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ON THE WIENER AND HYPER WIENER INDICES OF CERTAIN DOMINATING TRANSFORMATION GRAPHS OF KRAGUJEVAC TREES

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ABSTRACT

Let G be a graph. The distance $d_G(u,v)$ between the vertices u and v of the graph G is equal to the length of the shortest path that connects u and v. The Wiener index $W(G)$ is the sum of distances between all pairs of vertices of G, whereas the hyper-Wiener index $WW(G)$ is defined as $WW(G)=\frac{1}{2}\sum_{(u,v)\subseteq V(G)}[d_G(u,v)+d_G(u,v)^2]$. In this paper Wiener and hyper-Wiener indices are obtained for dominating transformation(d-transformations) graphs of Kragujevac tree T.

Keywords: *distance, Wiener index, hyper-Wiener index, d-transformation graphs.*

Subject Classification: *05C05, 05C07, 05C69.*

1. INTRODUCTION

In this paper we are concerned with simple graphs, having no directed or weighted edges, and no self loops. Let $G=(V,E)$ be such a graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)|=n$ and $|E(G)|=m$. By the *open neighborhood* of a vertex v of G we mean the set $N_G(v)=\{u\in V(G): uv\in E(G)\}$. The *d*istance $d_G(u,v)$ between the vertices u and v of the graph G is equal to the length of a shortest path that connects u and v. Let S be a finite set and let $F=\{S_1,S_2,\ldots,S_n\}$ be a partition of S. Then the *intersection graph* $\Omega(F)$ of F is the graph whose vertices are the subsets in F and in which two vertices S_i and S_j are adjacent if and only if $S_i\cap S_j\neq\Phi$. For terminology not defined here we refer the reader to [6].

A *topological index* of a graph is a single unique number characteristic of the graph and is mathematically invariant under graph automorphism. Usage of topological indices in biology and chemistry began in 1947 when H. Wiener [17, 18] introduced *Wiener index* which is denoted by W and is given by

$$W(G)=\sum_{(u,v)\subseteq V(G)} d_G(u,v)$$

The *hyper-Wiener index* of acyclic graphs was introduced by Milan Randic in 1993. Then Klein *et al.* [11], generalized Randic's definition for all connected graphs, as a generalization of the Wiener index. It is defined as

$$WW(G)=\frac{1}{2}\sum_{(u,v)\subseteq V(G)}[d_G(u,v)+d_G(u,v)^2]$$

Further, information about Wiener index and hyper-Wiener index are given in [2, 3, 4, 9, 10, 19].

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2. KRAGUJEVAC TREES

A connected acyclic graph is called a *tree.* A *rooted tree* is a tree in which one particular vertex is distinguished; this vertex is referred to as the root. The vertex of degree one is a *pendant vertex.* The vertex adjacent to pendant vertex is called *support vertex.* The formal definition of a Kragujevac tree was introduced in [5].

Definition 1. [5]: *Let P_3 be the 3-vertex tree,rooted at one of its terminal vertices.For $k = 2,3,...$, construct the rooted tree B_k by identifying the roots of k copies of P_3. The vertex obtained by identifying the roots P_3-trees is the root of B_k.*

Examples illustrating the structure of the rooted tree B_k depicted in Figure 1.

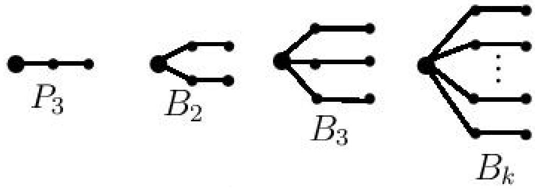


Figure 1 : The structure of the rooted tree Bk

Definition 2.[5]: *Let $d \geq 2$ be an integer. Let $\beta_1, \beta_2, \beta_3 ... \beta_d$ be rooted trees specified in Definition 1, i.e., $\beta_1, \beta_2, \beta_3 ..., \beta_d \in \{B_2, B_3, ...\}$. A Kragujevac tree T is a tree possessing a vertex of degree d, adjacent to the roots of $\beta_1, \beta_2, \beta_3 ..., \beta_d$. This vertex is said to be the central vertex of T, where d is the degree of T. The subgraphs $\beta_1, \beta_2, \beta_3 ..., \beta_d$ are the branches of T. Recall that some (or all) branches of T may be mutually isomorphic.*

A typical Kragujevac tree of degree d=5 is depicted in Figure 2.

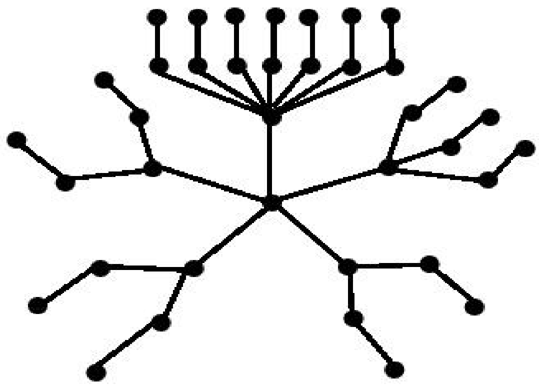


Figure 2: Kragujevac tree of degree d = 5

Clearly, the branch B_k has 2k+1 vertices. Therefore, the vertex set and the edge set of the Kragujevac tree T are:

$$V(T) = 1 + \sum_{i=1}^{d} (2k_i + 1) \quad and \quad E(T) = \sum_{i=1}^{d} (2k_i + 1)$$

We denote the vertices of Kragujevac trees as follows:

- pendant vertices by x_i
- support vertices by w_i
- vertices belongs to the set $N(w_i) - \{x_i\}_{i=1}^{k}$ by v_i
- the central vertex by u

Recent information on Kragujevac trees can be found in [1, 5].

3. d-TRANSFORMATION GRAPHS

The theory of domination has emerged as one of the most studied area in graph theory and its allied branch in mathematics. The wide variety of domination parameters have been defined and studied their applications in various fields [7]. A subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex of $V(G)\setminus D$ has a neighbor in D. The *domination number* of graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The dominating set D is called a *minimal dominating* set if no proper subset of D is a dominating set. For a comprehensive survey of domination in graphs, see [8].

In the mathematical literature several transformation graphs have been considered and constructed. Let $\mathcal{G}$ denote the set of simple undirected graphs. Various important results in graph theory have been obtained by considering some functions $\mathcal{F}: \mathcal{G} \rightarrow \mathcal{G}$ or $\mathcal{F}_s: \mathcal{G}_1 \times \ldots \times \mathcal{G}_s \rightarrow \mathcal{G}$ called transformations or operations (here each $\mathcal{G}_i = \mathcal{G}$) and by establishing how these operations affect certain properties or parameters of graphs. The complement, the k-th power of graph, line graph and the total graph are well known examples of such transformations or operations. The same concept of transformation has been applied to dominating sets and defined a variety of *dominating transformation(d-transformation)* graphs by Prof. V.R.Kulli and his students [12, 13, 14, 15, 16]. In what follows we define the best known representatives of such graphs.

The *minimal dominating graph MD(G)* of a graph G is the intersection graph defined on the family of all minimal dominating sets of vertices in G [12].

The *common minimal dominating graph CD(G)* of a graph G is the graph having same vertex set as G with two vertices are adjacent if and only if there exist a minimal dominating set in G containing them [13].

The *vertex minimal dominating graph* $M_vD(G)$ of a graph $G=(V,E)$ is a graph with $V(M_vD(G)) = V' = V \cup S$, where S is the collection of all minimal dominating sets of G with two vertices $u, v \in V'$ are adjacent if and only if they are adjacent in G or $v = D$ is a minimal dominating set of G containing u [14].

The *dominating graph D(G)* of a graph $G=(V,E)$ is a graph with $V(D(G)) = V \cup S$, where S is the set of all minimal dominating set of G and with two vertices $u, v \in V(D(G))$ are adjacent if $u \in V$ and $v = D$ is a minimal dominating set of G containing u [15].

In Figure 3, a graph G and its d-transformation graphs are shown.

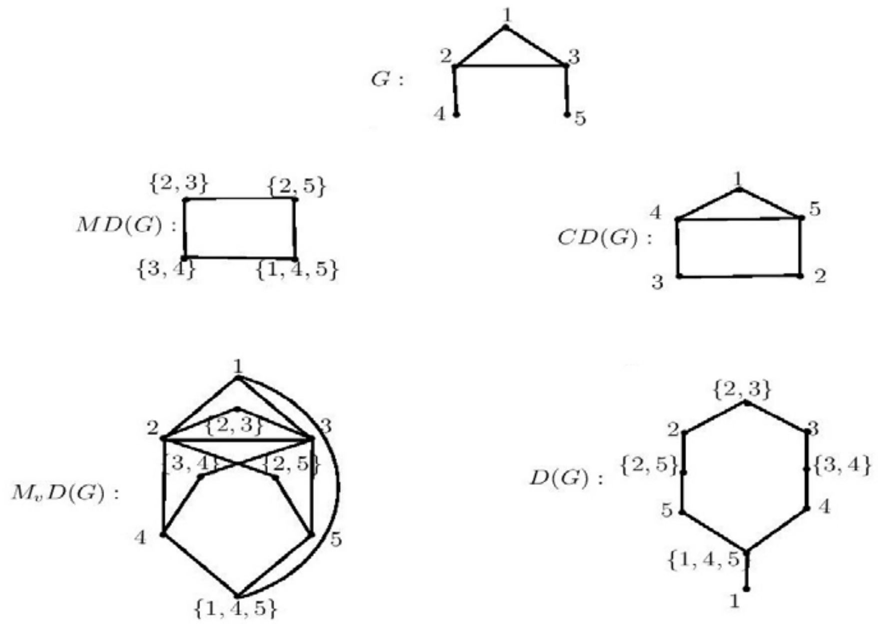
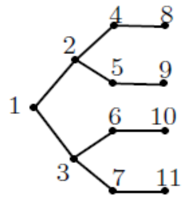


Figure 3 : A graph G and its d − transformation graphs MD(G), CD(G), $M_vD(G)$ and D(G)

In Figure 4, Kragujevac tree T and its d-transformation graphs $MD(T), CD(T), M_vD(T)$ and $D(T)$ are given.

$T:$



$pendant\ vertex\ set\ \{x_i\} = \{8, 9, 10, 11\}$

$support\ vertex\ set\ \{w_i\} = \{4, 5, 6, 7\}$

$vertices\ belonging\ to\ set\ \{N(w_i) - \{x_i\}\} = \{v_i\} = \{2, 3\}$

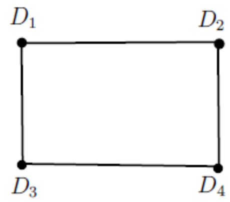
$central\ vertex\ \{u\} = \{1\}$

$Dominating\ sets:$

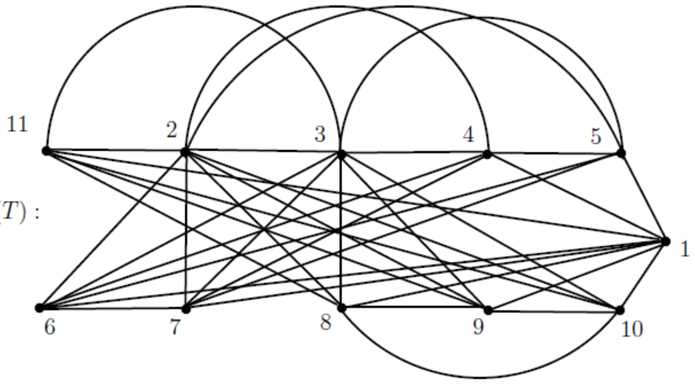
$D_1 = \{1, 4, 5, 6, 7\}$ $\quad D_2 = \{1, 8, 9, 10, 11\}$

$D_3 = \{2, 3, 4, 5, 6, 7\}$ $\quad D_4 = \{2, 3, 8, 9, 10, 11\}$

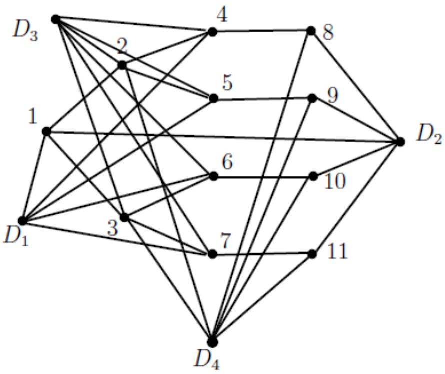
$MD(T):$



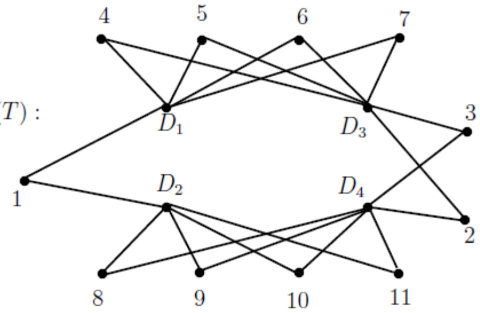
$CD(T):$



$M_vD(T):$



$D(T):$



$Figure\ 4:\ Kragujevac\ tree\ T\ and\ its\ d-transformation\ graphs\ MD(T), CD(T), M_vD(T)\ and\ D(T)$

In this paper, expressions for the Wiener and hyper-Wiener indices are obtained for dominating transformation(d-transformations) graphs of Kragujevac tree T.

4. COMPUTING WIENER INDEX FOR MD(T) ,CD(T), M_vD(T) AND D(T) OF KRAGUJEVAC TREE T.

We begin with the following straightforward, previously known auxiliary result.

Lemma 1: *The dominating sets of Kragujevac tree* T *are:*

$$D_1 = \left\{ u, \sum_{i=1}^{k} w_i \right\}$$

$$D_2 = \left\{ u, \sum_{i=1}^{k} x_i \right\}$$

$$D_3 = \left\{ \sum_{i=1}^{d} v_i, \sum_{i=1}^{k} w_i \right\}$$

$$D_4 = \left\{ \sum_{i=1}^{d} v_i, \sum_{i=1}^{k} x_i \right\}$$

Theorem 2: *The Wiener index of the minimal dominating graph of Kragujevac tree* T *is* $W(MD(T)) = 8.$

Proof: Let T be any Kragujevac tree T of degree $d \geq 2.$ Then by Lemma 1, there exist four minimal dominating sets namely D_1, D_2, D_3 *and* $D_4.$ Hence by the definition of $MD(G)$, the vertex set of the minimal dominating graph of Kragujevac tree two vertices. Therefore, $d_{MD(T)}(u,v) = 1$ if u and v are minimal dominating sets with common vertex, otherwise $d_{MD(T)}(u,v) = 2.$ Now by employing definition of Wiener index we get $W(MD(T)) = 8.$

Before going to prove our next result we need the following lemma.

Lemma 3: *The distance between each vertex in the common minimal dominating graph of Kragujevac tree T is:*

1. $d_{CD(T)}(u, v_i) = 2$
2. $d_{CD(T)}(u, x_i) = 1$
3. $d_{CD(T)}(u, w_i) = 1$
4. $d_{CD(T)}\left(x_i, \sum_{i=1}^{d} v_i \right) = 1$
5. $d_{CD(T)}\left(x_i, \sum_{i=1}^{k} w_i \right) = 2$
6. $d_{CD(T)}(x_i, x_j) = 1$
7. $d_{CD(T)}(w_i, w_j) = 1$
8. $d_{CD(T)}(v_i, v_j) = 1$
9. $d_{CD(T)}\left(w_i, \sum_{i=1}^{d} v_i \right) = 1.$

Theorem 4: *The Wiener index of the common minimal dominating graph of Kragujevac tree T is*

$$W[CD(T)] = 2[k^2 + kd + k + {}^k C_2 + d] + {}^d C_2.$$

Proof: By the definition of $CD(G)$ it is clear that $V(CD(T)) = V(T).$ By using the definition of Wiener index and Lemma 3, we have

$$W(CD(T)) = \sum_{(u,v)\in V} d_{CD(T)}(u,v)$$

$$= \sum_{i=1}^{d} d_{CD(T)}(u, v_i) + \sum_{i=1}^{k} d_{CD(T)}(u, x_i) + \sum_{i=1}^{k} d_{CD(T)}(u, w_i)$$

$$+ \sum_{i=1}^{k}\sum_{i=1}^{d} d_{CD(T)}(x_i, v_i) + \sum_{i=1}^{k} d_{CD(T)}(x_i, w_i) + \sum_{i<j} d_{CD(T)}(x_i, x_j) +$$

$$+ \sum_{i<j} d_{CD(T)}(w_i, w_j) + \sum_{i<j} d_{CD(T)}(v_i, v_j) + \sum_{i=1}^{k}\sum_{i=1}^{d} d_{CD(T)}(w_i, v_i)$$

$$W(CD(T)) = 2d + k(1) + k(1) + k(d) + k[k(2)] + {}^k C_2 + {}^k C_2 + {}^d C_2 + k(d)$$
$$= 2[k^2 + k + kd + {}^k C_2 + d] + {}^d C_2.$$

Thus, the result follows.

Lemma 5: *The distance between each vertex in the vertex minimal dominating graph of Kragujevac tree T is:*

1. $d_{M_vD(T)}(u, v_i) = 1$
2. $d_{M_vD(T)}(u, x_i) = 3$
3. $d_{M_vD(T)}(u, w_i) = 2$
4. $d_{M_vD(T)}\left(x_i, \sum_{i=1}^{k} w_i\right) = (3k-2)$
5. $d_{M_vD(T)}\left(x_i, \sum_{i=1}^{d} v_i\right) = 2d$
6. $d_{M_vD(T)}\left(w_i, \sum_{i=1}^{k} v_i\right) = 2d - 1$
7. $d_{M_vD(T)}(x_i, x_j) = 2$
8. $d_{M_vD(T)}(w_i, w_j) = 2$
9. $d_{M_vD(T)}(v_i, v_j) = 2$
10. $d_{M_vD(T)}\left(u, \sum_{i=1}^{4} D_i\right) = 6$
11. $d_{M_vD(T)}\left(x_i, \sum_{i=1}^{4} D_i\right) = 6$
12. $d_{M_vD(T)}\left(w_i, \sum_{i=1}^{4} D_i\right) = 6$
13. $d_{M_vD(T)}\left(v_i, \sum_{i=1}^{4} D_i\right) = 6$
14. $\sum_{i<j} d_{M_vD(T)}(D_i, D_j) = 14.$

Theorem 6: *The Wiener index of the vertex minimal dominating graph of Kragujevac tree T is*

$$W[M_v D(T)] = k[3k + 4d + 14] + 4\,{}^kC_2 + 2\,{}^dC_2 + 7d + 20.$$

Proof: By the definition of $M_v D(G), V(M_v D(T)) = V \cup S,$ where $S = \{D_1, D_2, D_3, D_4\}$. By using the definition of Wiener index and Lemma 5, we have,

$$W(M_v D(T)) = \sum_{(u,v)\in V} d_{M_vD(T)}(u, v)$$

$$= \sum_{i=1}^{d} d_{M_v D(T)}(u, v_i) + \sum_{i=1}^{k} d_{M_v D(T)}(u, x_i) + \sum_{i=1}^{k} d_{M_v D(T)}(u, w_i) + \sum_{i=1}^{k} d_{M_v D(T)}(x_i, \sum_{i=1}^{k} w_i)$$

$$+ \sum_{i=1}^{k} d_{M_v D(T)}\left(x_i, \sum_{i=1}^{d} v_i\right) + \sum_{i=1}^{k} d_{M_v D(T)}\left(w_i, \sum_{i=1}^{d} v_i\right) + \sum_{i<j} d_{M_v D(T)}(x_i, x_j)$$

$$+ \sum_{i<j} d_{M_v D(T)}(w_i, w_j) + \sum_{i<j} d_{M_v D(T)}(v_i, v_j) + d_{M_v D(T)}\left(u, \sum_{i=1}^{4} D_i\right) + \sum_{i=1}^{k} d_{M_v D(T)}\left(x_i, \sum_{i=1}^{4} D_i\right)$$

$$+ \sum_{i=1}^{k} d_{M_v D(T)}\left(w_i, \sum_{i=1}^{4} D_i\right) + \sum_{i=1}^{d} d_{M_v D(T)}\left(v_i, \sum_{i=1}^{4} D_i\right) + \sum_{i<j} d_{M_v D(T)}(D_i, D_j)$$

$$= d(1) + k(3) + k(2) + 6 + (3k-2)k + 2d(k) + k(6) + (2d-1)k + k(6) + 2\ {}^{k}C_2 + 2\ {}^{k}C_2$$

$$+ 2\ {}^{d}C_2 + 6d + 14$$

$$W[M_v D(T)] = k[3k + 4d + 14] + 4\ {}^{k}C_2 + 2\ {}^{d}C_2 + 7d + 20.$$

Lemma 7: *The distance between each vertex in the dominating graph of Kragujevac tree* T *is*

1. $d_{D(T)}(u, v_i) = 4$
2. $d_{D(T)}(u, x_i) = 2$
3. $d_{D(T)}(u, w_i) = 2$
4. $d_{D(T)}\left(x_i, \sum_{i=1}^{d} v_i\right) = 2$
5. $d_{D(T)}(x_i, w_i) = 4$
6. $d_{D(T)}(x_i, x_j) = 2$
7. $d_{D(T)}(w_i, w_j) = 2$
8. $d_{D(T)}(v_i, v_j) = 2$
9. $d_{D(T)}\left(w_i, \sum_{i=1}^{d} v_i\right) = 2$
10. $d_{D(T)}\left(u, \sum_{i=1}^{4} D_i\right) = 8$
11. $d_{D(T)}\left(x_i, \sum_{i=1}^{4} D_i\right) = 8$
12. $d_{D(T)}\left(w_i, \sum_{i=1}^{4} D_i\right) = 8$
13. $d_{D(T)}\left(v_i, \sum_{i=1}^{4} D_i\right) = 8$
14. $\sum_{i<j} d_{D(T)}(D_i, D_j) = 16.$

Theorem 8: *The Wiener index of the dominating graph of Kragujevac tree* T *is*

$$W[D(T)] = 4k(k + d + 5) + 12d + 4\ {}^{k}C_2 + 2\ {}^{d}C_2 + 24.$$

Proof: By the definition of $D(G), V(D(T)) = V \cup S,$ where $S = \{D_1, D_2, D_3, D_4\}$. By using the definition of Wiener index and Lemma 7, we have,

$$
\begin{aligned}
W(D(T)) &= \sum_{(u,v)\in V} d_{D(T)}(u,v) \\
&= \sum_{i=1}^{d} d_{D(T)}(u,v_i) + \sum_{i=1}^{k} d_{D(T)}(u,x_i) + \sum_{i=1}^{k} d_{D(T)}(u,w_i) + \sum_{i=1}^{k} d_{D(T)}(x_i,\sum_{i=1}^{d} v_i) \\
&+ \sum_{i=1}^{k} d_{D(T)}(w_i,\sum_{i=1}^{d} v_i) + \sum_{i=1}^{k} d_{D(T)}(x_i,w_i) + \sum_{i<j} d_{D(T)}(x_i,x_j) + \sum_{i<j} d_{D(T)}(w_i,w_j) \\
&+ \sum_{i<j} d_{D(T)}(v_i,v_j) + d_{D(T)}(u,\sum_{i=1}^{4} D_i) + \sum_{i=1}^{k} d_{D(T)}(x_i,\sum_{i=1}^{4} D_i) \\
&+ \sum_{i=1}^{k} d_{D(T)}(w_i,\sum_{i=1}^{4} D_i) + \sum_{i=1}^{d} d_{D(T)}(v_i,\sum_{i=1}^{4} D_i) + \sum_{i<j} d_{D(T)}(D_i,D_j) \\
&= 4d + 2k + 2k + 2kd + 2kd + 4(k)(k) + 2\ {}^{k}C_2 + 2\ {}^{k}C_2 + 2\ {}^{d}C_2 + 8 + 8k + 8k + 8d + 16 \\
&= 4k[k+d+5] + 12d + 4\ {}^{k}C_2 + 2\ {}^{d}C_2 + 24.
\end{aligned}
$$

5. COMPUTING HYPER-WIENER INDEX FOR MD(T) ,CD(T), M$_v$D(T) AND D(T) OF KRAGUJEVAC TREE T.

Using the definition of hyper-Wiener index and Lemma 1,3,5 and 7 we can deduce the expression for the hyper-Wiener index of $MD(T)$, $CD(T)$,$M_vD(T)$ and $D(T)$ of Kragujevac tree T.

Theorem 9: *The hyper-Wiener index of the minimal dominating graph of Kragujevac tree* T *is* $WW(MD(T)) = 10.$

Proof: By the definition of hyper-Wiener index

$$WW(G) = \frac{1}{2} \sum_{(u,v)\in V(G)} [d_G(u,v) + d_G(u,v)^2].$$

Then by Lemma 1 and using the definition of minimal dominating graph it is clear that $WW(MD(T)) = 10.$

By the definition of common minimal dominating graph $CD(G)$ and Lemma 3, we can deduce hyper-Wiener index for $CD(T)$ for Kragujevac tree in the following theorem.

Theorem 10: *The hyper-Wiener index of the common minimal dominating graph of Kragujevac tree* T *is*

$$WW(CD(T)) = \frac{1}{2}[2k^2(2k^2 + d^2 + 2) + 4d^2 + 2(d+k) + 2kd + 2\ {}^{k}C_2({}^{k}C_2 + 1) + {}^{d}C_2({}^{d}C_2 + 1)].$$

Proof: By the definition of $CD(G)$ it is clear that $V(CD(T)) = V(T).$ By using the definition of hyper-Wiener index and Lemma 3, we have

$$
\begin{aligned}
WW(CD(T)) &= \frac{1}{2} \sum_{(u,v)\in V(G)} [d_{CD(T)}(u,v) + d_{CD(T)}(u,v)^2] \\
&= \frac{1}{2}[[\sum_{i=1}^{d} d_{CD(T)}(u,v_i) + \sum_{i=1}^{d} d_{CD(T)}(u,v_i)^2] + [\sum_{i=1}^{k} d_{CD(T)}(u,x_i) \\
&+ \sum_{i=1}^{k} d_{CD(T)}(u,x_i)^2] + [\sum_{i=1}^{k} d_{CD(T)}(u,w_i) + \sum_{i=1}^{k} d_{CD(T)}(u,w_i)^2] \\
&+ [\sum_{i=1}^{k}\sum_{i=1}^{d} d_{CD(T)}(x_i,v_i) + \sum_{i=1}^{k}\sum_{i=1}^{d} d_{CD(T)}(x_i,v_i)^2] \\
&+ [\sum_{i=1}^{k} d_{CD(T)}(x_i,w_i) + \sum_{i=1}^{k} d_{CD(T)}(x_i,w_i)^2] + [\sum_{i<j} d_{CD(T)}(x_i,x_j) \\
&+ \sum_{i<j} d_{CD(T)}(x_i,x_j)^2] + [\sum_{i<j} d_{CD(T)}(w_i,w_j) + \sum_{i<j} d_{CD(T)}(w_i,w_j)^2]
\end{aligned}
$$

$$+[\sum_{i<j} d_{CD(T)}(v_i,v_j)+\sum_{i<j} d_{CD(T)}(v_i,v_j)^2]+[\sum_{i=1}^{k}\sum_{i=1}^{d} d_{CD(T)}(w_i,v_i)$$
$$+\sum_{i=1}^{k}\sum_{i=1}^{d} d_{CD(T)}(w_i,v_i)^2]]$$
$$=\frac{1}{2}[(2d+4d^2)+(k+k^2)+(k+k^2)+(kd+k^2d^2)+(2k^2+4k^4)+({}^kC_2+({}^kC_2)^2)$$
$$+({}^kC_2+({}^kC_2)^2)+({}^dC_2+({}^dC_2)^2)+(kd+(kd)^2)]$$
$$=\frac{1}{2}[2k^2(2k^2+d^2+2)+4d^2+2(d+k)+2kd+2\ {}^kC_2({}^kC_2+1)+{}^dC_2({}^dC_2+1)].$$

Thus, the result follows.

Theorem 11: *The hyper-Wiener index of the vertex minimal dominating graph of Kragujevac tree* T *is*

$$WW(M_vD(T))=\frac{1}{2}[k(9k^3-12k^2+93k+14)+d(37d+7)+4kd(2kd-k+1)+4\ {}^kC_2(2\ {}^kC_2)+1)$$
$$+2\ {}^dC_2({}^dC_2+1)+252].$$

Proof: By the definition of hyper-Wiener index and Lemma 5, we have

$$WW(M_vD(T))=\frac{1}{2}\sum_{(u,v)\in V(G)}[d_{M_vD(T)}(u,v)+d_{M_vD(T)}(u,v)^2]$$
$$=\frac{1}{2}[[\sum_{i=1}^{d} d_{M_vD(T)}(u,v_i)+\sum_{i=1}^{d} d_{M_vD(T)}(u,v_i)^2]+[\sum_{i=1}^{k} d_{M_vD(T)}(u,x_i)$$
$$+\sum_{i=1}^{k} d_{M_vD(T)}(u,x_i)^2]+[\sum_{i=1}^{k} d_{M_vD(T)}(u,w_i)+\sum_{i=1}^{k} d_{M_vD(T)}(u,w_i)^2]$$
$$+[\sum_{i=1}^{k} d_{M_vD(T)}(x_i,\sum_{i=1}^{k} w_i)+\sum_{i=1}^{k} d_{M_vD(T)}(x_i,\sum_{i=1}^{k} w_i)^2]$$
$$+[\sum_{i=1}^{k} d_{M_vD(T)}(x_i,\sum_{i=1}^{d} v_i)+\sum_{i=1}^{k} d_{M_vD(T)}(x_i,\sum_{i=1}^{d} v_i)^2]$$
$$+[\sum_{i=1}^{k} d_{M_vD(T)}(w_i,\sum_{i=1}^{d} v_i)+\sum_{i=1}^{k} d_{M_vD(T)}(w_i,\sum_{i=1}^{d} v_i)^2]+[\sum_{i<j} d_{M_vD(T)}(x_i,x_j)$$
$$+\sum_{i<j} d_{M_vD(T)}(x_i,x_j)^2]+[\sum_{i<j} d_{M_vD(T)}(w_i,w_j)+\sum_{i<j} d_{M_vD(T)}(w_i,w_j)^2]$$
$$+[\sum_{i<j} d_{M_vD(T)}(v_i,v_j)+\sum_{i<j} d_{M_vD(T)}(v_i,v_j)^2]+[d_{M_vD(T)}(u,\sum_{i=1}^{4} D_i)$$
$$+d_{M_vD(T)}(u,\sum_{i=1}^{4} D_i)^2]+[\sum_{i=1}^{k} d_{M_vD(T)}(x_i,\sum_{i=1}^{4} D_i)+\sum_{i=1}^{k} d_{M_vD(T)}(x_i,\sum_{i=1}^{4} D_i)^2]$$
$$+[\sum_{i=1}^{k} d_{M_vD(T)}(w_i,\sum_{i=1}^{4} D_i)+\sum_{i=1}^{k} d_{M_vD(T)}(w_i,\sum_{i=1}^{4} D_i)^2]$$
$$+[\sum_{i=1}^{d} d_{M_vD(T)}(v_i,\sum_{i=1}^{4} D_i)+\sum_{i=1}^{d} d_{M_vD(T)}(v_i,\sum_{i=1}^{4} D_i)^2]+[\sum_{i<j} d_{M_vD(T)}(D_i,D_j)$$
$$+\sum_{i<j} d_{M_vD(T)}(D_i,D_j)^2]]$$
$$=\frac{1}{2}[(d+d^2)+(3k+9k^2)+(2k+4k^2)+(6+36)+(3k-2)k+k^2(3k-2)^2+2kd$$
$$+4k^2d^2+6k+36k^2+(2d-1)k+(2d-1)^2k^2+6k+36k^2+2\ {}^kC_2+4\left({}^kC_2\right)^2+2\ {}^kC_2$$
$$+4\left({}^kC_2\right)^2+2\ {}^dC_2+4\left({}^dC_2\right)^2+6d+36d^2+14+(14)^2]$$

$$W[M_v D(T)] = \frac{1}{2}[k(9k^3 - 12k^2 + 93k + 14) + d(37d + 7) + 4kd(2kd - k + 1) + 4\ {}^kC_2(2\ {}^kC_2 + 1) + 2\ {}^dC_2(\ {}^dC_2 + 1) + 252].$$

Thus, the result follows.

In the following theorem we deduce the hyper-Wiener index of dominating graph $D(T)$ of Kragujevac tree T by using Lemma 7 and definition of hyper-Wiener index.

Theorem 12: *The hyper-Wiener index of the dominating graph of Kragujevac tree* T *is*

$$WW(D(T)) = \frac{1}{2}[4k[4k^3 + 35k + 5] + 4d[20d + 3] + 4kd[1 + 2kd] + 2\ {}^kC_2[1 + 2\ {}^kC_2] + 4\ {}^dC_2[1 + 2\ {}^dC_2] + 344].$$

Proof: By the definition of hyper-Wiener index and Lemma 7, we have

$$\begin{aligned}
WW(D(T)) &= \frac{1}{2}\sum_{(u,v)\in V}[d_{D(T)}(u,v) + d_{D(T)}(u,v)^2] \\
&= \frac{1}{2}[\sum_{i=1}^{d}[d_{D(T)}(u,v_i) + d_{D(T)}(u,v_i)^2] + \sum_{i=1}^{k}[d_{D(T)}(u,x_i) + d_{D(T)}(u,x_i)^2] \\
&\quad + \sum_{i=1}^{k}[d_{D(T)}(u,w_i) + d_{D(T)}(u,w_i)^2] + \sum_{i=1}^{k}[d_{D(T)}(x_i,\sum_{i=1}^{d}v_i) \\
&\quad + d_{D(T)}(x_i,\sum_{i=1}^{d}v_i)^2] + \sum_{i=1}^{k}[d_{D(T)}(w_i,\sum_{i=1}^{d}v_i) + d_{D(T)}(w_i,\sum_{i=1}^{d}v_i)^2] \\
&\quad + \sum_{i=1}^{k}[d_{D(T)}(x_i,w_i) + d_{D(T)}(x_i,w_i)^2] + \sum_{i<j}[d_{D(T)}(x_i,x_j) + d_{D(T)}(x_i,x_j)^2] \\
&\quad + \sum_{i<j}[d_{D(T)}(w_i,w_j) + d_{D(T)}(w_i,w_j)^2] + \sum_{i<j}[d_{D(T)}(v_i,v_j) + d_{D(T)}(v_i,v_j)] \\
&\quad + [d_{D(T)}(u,\sum_{i=1}^{4}D_i) + d_{D(T)}(u,\sum_{i=1}^{4}D_i)^2] + \sum_{i=1}^{k}[d_{D(T)}(x_i,\sum_{i=1}^{4}D_i) \\
&\quad + d_{D(T)}(x_i,\sum_{i=1}^{4}D_i)^2] + \sum_{i=1}^{k}[d_{D(T)}(w_i,\sum_{i=1}^{4}D_i) + d_{D(T)}(w_i,\sum_{i=1}^{4}D_i)^2] \\
&\quad + \sum_{i=1}^{d}[d_{D(T)}(v_i,\sum_{i=1}^{4}D_i) + d_{D(T)}(v_i,\sum_{i=1}^{4}D_i)^2] \\
&\quad + \sum_{i<j}[d_{D(T)}(D_i,D_j) + d_{D(T)}(D_i,D_j)^2]] \\
&= \frac{1}{2}[(4d + (4d)^2) + (2k + 4k^2) + (2k + 4k^2) + (2kd + 4k^2d^2) + (2kd + 4k^2d^2) \\
&\quad + (4(k)(k) + 16k^4) + (2\ {}^kC_2 + 4\ ({}^kC_2)^2) + (2\ {}^kC_2 + 4\ ({}^kC_2)^2) + (2\ {}^dC_2 + 4\ ({}^dC_2)^2) \\
&\quad + (8 + 64) + (8k + 64k^2) + (8k + 64k^2) + (8d + 64d^2) + (16 + 256)]
\end{aligned}$$

$$WW(D(T)) = \frac{1}{2}[4k[4k^3 + 35k + 5] + 4d[20d + 3] + 4kd[1 + 2kd] + 2\ {}^kC_2[1 + 2\ {}^kC_2] + 4\ {}^dC_2[1 + 2\ {}^dC_2] + 344].$$

Hence the proof.

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