M \setminus Z(M),$ then $e(M) \oplus f(M) = g(M),$ for some $g^2 = g \in S.$

Proof:

(1) Let $\sigma : e(mR) \to X$ be an $R-$isomorphism. Write $\sigma e(m) = x$ where $x \in X$ so $xR = X.$ We must show that $xR = g(mR),$ for some $g^2 = g \in S.$ Then by Lemma 2.3, we have $e(mR)$ is nonsingular PM-injective and hence xR is also nonsingular PM-injective. Since $\sigma e(m) \in M \setminus Z(M),$ $xR \subset^{\oplus} mR$ Lemma 2.4.

(2) Let $e(mR) \cap f(mR) = 0,\ e^2 = e \in S,\ f^2 = f \in S$ and $f(m) \in M \setminus Z(M).$ Then $e(mR) \oplus f(mR) = e(mR) \oplus (1-e)f(mR).$ If $(1-e)f(mR) = 0,$ then $e(mR) \oplus f(mR)$ is a direct summand of $M.$ If $(1-e)f(mR) \neq 0,$ then $(1-e)f(mR) \simeq f(mR),$ and hence $(1-e)f(mR) = g(mR)$ for some $g^2 = g \in S$ by (1). Let $h = e + g - ge,$ then $h^2 = h$ and $e(M) \oplus f(M) = h(M).$

Lemma 3.4: Let M be a nonsingular PQ-injective module and $S = End_R(M).$ If $\alpha \in S$ with $\alpha(M) \subset M \setminus Z(M),$ then $l_S(Ker(\alpha) \cap mR) = l_S(m) + S\alpha.$

Proof: Clearly, $l_S(m) + S\alpha \subset l_S(Ker(\alpha) \cap mR).$ Let $\beta \in l_S(Ker(\alpha) \cap mR).$ Then $r_R(\alpha(m)) \subset r_R(\beta(m)),$ so $l_M r_R(\beta(m)) \subset l_M r_R(\alpha(m)).$ Since $\alpha(m) \in M \setminus Z(M), S\beta(m) \subset l_M r_R(\beta(m)) \subset l_M r_R(\alpha(m)) = S\alpha(m)$ by Lemma 3.1, so $\beta(m) = s\alpha(m)$ for some $s \in S.$ It follows that $(\beta - s\alpha) \in l_S(m),$ and hence $\beta \in l_S(m) + S\alpha.$

Following [8], a right R-module M is called a *principal self-generator,* if every element $m \in M$ has the form $m = \gamma(m_1)$ for some $\gamma : M \to mR.$ If $uR \neq 0$ is uniform, we call u a *uniform element* of $M.$ We call a right R-module M is a *duo module* if every submodule of M is fully invariant.

Theorem 3.5: Let M be a principal module which is a principal self-generator. Then the following conditions are equivalent.

(1) M is nonsingular PQ-injective.

(2) $l_S(Ker(\alpha) \cap mR) = l_S(m) + S\alpha$ for all $m \in M$ and $\alpha \in S$ with $\alpha(M) \in M \setminus Z(M).$

(3) $l_S(Ker(\alpha)) = S\alpha$ for all $\alpha \in S$ with $\alpha(M) \in M \setminus Z(M).$

(4) $Ker(\alpha) \subset Ker(\beta),$ where $\alpha, \beta \in S$ with $\alpha(m) \in M \setminus Z(M),$ implies that $S\beta \subset S\alpha.$

Proof:

$(1) \Rightarrow (2)$: by Lemma 3.4.

$(2) \Rightarrow (3)$: If $M = m_0R,$ take $m = m_0$ in (2).

$(3) \Rightarrow (4)$: $\text{Ker}(\alpha) \subset \text{Ker}(\beta)$, then $l_S(\text{Ker}(\beta)) \subset l_S(\text{Ker}(\alpha))$. It follows that

$$S\beta \subset l_S(\text{Ker}(\beta)) \subset l_S(\text{Ker}(\alpha)) = S\alpha.$$

$(4) \Rightarrow (1)$: Let $m \in M \setminus Z(M)$, $\varphi : mR \to M$ be an $R-$homomorphism.

Since M is a principal self-generator, there exists $\beta \in S$ such that $\beta(m_1) = m$, so $\text{Ker}(\beta) \subset \text{Ker}(\varphi\beta)$ and $\beta(M) \subset M \setminus Z(M)$. Then by (4), $S\varphi\beta \subset S\beta$ hence $\varphi\beta = \hat{\varphi}\beta$ for some $\hat{\varphi} \in S$. This shows that $\hat{\varphi}$ is an extension of φ.

Theorem 3.6: Let M be a duo, nonsingular PQ-injective module. If u a *uniform element* of M with $u \in M \setminus Z(M)$, then $M_u = \{\alpha \in S \mid \text{Ker}(\alpha) \cap uR \neq 0\}$ is a unique maximal left ideal of S containing $l_S(u)$.

Proof: Since uR is uniform, M_u is a left ideal of S. It is clear that $l_S(u) \subset M_u \neq S$. Let X be a left ideal of S containing $l_S(u)$ and $X \neq S$. If $\alpha \in X - M_u$, then $\text{Ker}(\alpha) \cap uR = 0$. Since M is a duo module, $\alpha(u)R \subset M \setminus Z(M)$ and by Lemma 3.4 we have $S = l_S(\text{Ker}(\alpha) \cap uR) = l_S(u) + S\alpha \subset X$ a contradiction. Thus $X \subset M_u$.

Definition 3.7: Let M be a right R-module, $S = \text{End}_R(M)$. The module M is called *almost nonsingular* PQ-*injective* if, for each $m \in M \setminus Z(M)$, there exists an S-submodule X_m of M such that $l_M(r_R(m)) = Sm \oplus X_m$ as left S-modules.

Theorem 3.8: Let M be a right R-module, $S = \text{End}_R(M)$ and $m \in M \setminus Z(M)$.

(1) If $\text{Hom}_R(mR, M) = S \oplus Y$ as left S-modules, then $l_M(r_R(m)) = Sm \oplus X$ as left S- modules, where $X = \{f(m) : f \in Y\}$.

(2) If $l_M(r_R(m)) = Sm \oplus X$ for some $X \subset M$ as left S- modules, then we have $\text{Hom}_R(mR, M) = S \oplus Y$ as left S- modules, where $Y = \{f \in \text{Hom}_R(mR, M) : f(m) \in X\}$.

(3) Sm is a direct summand of $l_M(r_R(m))$ as left S- modules if and only if S is a direct summand of $\text{Hom}_R(mR, M)$ as left S- modules.

Proof: Define $\theta : \text{Hom}_R(mR, M) \to l_M(r_R(m))$ by $\theta(f) = f(m)$ for every $f \in \text{Hom}_R(mR, M)$. It is obvious that θ is an S-monomorphism. For $x \in l_M(r_R(m))$, define $g : mR \to M$ by $g(mr) = xr$ for every $r \in R$. Since $r_R(m) \subset r_R(x)$, g is well-defined, so it is clear that g is an R-homomorphism. Then $\theta(g) = g(m) = x$. Therefore θ is an S-isomorphism. Let $\alpha(m) \in Sm$. Since $\alpha(m) \in l_M(r_R(m))$, there exists $\varphi \in \text{Hom}_R(mR, M)$ such that $\theta(\varphi) = \alpha(m)$, so $\varphi(m) = \alpha(m)$. Define $\hat{\varphi} : M \to M$ by $\hat{\varphi}(x) = \alpha(x)$ for every $x \in M$. It is clear that $\hat{\varphi}$ is an R-homomorphism and is an extension of φ. Then $\alpha(m) = \hat{\varphi}(m) = \theta(\hat{\varphi})$. This shows that $Sm \subset \theta(S)$. The other inclusion is clear. Then $\theta(S) = Sm$ and $X = \theta(Y) = \{f(m) : f \in Y\}$. Then the Lemma follows.

Theorem 3.9: The following conditions are equivalent:

(1) M is almost nonsingular PQ-injective.

(2) There exists an indexed set $\{X_m : m \in M\}$ of S-submodules of M with the property that if $mR \subset M \setminus Z(M)$, $m \in M$, then $l_M(r_R(m) \cap aR) = (X_{ma} : a)_l + Sm$ and $(X_{ma} : a)_l \cap Sm \subset l_M(a)$ for all $a \in R$, where $(X_{ma} : a)_l = \{n \in M : na \in X_{ma}\}$ if $ma \neq 0$ and $(X_{ma} : a)_l = l_M(aR)$ if $ma = 0$.

Proof:
$(1) \Rightarrow (2)$ Let $m \in M \setminus Z(M)$. Then there exists an S-submodule X_m of M such that $l_M(r_R(m)) = Sm \oplus X_m$ as left S-modules. Let $a \in R$. If $ma = 0$, then $aR \subset r_R(m)$ so (2) follows. If $ma \neq 0$, then any $x \in l_M(r_R(m) \cap aR)$ we have $r_R(ma) \subset r_R(xa)$ and so $xa \in l_M(r_R(xa)) \subset l_M(r_R(ma)) = Sma \oplus X_{ma}$ because $ma \in M \setminus Z(M)$. Write $xa = \alpha(ma) + y$ where $\alpha \in S$ and $y \in X_{ma}$. Then $(x - \alpha(m))a = y \in X_{ma}$, so $x - \alpha(m) \in (X_{ma} : a)_l$. It follows that $x \in (X_{ma} : a)_l + Sm$.

This shows that $l_M(r_R(m) \cap aR) \subset (X_{ma} : a)_l + Sm$. Conversely, it is clear that $Sm \subset l_M(r_R(m) \cap aR)$. Let $y \in (X_{ma} : a)_l$. Then $ya \in X_{ma} \subset l_M(r_R(ma))$. If $as \in r_R(m) \cap aR$, then $mas = 0$ and so $yas = 0$. Hence $y \in l_M(r_R(m) \cap aR)$. This shows that $(X_{ma} : a)_l \subset l_M(r_R(m) \cap aR)$. Therefore $l_M(r_R(m) \cap aR) = (X_{ma} : a)_l + Sm$. If $\beta(m) \in (X_{ma} : a)_l \cap Sm$, then $\beta(m)a \in X_{ma} \cap Sma = 0$. Hence $\beta(m) \in l_M(a)$.

$(2) \Rightarrow (1)$ Let $m \in M \setminus Z(M)$. Then there exists an S-submodule X_m of M such that $l_M(r_R(m)) = l_M(r_R(m) \cap R) = (X_m : 1)_l + Sm$ and $(X_m : 1)_l \cap Sm \subset l_M(1) = 0$. Note that $(X_m : 1)_l = X_m$. Then (1) follows.

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REFERENCES

1. F. W. Anderson and K. R. Fuller, “Rings and Categories of Modules”, Graduate Texts in Math.No.13, Springer-verlag, New York, 1992.
2. Camillo V., *Commutative rings whose principal ideals are annihilators*, Portugal. Math., 46(1989), p. 33-37.
3. N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, “Extending Modules”, Pitman, London, 1994.
4. Lam T. Y., A First Course in Noncommutative Rings, Graduate Texts in Mathematics Vol. 131, Springer-Verlag, New York, 1991.
5. W. Junchao, *Jcp-Injective Rings* International Electronic Journal of Algebra, Volume 6 (2009) 1-22.
6. S. H. Mohamed and B. J. Muller, “Continuous and Discrete Modules”, London Math. Soc. Lecture Note Series 14, Cambridge Univ. Press, 1990.
7. W. K. Nicholson and M. F. Yousif, *Principally injective rings*, J. Algebra, 174(1995), 77--93.
8. W. K. Nicholson, J. K. Park and M. F. Yousif, *Principally quasi-injective modules*, Comm. Algebra, 27:4 (1999), 1683--1693.
9. R. Wisbauer, “Foundations of Module and Ring Theory”, Gordon and Breach London, Tokyo e.a., 1991.
10. S. Wongwai, *On the endomorphism ring of a semi-injective module*, Acta Math. Univ. Comenianae, Vol.71, 1(2002), pp. 27-33.
11. S. Wongwai, *Almost Mininjective Rings*, Thai Journal of Mathematics, 4(1) (2006), 245-249.
12. S. Wongwai, *Small PQ-Principally Injective Modules*, International Journal of Mathematics Archive – 3(3), 2012 , Page 962 - 967.
13. S. Wongwai and M. Kaewneam, *NPQ-Injective Modules*, International Journal of Mathematics Archive – 7(8), 2016, 148 - 154.

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