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SOLUTION OF ABEL'S INTEGRAL EQUATIONS USING LEGENDRE POLYNOMIALS AND FRACTIONAL CALCULUS TECHNIQUES

Z. Avazzadeh*, B. Shafiee and G. B. Loghmani

Department of Mathematics, Faculty of Science, Yazd University,
P. O. Box: 89195-741, Yazd, Iran

E-mail: *z.avazzedeh@yahoo.com*

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ABSTRACT

In this paper, we offer a new approach for solving Abel's integral equations as singular integral equation. Since Abel's integral equation can be considered the fractional integral equation, we use fractional integral for solving it. The fractional operator is considered in the sense of Riemann- Liouville. Computation of fractional integral for arbitrary function are directly hard and cost, hence we approximate the integrand function by Legendre polynomials. Although Abel's integral equations as convolution and singular integral equation are heavily and difficult in computation, some numerical examples shows high accurate and good efficiency.

Keywords: *Abel's integral equation; Fractional calculus; Legendre polynomial; Collocation method.*

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1. INTRODUCTION

Abel's integral equations are investigated by NielHenrik Abel in 1823 and Liouville in 1832 as a fractional integral equation. Fractional calculus as an important subject in mathematical analysis can be used for solving some singular integral equations. Reader can be find the more detail about fractional calculus [13, 15, 18, 20].

Abel's integral equations appear in many different problems of basic and engineering sciences such as physics, chemistry, biology, electronics and mechanics [11]. Some new applications of Abel's integral equations can be found in [7, 8, 16].

Abel's integral equations often are expressed in two forms the first and second kind as follow respectively.

$$f(x)=\int_0^x \frac{u(t)}{(x-t)^{\alpha}}dt, \tag{1}$$

and

$$u(x)=f(x)+\int_0^x \frac{u(t)}{(x-t)^{\alpha}}dt, \tag{2}$$

where$0<\alpha<1$, $f(x)\in C[0,T]$, $0\le x,t\le T$ and T is constant.

There are many different methods for solving integral equations [3,21,22] which only some of them are efficient for singular integral equations [2,4,9]. In particular, Abel's integral equations of the first and second kind with singularity properties cause hard and heavy computations [1, 5, 6, 11, 14, 19, 23, 24].

In this study, we use fractional calculus properties for solving of these singular and convolution integral equations. Since fractional operators are not efficient for most of functions, we approximate the integrand function by Legendre polynomials. Finally, by using the collocation method we obtain the system of linear equations. In fact, directly using of the collocation method leads to ill-conditioned systems while fractional operators can reduce ill-conditioning and improve the solutions.

The structure of paper is in the following way: In section 2, we present the fractional integral operators and some of its properties. In section 3, we apply fractional integral and Legendre polynomials. Then we will obtain the solution of

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**Corresponding author: Z. Avazzadeh*, *E-mail: z.avazzedeh@yahoo.com*

Abel's integral equation by using collocation method. Section 4 includes some examples solved by the method. The numerical results confirm ability and effectiveness of the method.

2. BASIC DEFINITION

Definition: 2.1 A real function $u(x), x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p(> \mu)$, such that $u(x) = x^p v(x)$, where $v(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $u^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition: 2.2 The Riemann-Liouville fractional integral operator of order$\alpha \geq 0$, of a function $u(x) \in C_\mu$, $\mu \geq -1$ is defined as

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt, \alpha > 0, \quad x > 0, \tag{3}$$

such that $J^0 u(x) = u(x)$.

Proposition: 2.1 The operator J^α in above definition satisfy in the following properties for $u_i \in C_\mu$, $i = 0 \ldots n$, $\mu \geq -1$

1. $J^\alpha (\sum_{i=0}^n u_i(x),) = \sum_{i=0}^n J^\alpha u_i(x),$
2. $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}, \qquad \beta > -1.$

More properties of the fractional integral can be found in [13, 20].

In following, we introduce briefly orthogonal Legendre polynomials as a suitable tool for approximation in the method.

Definition: 2.3 The Legendre polynomials, denoted by L_n, constitute a family of orthogonal polynomials on $[-1,1]$. The successive polynomials can be constructed by the following recurrence relation.

$$\begin{cases} L_0(x) = 1, \\ L_1(x) = x, \\ (n+1) L_{n+1}(x) = (2n+1) x L_n(x) - n L_{n-1}(x). \end{cases} \tag{4}$$

3. SOLUTION OF ABEL'S INTEGRAL EQUATIONS

In the method, we use the Legendre polynomials and fractional calculus techniques for approximating the solution of Abel's integral equations. We describe the method in detail for the first and second kind in two subsections.

3.1. The first kind

According to (1) and (3), Abel's integral equation of the first kind can be rewritten as follow

$$f(x) = \Gamma(1-\alpha) J^{1-\alpha} u(x). \tag{5}$$

Since calculating of $J^{1-\alpha} u(x)$ is directly cost and inefficient, we will use Legendre polynomials to approximate $u(x)$. We assume $u(x)$ on interval $[-1,1]$ can be written as a infinite series of Legendre basis

$$u(x) = \sum_{i=0}^{\infty} a_i L_i(x). \tag{6}$$

For interval$[a, b]$, we can use suitable change of variable to obtain the interval. So we express $u(x)$ as a truncated Legendre series as follow

$$u_n(x) = \sum_{i=0}^{n} a_i L_i(x), \tag{7}$$

such that $u_n(x)$ will be approximated solution of Abel's integral equation. Now, we can write (5) in the form

$$f(x) = \Gamma(1-\alpha) J^{1-\alpha} \left(\sum_{i=0}^{n} a_i L_i(x) \right). \tag{8}$$

Clearly, we can reorder Legendre series as follow

$$\sum_{i=0}^{n} a_i\, L_i(x) = \sum_{i=0}^{n} c_i\, x^i, \tag{9}$$

where c_i is linear combination of a_i, $i = 0,..,n$. According to (9), (8) is transformed to the following form correspondingly

$$f(x) = \Gamma(1-\alpha) J^{1-\alpha}\left(\sum_{i=0}^{n} c_i\, x^i\right). \tag{10}$$

Note that, with applying the linear combination property of fractional integral according to proposition 2.1. we will have

$$f(x) = \Gamma(1-\alpha) \sum_{i=0}^{n} c_i\, J^{1-\alpha} x^i. \tag{11}$$

Now, we substitute the roots of Legendre polynomial of degree $n+1$ as collocation points in (11). It leads to the system of linear equations. By solving of the obtained system we will have the approximated solution of Abel's integral equation as the mentioned form in (7).

We emphasize for more efficiency of the method, we reordered Legendre series as (9). This reformation leads to reduce computation the term $J^{1-\alpha}x^i$ from n^2 to n times. We remind using directly $\{1, x, \dots, x^n\}$ as basis instead of Legendre polynomials leads to an ill-condition system.

3.2. The second kind

We can rewrite (2) with consideration (3) in the form

$$u(x) = f(x) + \Gamma(1-\alpha) J^{1-\alpha} u(x). \tag{12}$$

Similarly, we replace (7) to (12). So we have

$$\sum_{i=0}^{n} a_i\, L_i(x) = f(x) + \Gamma(1-\alpha) J^{1-\alpha}\left(\sum_{i=0}^{n} a_i\, L_i(x)\right), \tag{13}$$

or equivalently by using (9) and proposition 2.1 we get

$$\sum_{i=0}^{n} a_i\, L_i(x) = f(x) + \Gamma(1-\alpha) \sum_{i=0}^{n} c_i\, J^{1-\alpha} x^i. \tag{14}$$

After computing $J^{1-\alpha}x^i$ and substitute the collocation points we have a system of linear equations. Solution of the system leads to the approximated solution of Abel's integral equation. We solve some examples by presented method and assess theaccuracy of the method in the next section.

4. NUMERICAL RESULTS

In this section, the illustrated examples are given to show efficiency of the proposed method in section 3. The criteria of error is root of mean square of error(RMSE) that obtained as follow

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N} (u(x_i) - u_n(x_i))^2}{N}}, \tag{15}$$

where x_i, $i = 1, \dots, N$, are chosen uniformly in interval [0,T]. All of the computations have been done by using Maple 13 with 100 digits precision.

Example: 1 Consider Abel's integral equation [24]

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3),$$

with the exact solution $x^3 - x^2 + 1$. Since the exact solution is a polynomial of degree 3, this method leads to the exact solution for $n \geq 3$.

Example: 2 Consider Abel's integral equation [23]

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt = \frac{3\pi}{8} x^2,$$

with the exact solution $x\sqrt{x}$. A comparison between the exact and approximate solutions at 10 points with uniform mesh in [0, 1] is demonstrated for $n = 10,20,30$ in Table 1. Also, we report RMSE for these points. From Table 1, it can be found that the obtained approximations are fast and accurate.

Example: 3 Abel-type integral equation as [12]

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt = e^x - 1,$$

with the exact solution $\frac{1}{\sqrt{\pi}} e^x \mathrm{erf}(\sqrt{x})$ is considered, which $\mathrm{erf}(x)$ is error function and defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\tau^2}\, d\tau.$$

We report the numerical results in Table 2.

Example: 4 Consider Abel's integral equation as[23]

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt = \sinh(x),$$

with the analytical exact solution as

$$\frac{1}{\pi} \int_0^x \frac{cosh\,(t)}{\sqrt{x-t}}.$$

The numerical results are shown in Table 3.

Example 5. Consider singular integral equation of Abel type the [12]

$$\int_0^x \frac{u(t)}{(x-t)^{\frac{4}{5}}}\, dt = x + 1,$$

and its exact solution is as follows

$$u(t) = \frac{(1 + 1.25x)\sin(0.8\pi)}{\pi \sqrt[5]{x}}.$$

Table 4 presents valuesof $u_n(x)$, exact solution $u(x)$ and RMSE value.

Example: 6 Consider Abel's integral equation of the second kind as follow [10]

$$u(x) = x - \frac{4}{3} x^{\frac{3}{2}} + \int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt,$$

with the exact solution x. Since the exact solution is a polynomial of degree 1, this method gives the exact solution for $n \geq 1$.

Example: 7 Let following Abel's integral equation of the second kind [19, 24]

$$u(x) = x^2 + \frac{16}{15} x^{\frac{5}{2}} - \int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt,$$

with the exact solution x^2. Similarly the previous example, since the exact solution is a polynomial of degree 2, this method leads to the exact solution for $n \geq 2$.

Example: 8 Consider Abel's integral equation of the second kind [1]

$$u(x) = 1 - \int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt,$$

with the exact solution $e^{\pi x}\mathrm{erf}c(\sqrt{\pi x})$, which $\mathrm{erf}c(\sqrt{\pi x})$ is complementary error function and defined by

$$\mathrm{erf}c(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-\tau^2}\, d\tau.$$

Numerical results are shown in Table 5 for some different n.

Example: 9 Consider Abel's integral equation of second kind [19,24]

$$u(x) = \frac{1}{x+1} + \frac{2\, arcsinh(\sqrt{x})}{\sqrt{1+x}} - \int_0^x \frac{u(t)}{\sqrt{x-t}}\, dt,$$

with the exact solution $u(x) = \frac{1}{x+1}$. RMSE values are $5.90 \times 10^{-9}, 1.56 \times 10^{-16}$ and 3.83×10^{-24} for $n = 10{,}20$ and 30 respectively. Error function for $n = 30$ is illustrated in Figure 1.

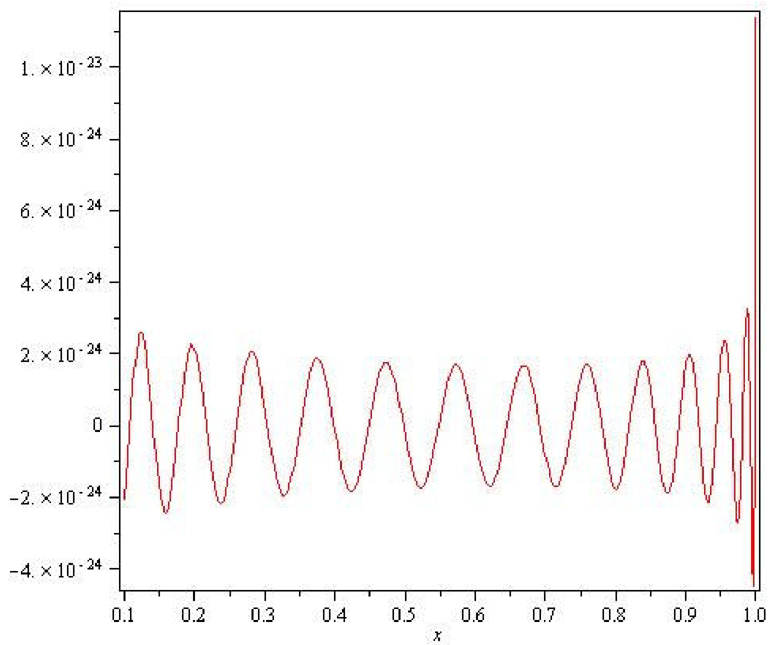

Figure: 1 Error function of Example 8 for *n=30***.**

5. CONCLUSION

In this method, we apply the approximation by orthogonal Legendre polynomials and the fractional technique for solving Abel's integral equations. This method is flexible and extensible for more generalized forms, i. e.

$$f(x) = \int_0^x \frac{u(t)^n}{(x-t)^\alpha} dt, \tag{16}$$

and

$$u(x) = f(x) + \int_0^x \frac{u(t)^n}{(x-t)^\alpha} dt, \tag{17}$$

where $0 < \alpha < 1$, $f(x) \in C[0,T]$, $0 \le x, t \le T$and T is constant. Replacing(7) in the integral equation leads to a system of nonlinear equations with respect to $a_i, i = 0,1, \dots, n$.

We note that this method is easy to computation and running. Also, ability and efficiency of the method are great. In particular, when the solution of problem is in a power series form, the method evaluates the exact solution. It is observed in Example 1, 6 and 7.

Table 1: Estimated and exact solution of Example 2

x	n=10	n=20	n=30	exact
0.1	0.031608	0.0316222	0.03162273	0.03162277
0.2	0.089449	0.0894422	0.08944281	0.08944271
0.3	0.164304	0.1643159	0.16431672	0.16431676
0.4	0.252996	0.2529824	0.25298214	0.25298221
0.5	0.353544	0.3535538	0.35355332	0.35355339
0.6	0.464757	0.4647575	0.46475794	0.46475800
0.7	0.585670	0.5856619	0.58566194	0.58566201
0.8	0.715529	0.7155422	0.71554168	0.71554175
0.9	0.853830	0.8538156	0.85381496	0.85381496
1.0	1.000111	1.0000089	1.00000193	1.00000000
RMSE	3.69×10^{-5}	2.68×10^{-6}	6.1×10^{-7}	

Table 2: Estimated and exact solution of Example 3

x	n=10	n=20	n=30	exact
0.1	0.21574	0.21534	0.21529	0.21529
0.2	0.32569	0.32593	0.32586	0.32588
0.3	0.42790	0.42764	0.42757	0.42756
0.4	0.52893	0.52930	0.52934	0.52933
0.5	0.63528	0.63499	0.63504	0.63503
0.6	0.74704	0.74708	0.74705	0.74704
0.7	0.86696	0.86719	0.86720	0.86718
0.8	0.99742	0.99704	0.99710	0.99708
0.9	1.13789	1.13823	1.13827	1.13829
1.0	1.28942	1.29153	1.29198	1.29238
RMSE	9.8×10^{-4}	2.8×10^{-4}	1.9×10^{-4}	

Table 3: Estimated and exact solution of Example 4

x	n=10	n=20	n=30	exact
0.1	0.20231	0.20190	0.20186	0.20185
0.2	0.28755	0.28779	0.28772	0.28774
0.3	0.35745	0.35718	0.35711	0.35710
0.4	0.41958	0.41996	0.42000	0.41998
0.5	0.48090	0.48060	0.48066	0.48064
0.6	0.54156	0.54160	0.54156	0.54155
0.7	0.60419	0.60443	0.60444	0.60442
0.8	0.67095	0.67056	0.67062	0.67060
0.9	0.74085	0.74119	0.74124	0.74126
1.0	0.81445	0.81661	0.81706	0.81747
RMSE	1.0×10^{-3}	2.7×10^{-4}	1.2×10^{-4}	

Table 4: Estimated and exact solution of Example 5

x	n=10	n=20	n=30	exact
0.1	0.33033	0.33266	0.33322	0.33359
0.2	0.32447	0.32214	0.32302	0.32268
0.3	0.32595	0.32695	0.32733	0.32730
0.4	0.33824	0.33739	0.33705	0.33709
0.5	0.34899	0.34935	0.34920	0.34924
0.6	0.36228	0.36239	0.36259	0.36264
0.7	0.37753	0.37682	0.37667	0.37674
0.8	0.39052	0.39152	0.39116	0.39127
0.9	0.40695	0.40632	0.40617	0.40604
1.0	0.42469	0.42274	0.42209	0.42097
RMSE	1.8×10^{-3}	6.8×10^{-4}	3.9×10^{-4}	

Table 5: Estimated and exact solution of Example 8

x	$n=10$	$n=20$	$n=30$	exact
0.1	0.58427	0.58574	0.58589	0.58594
0.2	0.49256	0.49154	0.49168	0.49164
0.3	0.43505	0.43564	0.43570	0.43569
0.4	0.39715	0.39671	0.39665	0.39665
0.5	0.36707	0.36714	0.36713	0.36713
0.6	0.34347	0.34363	0.34367	0.34367
0.7	0.32477	0.32441	0.32439	0.32439
0.8	0.30783	0.30819	0.30814	0.30815
0.9	0.29460	0.29425	0.29422	0.29421
1.0	0.28336	0.28225	0.28211	0.28205
RMSE	8.0×10^{-4}	9.0×10^{-5}	2.0×10^{-5}	

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