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STUDY ON INTUITIONISTIC α-OPEN SETS AND α-CLOSED SETS

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ABSTRACT

The main objective of this paper is to introduce and investigate the concept of intuitionistic α-open Sets. Further, we define intuitionistic α-interior and intuitionistic α-closure and the properties of intuitionistic α-interior and intuitionistic α- closure operators in the intuitionistic topological spaces.

Keywords: *intuitionistic set, intuitionistic topology, intuitionistic topological space, intuitionistic semiopen set, intuitionistic semiclosed set, intuitionistic points, intuitionistic α-open sets intuitionistic α-closed sets.*

1. INTRODUCTION

In1983, the idea of intuitionistic fuzzy set was first given by Krassimir T.Atanasav [1]. After the study of the concept of fuzzy set by Zadeh [7], several researches were conducted on the generalizations of the notion of fuzzy set. D.Coker [2, 3, 4, 5, 6] defined and studied intuitionistic topological spaces, intuitionistic open sets, intuitionistic closed sets, intuitionistic fuzzy topological spaces and using intuitionistic sets,he defined closure and interior operators in ITS. Levine N [8] has introduced the concept of semiopen sets in topological spaces. In [9], we defined intuitionistic semiopen sets and intuitionistic semiclosed sets and established their properties and characterizations. In this paper, we shall give a brief introduction to intuitionistic α-open sets, α-closed sets and also discuss the properties of the sets. The following definitions and results are essential to proceed further.

Definition 1.1: Let X be a nonempty fixed set. An intuitionistic set (IS for short) [1] A is an object having the form $A= < X, A^1, A^2 >$ where A^1 and A^2 are subsets of X such that $A^1 \cap A^2 = \phi$. The set A^1 is called the set of member of A, while A^2 is called the set of non member of A. Every subset A of a nonempty set X is obviously an IS having the form $< X, A, A^C >$. Several relations and operations between IS's are given below.

Definition 1.2 [1]: Let X be a non empty set, $A= < X, A^1, A^2 >$ and $B=< X, B^1, B^2 >$ be IS on X and let $\{A_i, i \in j\}$ be an arbitrary family of IS in X, where $A_i = < X, A_i^1, A_i^2 >$. Then

(a) $A \subseteq B$ if and only if $A^1 \subseteq B^1$ and $B^2 \subseteq A^2$.
(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,
(c) $A \sqsubset B$ if and only if $A^1 \cup A^2 \sqsupseteq B^1 \cup B^2$.
(d) $\bar{A}= < X, A^2, A^1 >$ is called the complement of A. We also denote it by A^C
(e) $\cup A_i = < X, \cup A_i^1, \cap A_i^2 >$.
(f) $\cap A_i = < X, \cap A_i^1, \cup A_i^2 >$.
(g) $A - \overline{B.} = A \cap \overline{B.}$
(h) $[\ ]A =< X, A^1, (A^1)^C >$.
(i) $< > A =< X, (A^2)^C, A^2 >$.
(j) $\tilde{\phi} = < X, \phi, X >$ and $\tilde{X} =< X, X, \phi >$. Clearly for every $A=< X, A^1, A^2 >$, $A \subset \tilde{X}$.

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Definition 1.3 [1]: An intuitionistic topology (IT for short) on a nonempty set X is a family τ of IS's satisfying the following axioms.

(a) $\tilde{\phi}, \tilde{X} \in \tau$
(b) $G_1 \cap G_2 \in \tau$ for every $G_1, G_2 \in \tau$, and
(c) $\cup G_i \in \tau$ for every arbitrary family $\{G_i : i \in J\} \subseteq \tau$.

The pair (X,τ) is called an intuitionistic topological space (ITS for short) and any IS G in τ is called an intuitionistic open set (IOS for short) in X. The complement $\bar{A}$ of an IO set A in an ITS (X, τ) is called an intuitionistic closed set[1] (ICS for short). Also, in [1], it is stated that if (X, τ) is an ITS on X, then the following families are also ITS's on X.

(a) $\tau_{0,1} = \{[\ ]G \mid G \in \tau\}$.
(b) $\tau_{0,2} = \{<>G \mid G \in \tau\}$.

Now we state the definition for the closure and interior operators in ITS's.

Definition 1.4 [1]: Let (X, τ) be an ITS and $A = < X, A^1, A^2 >$ be an IS in X. Then the interior and the closure of A are denoted by int(A) and cl(A), respectively and are defined as follows.
$cl(A) = \cap\{K \mid K$ is an ICS in X and $A \subseteq K\}$ and
$int(A) = \cup\{G \mid G$ is an IOS in X and $G \subseteq A\}$.

Also, it can be established that cl(A) is an ICS and int(A) is an IOS in X and cl and int are monotonic and idempotent operators. Moreover, cl is increasing and int is decreasing. Moreover, A is an ICS in X if and only if cl(A) = A and A is an IOS in X if and only if int(A) = A.

Definition 1.5 [2]: Let X be a nonempty set and $p \in X$ a fixed element in X. Then the IS $\tilde{p}$ defined by $\tilde{p} = < X, \{p\}, \{p\}^C >$ is called an intuitionistic point (IP for short) in X.

Definition 1.6 [9]: A subset $A = < X, A^1, A^2 >$ of $\tilde{X} = < X, X, \phi >$ is said to be an intuitionistic semiopen set (in short, ISO set) in an intuitionistic topological space (X,τ) if there is an intuitionistic open set (IO set) $G \neq < X, \phi, X >$ such that $G \subset A \subset cl(G)$. Clearly, every IO set is an ISO set, $\tilde{\phi}$ and $\tilde{X}$ are ISO sets. Also, from the definition, it follows that the closure of every IO set is an intutionistic semiopen set. The complement of every intuitionistic semiopen is said to be an intuitionistic semiclosed (ISC) sets. The family of all ISO set is denoted by $\tau_{\beta.}$

2. INTUITIONISTIC α-OPEN SET AND INTUITIONISTIC α-CLOSED SET

In this section, we define Intuitionistic α-open set and Intuitionistic α-closed set and discuss their properties and characteristics

Definition 2.1: Let (X,τ) be an ITS and A be a IS in X. A is said to be an intuitionistic α-open set (in short IαO set) *if* $A \subseteq int(cl(int(A)))$. The intuitionistic complement of an IαO set is called an Intuitionistic α-closed set (in short IαC set). Clearly, every IO set is an IαO set but not the converse as the following Example 2.2 shows.

Example 2.2: Let X = {1, 2, 3, 4, 5} and consider the family $\tau = \{\tilde{\phi}, \tilde{X}, A_1, A_2, A_3, A_4\}$ where
$A_1 = < X, \{1, 2, 3\}, \{4\} >$, $A_2 = < X, \{3, 4\}, \{5\} >$, $A_3 = < X, \{3\}, \{4, 5\} >$,
$A_4 = < X, \{1, 2, 3, 4\}, \{\phi\} >$, $\tilde{X} = < X, X, \phi >$ and $\tilde{\phi} = < X, \phi, X >$.

Now, let $A_{11} = < X, \{1, 2, 3\}, \{\phi\} >$, Then A_{11} is an IαO set in (X,τ_α). Since $A \subset A_{12} \subset I\alpha cl(A)$, $I\alpha int\ A_{11} = A_4$.
$I\alpha cl(A_4) = X$ and $I\alpha int(X) = X$ which implies $A_{11} \subset I\alpha int(I\alpha cl(I\alpha int(A_{11}))) = X$.

The following Theorem 2.3 shows that the arbitrary union of IαO sets is an IαO set and Theorem 2.5 below shows that the intersection of two IαO sets is an IαO set.

Theorem 2.3: Let (X,τ) be an ITS and $\{A_\alpha : \alpha \in I\}$ be a family of intuitionistic α-open sets in (X,τ). Then $\cup_{\alpha\in I} A_\alpha$ is also an intuitionistic α- open set.

Proof: Let $\{A_\alpha : \alpha \in I\}$ be a family of intuitionistic α-open sets in (X, τ). Then for each A_α.
$A_\alpha \subset int(cl(int(A_\alpha)))$. Since $A_\alpha \subset \cup A_\alpha$, for each α, $A_\alpha \subset int(cl(int(A_\alpha))) \subset int(cl(int(\cup A_\alpha)))$,
and so $\cup A_\alpha \subset int(cl(int(\cup A_\alpha)))$.

Therefore, $\cup A_\alpha$ is an intuitionistic α-open set.

The following Lemma 2.4 is essential to prove the following Theorem 2.5.

Lemma 2.4: Let (X, τ) be an ITS, and A and B be two IS in (X, τ). If B is an IO set, then $cl(A) \cap B \subseteq cl(A \cap B)$.

Proof: Let $p \in cl(A) \cap B$. Then $p \in cl(A)$ and $p \in B$. If C is an intuitionistic open set containing p then $B \cap C$ is an IO set containing p. Therefore, $p \in cl(A)$ implies that $(B \cap C) \cap A \neq \phi$ and so $C \cap (A \cap B) \neq \phi$. Therefore, $p \in cl(A) \cap B$ implies that $p \in cl(A \cap B)$. Hence $cl(A) \cap B \subseteq cl(A \cap B)$.

Theorem 2.5: Let (X,τ) be an ITS. If A and B are any two intuitionistic α-open sets in (X,τ), then $A \cap B$ is an intuitionistic α-open set.

Proof: If A and B are any two intuitionistic α-open sets *in* (X,τ), then $A \subseteq int(cl(int(A)))$ and $B \subseteq int(cl(int(B)))$.

Now $A \cap B \subseteq int(cl(int(A))) \cap int(cl(int(B))) = int[cl(int(A)) \cap int(cl(int(B)))] \subseteq int(cl(int(B)))]]$, by Lemma 2.4
by Lemma 2.4 $B \subseteq int[cl[int(A) \cap cl(int(B))]] \subseteq int[cl[cl[int(A) \cap int(B)]]] = int(cl(int(A \cap B)))$

Therefore, $A \cap B$ is an intuitionistic α-open set.

The proof of the following Theorem 2.6 follows from Theorems 2.3 and 2.5.

Theorem 2.6: Let (X, τ) be an ITS. If τ_α is the family of all intuitionistic α-open sets, then τ_α is an intuitionistic topology.

Theorem 2.7: Let A be a subset of an intuitionistic topological space (X, τ). Then A is an intuitionistic α-open set if and only if there exists an intuitionistic open set B such that $B \subseteq A \subseteq int(cl(B))$.

Proof: Suppose that there exists an intuitionistic open set B such that $B \subseteq A \subseteq int(cl(B))$. Since $A \subseteq int(cl(B)) = int(cl(int(B))) \subseteq int(cl(int(A)))$, by hypothesis and so by definition, A is intuitionistic α-open set. On the other hand, let A be an intuitionistic α-open set.Then $A \subseteq int(cl(int(A)))$. Let $int(A) = B$. Since $int(A) \subseteq A, B \subseteq A$ and also $A \subseteq int(cl(B))$. Hence there exists an intuitionistic open set B such that $B \subseteq A \subseteq int(cl(B))$.

The easy proof of the following Theorem 2.8 is omitted.

Theorem 2.8: Let (X, τ) be an intuitionistic topological space. Then the following hold.

(i) Arbitrary intersection of intuitionistic α-closed sets is an intuitionistic α-closed set.
(ii) Finite union of intuitionistic α-closed sets is an intuitionistic α-closed set.
(iii) $\tilde{X}$ and $\tilde{\phi}$ are intuitionistic α-closed sets.

Theorem 2.9: Let A be a subset of an intuitionistic topological space (X, τ). Then A is an intuitionistic α-closed set if and only if there exists an intuitionistic closed set B such that $int(cl(B)) \subseteq A \subseteq B$.

Proof: Let A be an intuitionistic α-closed set. Then $cl(int(cl(A))) \subseteq A$. Let $cl(A) = B$. Then B is an intuitionistic closed set. Since $A \subseteq cl(A)$, $A \subseteq B$ and $cl(int(cl(A))) \subseteq A \Rightarrow cl(int(B)) \subseteq A$. Thus there exists an intuitionistic closed set B such that $cl(int(B)) \subseteq A \subseteq B$. On the other hand, suppose there exists an intuitionistic closed set B such that $cl(int(B)) \subseteq A \subseteq B$. Since B is an intuitionistic closed set, $cl(B) = B$, by hypothesis, $cl(int(B)) \subseteq A \Rightarrow cl(int(cl(B))) \subseteq A$. since $A \subseteq B \Rightarrow cl(A) \subseteq cl(B) \Rightarrow cl(int(cl(A))) \subseteq cl(int(cl(B))) \subseteq A \Rightarrow A$ is an intuitionistic α-closed set.

Theorem 2.10: Let A be a subset of an intuitionistic topological space (X, τ). Then A is an intuitionistic α-closed set if and only if $cl(int(cl(A))) \subseteq A$.

Proof: Let A be a intuitionistic α-closed set. So A^C is an intuitionistic α-open set. Therefore, by definition, $A^C \subseteq int(cl(int(A^C))) = int(cl(cl(A^C))) = int[int(cl(A))]^C = [cl(int(cl(A)))]^C$ which gives $cl(int(cl(A))) \subseteq A$. The proof of the converse part is similar.

Definition 2.11: Let (X,τ) be an intuitionistic topological space and A be an intuitionistic set in X .The closure and interior of A in (X, τ_α) are defined as follows.

(i) $I\alpha int(A) = \cup\{G : G$ is an IαO set and $G \subseteq A\}$.
(ii) $I\alpha cl(A) = \cap\{K : K$ is an IαC set and $A \subseteq K\}$.

Also, it can be established that $I\alpha cl(A)$ is an IαC set and $I\alpha int(A)$ is an IαO set in X. Moreover, A is an IαC set in X if and only if $I\alpha cl(A) = A$ and A is an IαO set in X if and only if $I\alpha int(A) = A$.

Remarks 2.12: Let (X, τ) be an intuitionistic topological space and let A be an intuitionistic set in X. Then (X, τ_α) is also an intuitionistic topological space. Hence the proof of the following Theorems 2.13, 2.14 and 2.15 follow from the similar already established results for intuitionistic topological spaces.

Theorem 2.13: Let (X, τ) be an intuitionistic topological space and let A be an intuitionistic set in X. Then the following hold.

(i) A is an IαC set in (X, τ) if and only if $A = I\alpha cl(A)$.
(ii) A is an IαO set in (X, τ) if and only if $A = I\alpha int(A)$.
(iii) $I\alpha cl(\tilde{\phi}) = \tilde{\phi}$ and $I\alpha cl(\tilde{X}) = \tilde{X}$.
(iv) $I\alpha int(\tilde{\phi}) = \tilde{\phi}$ and $I\alpha int(\tilde{X}) = \tilde{X}$.
(v) $I\alpha cl(I\alpha cl(A)) = I\alpha cl(A)$.
(vi) $I\alpha int(I\alpha int(A)) = I\alpha int(A)$.
(vii) $(I\alpha cl(A))^C = I\alpha int(A^C)$.
(viii) $(I\alpha int(A))^C = I\alpha cl(A^C)$.

Theorem 2.14: Let (X, τ) be an intuitionistic topological space and let A and B be two intuitionistic sets in X. Then the following hold.

(i) $A \subseteq B \Rightarrow I\alpha int(A) \subseteq I\alpha int(B)$.
(ii) $A \subseteq B \Rightarrow I\alpha cl(A) \subseteq I\alpha cl(B)$.
(iii) $I\alpha cl(A \cup B) = I\alpha cl(A) \cup I\alpha cl(B)$.
(iv) $I\alpha int(A \cap B) = I\alpha int(A) \cap I\alpha int(B)$.
(v) $I\alpha cl(A \cap B) \subseteq I\alpha cl(A) \cap I\alpha cl(B)$.
(vi) $I\alpha int(A \cup B) \supseteq I\alpha int(A) \cup I\alpha int(B)$.

The proof of the following Theorem 2.15 is similar to the proof of Lemma 2.4.

Theorem 2.15: Let (X, τ) be an intuitionistic topological space and A and B be any two intuitionistic sets in (X, τ). If B is an intuitionistic α-open set, then $B \cap I\alpha cl(A) \subseteq I\alpha cl(A \cap B)$.

The following Thorem 2.16 gives a characterization of IαO-set and Corollary2.17follows fromTheorem2.16.

Theorem 2.16: Let (X, τ) be an intuitionistic topological space and τ_α be the family of IαO sets. Then τ_α consists of exactly those sets A for which $A \cap B \in \tau_\beta$ for all $B \in \tau_\beta$.

Proof: Let $A \in \tau_\alpha$, $B \in \tau_\beta$, $x \in A \cap B$ and let U be an open neighborhood of x.

Clearly, $U \cap int(cl(int(A)))$ is an open neighborhood of x.

Since $x \in B \subseteq cl(int(B))$, $(U \cap int(cl(int(A)))) \cap int(B)$ is non-empty and so $V = (U \cap int(cl(int(A)))) \cap B$ is non-empty.

Since $V \subset int(cl(A))$, it follows that $U \cap (int(A) \cap int(B)) = V \cap int(A) \neq \phi$.

If follows that $A \cap B \subset cl(int(A) \cap int(B)) = cl(int(A \cap B))$. Therefore, $A \cap B \in \tau_\beta$.

Conversely, let $A \cap B \in \tau_\beta$ for all $B \in \tau_\beta$. Then in particular, $A \in \tau_\beta$. Assume that $x \in A \cap (int(cl(int(A))))^C$

Then $x \in cl(B)$ where $B = (cl(int(A)))^C$. Clearly $\{x\} \cup B \in \tau_\beta$, and consequently,

$A \cap (\{x\} \cup B) \in \tau_\beta$ but $A \cap (\{x\} \cup B) = \{x\}$.

Hence $\{x\}$ is open as $x \in int(cl(A)) \Rightarrow x \in int(cl(int(A)))$, a contradiction to our assumption.

Thus $x \in A \Rightarrow x \in int(cl(int(A)))$ and so $A \in \tau_\alpha$.

Corollary 2.17: Let (X, τ) be an intuitionistic topological space. Then the intersection of an IαO set and an ISO set is an ISO set.

The following Theorem 2.18 shows that the ISO sets of the IT's τ_α and τ are identical.

Theorem 2.18: Let (X, τ) be an ITS. Then $\tau_\beta = \tau_{\alpha\beta}$.

Proof: Let $B\in\tau_\beta$. If $x\in B\in\tau_\beta$ and $x\in A\in\tau_\alpha$, then $x\in cl(int(B))$ and $x\in int(cl(int(A)))$. Therefore $int(cl(int(A)))\cap int(B) \neq \phi$, since $int(cl(int(A)))$ is a neighbourhood of x and so $int(B)\cap int(A)\neq\phi$.

Hence $A\cap int(B)\neq\phi$ which implies that $A\cap I\alpha int(B)\neq\phi$ which implies that $x\in I\alpha cl(I\alpha int(B))$.Therefore $B\subset I\alpha cl(I\alpha int(B))$ which show that $B\in(\tau_\alpha)_\beta$. Hence $\tau_\beta \subset \tau_{\alpha\beta}$. On the other hand, let $A\in(\tau_\alpha)_\beta$, $x\in A$ and $x\in V\in\tau$.

As $V\in\tau_\alpha$, and $x\in I\alpha cl(I\alpha int(A))$, we have $V\cap I\alpha int(A)\neq\phi$ and there exist a nonempty set $W\in\tau$ such that $W\subset V\cap I\alpha int(A)\subset A$. In other words $V\cap int(A)\neq\phi$ and $x\in cl(int(A))$ which shows that $B\subset cl(int(B))$. This gives $\tau_{\alpha\beta}\subset\tau_\beta$.

This completes the proof.

The following Theorem 2.19 shows that in any ITS the family of all IαO sets of the topologies τ and τ_α are the same.

Theorem 2.19: Let (X, τ) be an ITS. Then $\tau_\alpha = \tau_{\alpha\alpha}$.

Proof: By Theorem 2.16, $\tau_{\alpha\alpha} = \{A \mid A\cap B\in\tau_{\alpha\beta} \text{ for every } B\in\tau_{\alpha\beta}\}$
$= \{A \mid A\cap B\in\tau_\beta \text{ for every } B\in\tau_\beta\}$ by Theorem 2.18,
$= \tau_\alpha$, by Theorem 2.16.

Definition 2.20: A subset A of an intuitionistic space (X, τ) is said to be a intuitionistic nowhere dense set if $int(cl(A))=\phi$. The following Theorem 2.21 gives a characterization of IαO set in an ITS.

Theorem 2.21: Let (X, τ) be an ITS. Then $\tau_\alpha = \{A \mid A = G - N$ where G is an IO set and N is intuitionistic nowhere dense$\}$.

Proof: If $A\in\tau_\alpha$ we have $A= int(cl(int(A))) - (int(cl(int(A))) - A)$ where $int(cl(int(A)))-A$ clearly is an Intuitionistic nowhere dense. Conversely, if $A= B - N$, $B\in\tau$, N intuitionistic nowhere dense, we easily see that $B\subset cl(int(A))$ and consequently, $A \subset B \subset int(cl(int(A)))$. Therefore, $A \in \tau_\alpha$.

The proof of the following Corollary 2.22 follows from the above Theorem 2.21.

Corollary 2.22: Let (X, τ) be an ITS. Then $\tau = \tau_\alpha$ if and only if all intuitionistic nowhere dense sets are closed.

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