PULSATILE FLOW OF BLOOD THROUGH AN INCLINED BELL SHAPE STENOSED TUBE UNDER PERIODIC BODY ACCELERATION WITH MAGNETIC EFFECT

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ABSTRACT

This is about the mathematical model for blood flow through an inclined bell shape stenosed tube under periodic body acceleration and magnetic field. Using appropriate boundary conditions, analytical expressions for the velocity, the volumetric flow rate, the fluid acceleration have been derived. These expressions are computed numerically and the computational results are presented graphically.

Key Words: Pulsatile flow, Stenosis, Bell shaped Stenosis, Periodic body acceleration, magnetic field, Inclined tubes.

INTRODUCTION


MATHEMATICAL FORMULATION

Let us consider a one-dimensional pulsatile flow of blood through a uniform straight and stenosed inclined cylindrical tube using magnetic field by considering blood as a couple stress fluid. The flow is considered as axially symmetric, pulsatile and fully developed. The pressure gradient and body acceleration G are given by the expressions

\[
\frac{\partial P}{\partial z} = A_0 + A_1 \cos(\omega t)
\]

\[
G = a_0 \cos(\omega t + \phi)
\]
where $A_0$ and $A_1$ are pressure gradient of steady flow and amplitude of oscillatory part respectively, $a_0$ is the amplitude of body acceleration, $\omega_p = 2\pi f_p$, $\omega_b = 2\pi f_b$, with $f_p$ is the pulse frequency and $f_b$ is body acceleration frequency, $\phi$ is the phase angle of body acceleration $G$ with respect to pressure gradient and time, $t$.

The pulsatile couple stress equation (Stokes), in cylindrical polar coordinates under the periodic body acceleration with inclined angle and magnetic field can be written in the form:

$$\frac{\rho \partial u}{\partial t} = -\frac{\partial p}{\partial z} + \rho G + \mu \nabla^2 u - \eta \nabla^2 \left(\nabla^2 u\right) + \rho g \sin \theta - \sigma B_0^2 u$$

Where $\nabla^2 = \frac{1}{\xi} \left(\frac{\partial}{\partial \xi} \left(\xi \frac{\partial}{\partial \xi}\right)\right)$

(3)

where $u(\xi,t)$ is axial velocity, $\rho$ and $\mu$ are the density and viscosity of blood, $\eta$ is the couple stress parameter and $\xi$ is the radial coordinate.

Let us introduce the following dimensionless quantities:

$$u^* = \frac{u}{\omega R}, t^* = t \omega, A_0^* = \frac{R}{\mu \omega} A_0, A_1^* = \frac{R}{\mu \omega} A_1, a_0^* = \frac{\rho R}{\mu \omega} a_0, z^* = \frac{z}{R}, g^* = \frac{\rho R}{\mu \omega} g$$

(4)

Consider the axisymmetric flow of blood through a bell shaped stenosis, specified at the location as shown in Figure 1, in an artery. The geometry of the stenosis, assumed to be manifested in the arterial wall segment, is described (Srivastava et al. 2012) as

$$\frac{R(z)}{R_0} = \begin{cases} 1 - \delta \exp \left(-\frac{\omega^2 \epsilon^2 z^2}{R_0^2}\right); |z| \leq L_0 \\ 1; otherwise \end{cases}$$

(5)

$Figure-1$: The geometry of a bell shaped stenosis in an artery

where $R_0$ is the radius of the arterial segment in the non-stenotic region, $R(z)$ is the radius of the stenosed portion located at the axial distance $z$ from the left end of the segment, $\delta$ is the depth of stenosis at the throat and, $\omega$ is a parametric constant, $\epsilon$ is the relative length of the constriction, defined as the ratio of the radius to half length of the stenosis, i.e., $\epsilon = R_0 / L_0$ and let us introduce a radial coordinate transformation given by:

$$\xi = \frac{r}{R(z)}$$

In terms of these variables, equation (3) [after dropping stars] becomes:

$$\tilde{\alpha}^2 \alpha^2 \frac{\partial u}{\partial t} = \tilde{\alpha}^2 A_0 + \tilde{\alpha}^2 A_1 \cos \left(bt + \phi\right) + \tilde{\alpha}^2 a_0 \cos \left(ct + \phi\right)$$

$$+ \tilde{\alpha}^2 \left(1 \frac{\partial}{\xi} \left(\xi \frac{\partial}{\xi}\right)\right) - \frac{1}{\xi} \left(\xi \frac{\partial}{\xi}\right) \left(\xi \frac{\partial}{\xi}\right) - \frac{1}{\xi} \left(\xi \frac{\partial}{\xi}\right) \left(\xi \frac{\partial}{\xi}\right) + \tilde{\alpha}^2 g \sin \theta - \tilde{\alpha}^2 H^2 u$$

(6)
where \( \tilde{\alpha}^2 = R^2 \left( \frac{\mu}{\eta} \right) \) -- Couple stress parameter, \( \alpha^2 = R^2 \left( \frac{\omega \rho}{\mu} \right) \) -- Womersley parameter,

\[
b = \frac{\alpha}{\omega}, \quad c = \frac{\alpha}{\omega}, \quad H = B R \left( \frac{\sigma}{\mu} \right)^{\frac{1}{2}}
\]

-- Hartman number and \( R \) -- radius of the tube

The initial and boundary conditions for this problem are:

\[
u(\xi, 0) = 2 \sum_{n=1}^{\infty} \frac{J_0(\xi \lambda_n)}{\lambda_n J_1(\lambda_n)} \left[ A_n + A_0 + a_0 \cos \Phi + g \sin \theta \right]
\]

\[
\left[ \lambda_n^2 (\lambda_n^2 + \alpha^2) + \tilde{\alpha}^2 H^2 \right]
\]

\( u \) and \( \nabla^2 u \) are all finite at \( \xi = 0 \)

\( u = 0, \nabla^2 u = 0 \) at \( \xi = 1 \)

\( (7a) \)

\( (7b) \)

\( (7c) \)

**INTEGRAL TRANSFORMS:**

If \( f(\xi) \) satisfies Dirichlet conditions in closed interval \((0, 1)\) then its finite Hankel transform, Sneddon, is defined as

\[
f^*(\lambda_n) = \int_{0}^{1} \xi f(\xi) J_0(\xi \lambda_n) d\xi
\]

Where \( \lambda_n \) are the roots of \( J_0(\lambda_n) = 0 \). Then at each point of the interval at which \( f(\xi) \) is continuous:

\[
f(\xi) = 2 \sum_{n=1}^{\infty} f^*(\lambda_n) \frac{J_0(\xi \lambda_n)}{J_1(\lambda_n)}
\]

Where the sum is taken over all positive roots of \( J_0(\xi) = 0, J_0 \) and \( J_1 \) are Bessel functions of first kind.

The Laplace transform of any function is defined as:

\[
\tilde{f}(s) = \int_{0}^{\infty} e^{-st} f(t) dt, \quad s > 0
\]

\( (10) \)

**SOLUTIONS:**

Employing the Laplace transforms (10) to equation (6) in the light (7b), we get:

\[
\tilde{\alpha}^2 \alpha^2 \left( \tilde{s} u - u(\xi, 0) \right) = \tilde{\alpha}^2 \frac{A_0}{s} + \tilde{\alpha}^2 \frac{A_s}{s^2 + b^2} + \tilde{\alpha}^2 \left( \frac{a_0 (s \cos \varphi - c \sin \varphi)}{s^2 + c^2} + \frac{g \sin \theta}{s} \right) + \tilde{\alpha}^2 \left( \frac{1}{\xi} \frac{\partial}{\partial \xi} \left[ \xi \tilde{u} \right] \right)
\]

\[
- \left( \frac{1}{\xi} \frac{\partial}{\partial \xi} \left[ \xi \frac{\partial}{\partial \xi} \xi \frac{\partial}{\partial \xi} \right] \right) + \tilde{\alpha}^2 g \sin \theta - \tilde{\alpha}^2 H^2 u
\]

\( (11) \)

Now applying the finite Hankel transform (8) to (11) and using (7c) we obtain:

\[
\tilde{u}(\lambda_n, s) = \frac{J_1(\xi \lambda_n) \tilde{\alpha}^2}{\lambda_n} \left[ \frac{A_0}{s} + \frac{A_s}{s^2 + b^2} + \frac{a_0 (s \cos \varphi - c \sin \varphi)}{s^2 + c^2} + \frac{g \sin \theta}{s} \right] \times \frac{1}{\left[ sm + \lambda_n^2 (\lambda_n^2 + \alpha^2) + \tilde{\alpha}^2 H^2 \right]}
\]

\( m = \tilde{\alpha}^2 \alpha^2 \)

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Now rearranging the terms for taking the inverse Laplace transform,

$$
\tilde{u}^*(\lambda_n, s) = \frac{J_1(\xi \lambda_n) \alpha^2}{\lambda_n} \left[ \frac{A_0}{[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2]} \left( \frac{1}{s} \right) + \frac{b^2 m}{(\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2)^2 + b^2 m^2} \right] \\
+ \frac{A_1[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2] \cos(bt) + m \sin(bt)}{(\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2)^2 + b^2 m^2} \\
+ a_0[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2] \cos(ct + \varphi) + c \sin(ct + \varphi)] \left[ \frac{1}{s} \right] + \sum_{n=1}^{\infty} \frac{A_0 + A_1 + a_0 \cos \varphi + g \sin \theta}{(s + h)[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2]} \\
- \frac{g \sin \theta}{\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2} \left[ \frac{1}{s} \right] + \sum_{n=1}^{\infty} \frac{A_0 + A_1 + a_0 \cos \varphi + g \sin \theta}{(s + h)[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2]} \\
- \frac{g \sin \theta}{\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2} \left[ \frac{1}{s} \right] + \sum_{n=1}^{\infty} \frac{A_0 + A_1 + a_0 \cos \varphi + g \sin \theta}{(s + h)[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2]} \\
(13)
$$

where

$$
h = \frac{\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2}{m} \\
(14)
$$

Now taking the inverse Laplace transform of (13) gives

$$
u^*(\lambda_n) = \frac{J_1(\xi \lambda_n) \alpha^2}{\lambda_n} \left[ \frac{1}{[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2]} \\
+ \frac{A_0[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2] \cos(bt) + m \sin(bt)}{(\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2)^2 + b^2 m^2} \\
+ a_0[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2] \cos(ct + \varphi) + c \sin(ct + \varphi)] \\
- \frac{g \sin \theta}{\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2} \left[ \frac{1}{s} \right] + \sum_{n=1}^{\infty} \frac{A_0 + A_1 + a_0 \cos \varphi + g \sin \theta}{(s + h)[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2]} \\
(15)
$$

The finite Hankel inversion of (15) gives the final solution as:

$$
u(\xi, t) = 2 \sum_n \nu^*(\lambda_n) \frac{J_0(\xi \lambda_n)}{J_1(\lambda_n)} \\
(16)
$$

$$
u(\xi, t) = 2 \sum_{n=1}^{\infty} A_0 J_0(\xi \lambda_n) \left[ \frac{1}{\lambda_n J_1(\lambda_n) \left[ \frac{\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2]}{(\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2)^2 + b^2 m^2} \\
+ \frac{A_1[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2] \cos(bt) + m \sin(bt)}{(\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2)^2 + b^2 m^2} \\
+ a_0[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2] \cos(ct + \varphi) + c \sin(ct + \varphi)] \\
\left[ \frac{1}{s} \right] + \sum_{n=1}^{\infty} \frac{A_0 + A_1 + a_0 \cos \varphi + g \sin \theta}{(s + h)[\lambda_n^2 + (\alpha^2 + \alpha^2) + \alpha^2 H^2]} \\
\right) \\
(17)
$$

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The expression for the flow rate $Q$ can be written as:

$$Q = 2\pi \int_0^\xi u \, d\xi$$  \hspace{1cm} (17)

Then

$$Q(\xi, t) = 4\pi \sum_{n=1}^{\infty} \frac{A_0\alpha^2}{\lambda_n^2} \left\{ -e^{-ht} \left[ \frac{1}{\lambda_n^2(\lambda_n + \alpha^2) + \alpha^2 H^2} \right] + \frac{A_1/ A_0 [\lambda_n^2(\lambda_n + \alpha^2) + \alpha^2 H^2] \cos \phi + c m \sin \phi}{\lambda_n^2(\lambda_n + \alpha^2) + \alpha^2 H^2} \right\} + \frac{(1/A_0) g \sin \theta}{\lambda_n^2(\lambda_n + \alpha^2) + \alpha^2 H^2}$$

Similarly the expression for fluid acceleration $F$ can be obtained from:

$$F(\xi, t) = \frac{\partial u}{\partial t}$$  \hspace{1cm} (19)

Then we have

$$F(\xi, t) = 2 \sum_{n=1}^{\infty} \frac{A_0 J_0(\xi \lambda_n^2) e^{\alpha^2 t}}{\lambda_n J_1(\lambda_n)} \left\{ -e^{-ht} \left[ \frac{1}{\lambda_n^2(\lambda_n + \alpha^2) + \alpha^2 H^2} \right] + \frac{A_1/ A_0 [\lambda_n^2(\lambda_n + \alpha^2) + \alpha^2 H^2] \cos \phi + c m \sin \phi}{\lambda_n^2(\lambda_n + \alpha^2) + \alpha^2 H^2} \right\} + \frac{(1/A_0) g \sin \theta}{\lambda_n^2(\lambda_n + \alpha^2) + \alpha^2 H^2}$$

$$(20)$$
RESULTS

Fig. 2. Variation of velocity profiles for different $\delta/R_0$ against $\xi$ with $\phi=30^0, \theta=30^0, H=1, g=9.8$

Fig. 3. Variation of velocity profiles for different $\delta/R_0$ against $\xi$ with $\phi=30^0, \theta=30^0, t=1, g=9.8$

Fig. 4. Variation of flow rate against $t$ with $\phi=30^0, \theta=30^0, t=1, g=9.8$
In Fig 2, velocity u versus $\xi$ for different time t is plotted. It is observed that velocity u decreases with increase in $\xi$. As t increases, velocity u decreases.

In Fig 3, velocity u versus $\xi$ for different H is plotted. Velocity u decreases with increase in $\xi$ and as H increases u decreases.

In Fig 4, flow rate Q versus $\xi$ for different H is plotted. As H increases, flow rate Q decreases and as t increases, Q decreases.

In Fig 5, fluid acceleration F versus $\xi$ for different time t is plotted. As $\xi$ increases, F increases. Also, as t increases, F decreases.

In Fig 6, fluid acceleration F versus $\xi$ for different time H is plotted. As $\xi$ increases, F increases and as H increases, F increases.
REFERENCES


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