## APPROXIMATIONS OF FIXED POINT MAPPINGS IN BANACH SPACES

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### ABSTRACT

*In this paper we prove some results concerning approximations of Fixed points of pair of mappings in Banach **spaces.***

## 1 INTRODUCTION

Let X be a Banach space with norm ∥.∥ and C be closed, convex and bounded subset of X. In [9] Dotson introduced the concept of quasi-nonexpansive mappings which is as follows:

**Definition 1.1:** A self-mapping T of C is said to be quasi-nonexpansive provided T has a fixed point in C and if
$$\|T(x) - p\| \leq \|x - p\| \tag{1.1}$$
is true for all x Є C

In the following we consider a new class of contraction mappings.

**Definition 1.2:** A self-mapping T of C with nonempty fixed point set F(T) in C is said to be a F-contraction if $p \in F(T)$, then for some q, $0 \leq q < 1$;
$$\|T(x) - p\| \leq q \|x - p\| \tag{1.2}$$
is true for all x Є C

It is clear that every F-contraction mapping is contained in the class of quasi-nonexpansive mappings.

In a uniformly convex Banach space, Senter and Dotson [8] have given condition under which Mann type iterates of quasi-nonexpansive mappings converge to fixed point of the mapping. Here we consider a pair of mappings in Banach space satisfying quasi-nonexpansive type conditions of Ciric [1] and using an approach similar to Senter and Dotson [8], we prove some results concerning the approximation of fixed points of such pairs of mappings. For this we need the following conditions given by Senter and Dotson [8].

**Condition 1:** Let T: C → C be with a nonempty fixed point set F(T) in C such that there is a non-decreasing function f: [0;∞) → [0;∞) with f(0) = 0; f(t) > o for t Є(0; ∞), satisfying
$$\|x - Tx\| > f(d(x; F(T))) \tag{1.3}$$

For x ЄC where d(x, F(T)) = inf{ ∥x-p∥: p ЄF(T)}.

**Condition 2:** Let T: C →C be with a nonempty fixed point set F(T) in C such that there is a real number α > 0 for which
$$\|x - Tx\| > \alpha(d(x; F(T))) \tag{1.4}$$
holds for all x Є C

It can be easily seen that the mapping satisfying condition 2 also satisfies condition 1.

We need the following Theorem [10]

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**Theorem 1.1:** Let C be a nonempty, closed convex and bounded subset of normed linear space X and let $T_1$, $T_2$:C$\rightarrow$ C be two mappings satisfying conditions:

i) $\|T_1x-T_2y\| \leq a(\|x-T_1x\|+\|y-T_2y\|)+b \max(\|x-y\|, \|x-T_1x\|,\|y-T_2y\|, 1/2(\|x-T_1y\|+\|y-T_1x\|)\}$        (1.5)
         for x, y $\in$ C, where a and b are nonnegative real numbers satisfying a + b < 1.

and

ii) for some $x_0 \in$ C the sequence $\{x_n\}$ where
$$x_{2n+1} = (1 - t)x_{2n} + t\, T_1x_{2n},$$
$$x_{2(n+1)} = (1 - t)x_{2n+1} + t\, T_2x_{2n+1}, \; n = 0; 1, 2,\ldots \text{ and } t \in (0, 1) \text{ converges to a point u.}$$

Then u is a common fixed point of $T_1$ and $T_2$.

## 2. MAIN RESULT

**Theorem 2.1:** Let X be a uniformly convex Banach space. Let C be a nonempty, closed, convex and bounded subset of X. If $T_1$ and $T_2$ are two mappings on C into itself satisfying condition (1.5) with a+ b < 1 / 2. Then for arbitrary $x_1$, $x'_1 \in$ C, $M(x_1, t_n, T_1)$ and $M(x'_1, t_n, T_2)$ converge to a unique common fixed point of $T_1$ and $T_2$.

**Proof:** By Theorem 1.1, $T_1$ and $T_2$ have a unique fixed point, say p $\in$ C. If u and w are fixed points of $T_1$ and $T_2$ respectively, then v = w = p. For suppose that v $\neq$ w, then by condition (1.5), we get
$$\|v - w\| = \|T_1v - T_2w\| \leq a(\|v - T_1v\| + \|w - T_2w\|) + b \max\{\|v - w\|, \|v - T1v\|, \|w - T_2w\|,$$
$$1/2(\|v - T_1w\| + \|w - T_1v\|)\}$$
$$= a(\|v - v\| + \|w - w\|) + b \max\{\|v - w\|, \|v - v\|, \|w - w\|, 1/2(\|v - w\| + \|w - v\|\}$$
$$= b \max(0; \|v - w\|)$$
$$= b\|v - w\|$$

which is contradiction since b < 1. Hence v = w is common fixed point of $T_1$ and $T_2$. By Uniqueness of the common fixed point p of $T_1$ and $T_2$, we have v = w = p.

Therefore, to prove the Theorem, we only show that the sequences $M(x_1, t_n, T_1)$, $M(x'_1, t_n, T_2)$ Converge to the fixed points of $T_1$ and $T_2$ respectively. Clearly, each of $T_1$ and $T_2$ is quasi-nonexpansive on C.

For, let x $\in$ C, then
$$\|T_1x - p\| = \|T_1x - T_2p\| \leq a(\|x - T_1x\| + \|p - T_1p\|) + b \max\{\|x - p\|, \|x - T_1x\|, \|p - T_2p\|,$$
$$1/2(\|x - T_2p\| + \|p - T_1x\|)\}$$
$$= a\|x - T_1x\| + b \max\{\|x - p\|, \|x - T_1x\|, 1/2(\|x - p\| + \|x - T_1x\|)\} \quad\quad (2.1)$$

Now there are three cases:

**Case-1:** If max $\{\|x-p\|; \|x- T_1x\|; 1/2(\|x-p\|; \|p- T_1x\|)\} = \|x-p\|$
$$\|T_1x-p\| \leq a\|x- T_1x\| + b\|x-p\|$$
$$\leq a\|x - p\| + a\|T_1x - p\| + b\|x - p\|$$

i:e,          $\|Tx_1 - p\| \leq \Gamma_1\|x - p\|$                                       (2.2)

where $\Gamma_1 = (a+b)/(1-a) < 1$ Since, a + b < 1/2 .

**Case-2:** If max $\{\|x-p\|; \|x-T_1 x\|; 1/2(\|x-p\|; \|p-T_1x\|)\} = \|x-T_1x\|$
$$\|T_1x-p\| \leq a\|x-T_1x\|+b\|x-T_1x\|$$
$$= (a + b)\|x - T_1x\|$$
$$\leq (a + b)\|x - p\| + (a + b)\|T_1x - p\|$$

$$\|T_1x - p\| \leq \Gamma_2\|x - p\| \quad\quad (2.3)$$
where $\Gamma_2 = (a+b)/1-(a+b) < 1$

Since, a + b < 1.

**Case-3:** If max $\{\|x - p\|, \|x - T_1x\|, 1/2(\|x - p\| + \|p - T_1x\|)\} = 1/2(\|x - p\| + \|p - T_1x\|)$
$$\|T_1x - p\| \leq a\|x - T_1x\| + b\, 1/2(\|x - p\| + \|p - T_1x\|)$$
$$\leq a\|x-p\|+a\|T_1x-p\|+b/2\|x-p\|+ b/2\|p-T_1x\|$$
$$= (a+b/2)\|x-p\| + (a+b/2)\|T_1x-p\|$$
$$\leq (a+b/2)/(1-(a+b/2))\|x-p\|$$
$$\|T_1x - p\| \leq \Gamma_3\|x-p\|$$

Where $\Gamma_3 = (a+b/2) / (1-(a+b/2)) < 1$ since $a+b<1/2$.

Let $\Gamma = \max (\Gamma_1, \Gamma_2, \Gamma_3)$ then from (2.2), (2.3) and (2.4) we obtain
$$\| T_1 x - p \| \leq \alpha \|x - p\| \quad \text{for all } x \in C.$$

This shows that $T_1$ is F-contraction and hence $T_1$ is quasi-nonexpansive on C. Similarly it can be shown on same line that $T_2$ is quasi-non expansive on C.

Next we show that $T_1$ and $T_2$ satisfy condition 2 on C.

For this we see again three cases

**Case-1:** If $\max \{\|x-p\|, \|x-T_1x\|, 1/2 (\|x-p\|+ \|p-T_1x\|)\} = \|x-p\|$. Then from (2.1) we obtain
$$\|T_1x-p\| \leq a\|x-T_1x\|+b \|x-p\|$$

$$\| x-p \| - \|x-T_1x \| \leq \|T_1x-p\|$$
$$\leq a \| x-T_1x \| + b \|x-p \|$$

$$(1-b) \| x- p\| \leq (1+a) \|x-T_1x \|$$

$$\| x-T_1x \| \geq (1-b)/ (1+a) \| x-p \|$$

$$\| x-T_1x \| \geq \alpha_1 \| x-p \|$$

For all $x \in C$ where $\alpha_1 = (1-a)/ (1+a) > 0$

**Case-2:** If $\max \{\|x-p\|, \|x-T_1x\|, 1/2 (\|x-p\|+ \|p-T_1x\|)\} = \|x-T_1x\|$

By (2.1) we obtain
$$\| T_1x-p\| \leq a\|x-T_1x\| +b \| x-T_1x \|$$
$$= (a +b) \| T_1x-p \|$$
$$\| x-p \| - \| x-T_1x\| \leq \| x-T_1x \|$$

$$\| x-p \| - \| x-T_1x\| \leq (a +b) \| x-T_1x \|$$

$$\| x-p \| \leq (1+a+b) \| x-T_1x \|$$

$$\| x-T_1x \| \geq 1/ (1+a+b) \| x-p \|$$

$$\| x-T_1x \| \geq \alpha_2 \| x-p \|$$
For all $x \in C$ where $\alpha_2 = 1/ (1+a+b)>0$.

**Case-3:** If $\max \{\|x - p\|, \|x - T_1x\|, 1/2 (\|x - p\|+ \|p - T_1x\|)\} = 1/2 (\|x - p\|+ \|p - T_1x\|)$

$$\|T_1x - p\| \leq a \|x - T_1x\| + b \ 1/2 (\|x - p\| + \|p - T_1x\|)$$

$$(1-b/2) \| T_1x-p \| \leq a \| x-T_1x\|+b/2 \|x-p \|$$

$$(1-b/2) \{\|x-p \| -\|x-T_1x\|\} \leq a \| x-T_1x\|+b/2 \ \|x-p \|$$

$$(a+ (2-b)/2)\| x-T_1x\| \geq ((2-b)/ (2 - b/2) \|x-p \|$$

$$\| x-T_1x\| \geq (2-2b)/ (2a-b+2) \|x-p \|$$

$$\| x-T_1x\| \geq \alpha_3 \|x-p \| \text{ for all } x \in C \text{ where } \alpha_3 = (2-2b)/(2a-b+2 ) > 0.$$

Let $\alpha = \min (\alpha_1, \alpha_2, \alpha_3)$ then
$$\| x-T_1x\| \geq \alpha \|x-p \|, \alpha >0, \text{ for all } x \in C.$$

Thus $T_1$ satisfies condition 2 and hence condition 1 on C. Repeating the similar arguments, it can be shown that $T_2$ satisfies the condition 2 and hence 1 on C. Thus by theorem 3[8], for arbitrary $x_1, x'_1 \in C$, the sequences $M(x_1, t_n , T_1)$ and $M(x'_1, t_n, T_2)$ converge to a fixed point $T_1$ and $T_2$ respectively. Consequently the sequences $M(x_1, t_n, T_1)$ and $M(x'_1, t_n, T_2)$ converge to a unique common fixed point of $T_1$ and $T_2$. This complete the proof.

We note that by taking $T_1 = T_2 = T$ in the above Theorem we get the following result as a corollary.

**Corollary 2.1:** Let C be non-empty, closed, convex and bounded subset of uniformly convex Banach Space X. If T is a mapping on C into itself satisfying the condition (1.5), with a +b < 2, then for arbitrary $x_1 \in C$, the sequence $M(x_1, t_n, T)$ converges to a unique fixed point of T.

**Theorem 2.2:** Let C be a nonempty, closed, convex and bounded subset of uniformly convex Banach spaces X and let $T_1 \ T_2 : C \rightarrow C$ be two mappings such that there exists r and t satisfying condition

$$\|T^r_1 x - T^s_2 y\| \le a(\|x - T^r_1 x\| + \|y - T^s_2 y\|) + b \max\{\|x-y\|, \|x - T^r_1 x\|, \|y - T^s_2 y\|, \tfrac{1}{2}(\|x - T^s_2 y\| + \|y - T^r_1 x\|)\} \qquad (2.5)$$

For every x, y $\in$ C and a+b<1/2. Then for arbitrary $x_1$, $x'_1 \in C$ the sequences $M(x_1, t_n, T^r_1)$ and $M(x'_1, t_n, T^s_2)$ converge to a unique common fixed point of $T_1$ and $T_2$.

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