INDEPENDENT VERTEX-EDGE DOMINATION IN GRAPHS

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ABSTRACT

Let the vertices and edges of a graph $G$ be called the elements of $G$. A set $X$ of elements in $G$ is a vertex-edge dominating set of $G$ if every element not in $X$ is either adjacent or incident to at least one element in $X$. An vertex-edge dominating set $X$ of elements in $G$ is an independent vertex-edge dominating set of $G$ if any two elements in $X$ are neither adjacent nor incident. The independent vertex-edge domination number $\gamma_{ve}(G)$ of $G$ is the smallest cardinality of an independent vertex-edge dominating set of $G$. In this paper, we obtained bounds for $\gamma_{ve}(G)$.

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1. INTRODUCTION

All graphs considered here are simple, finite, connected and nontrivial. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the vertex set and $E(G)$ be the edge set of $G$. The vertex $v \in V$ is called a pendant vertex, if $\text{deg}_G(v) = 1$ and an isolated vertex if $\text{deg}_G(v) = 0$, where $\text{deg}_G(x)$ is the degree of a vertex $x \in V(G)$. A vertex which is adjacent to a pendant vertex is called a support vertex. We denote $\delta(G)(\Delta(G))$ as the minimum (maximum) degree and $p = |V(G)|$, $q = |E(G)|$ the order and size of $G$ respectively. A spanning subgraph is a subgraph containing all the vertices of $G$. A shortest $u-v$ path is often called a geodesic. The diameter $\text{diam}(G)$ of a connected graph $G$ is the length of any longest geodesic. The neighborhood of a vertex $u$ in $V$ is the set $N(u)$ consisting of all vertices which are adjacent with $u$. A claw is another name for the complete bipartite graph $K_{1,3}$. A claw-free graph is a graph that does not have a claw as an induced subgraph.

A subset $D \subseteq V$ is said to be a dominating set of $G$ if every vertex $V - D$ is adjacent to at least one vertex in $D$. The minimum cardinality of a minimal dominating set is called the domination number $\gamma(G)$ of $G$ [2].

A subset $D$ of $V(G)$ is an independent set if no two vertices in $D$ are adjacent. A dominating set $D$ which is also an independent dominating set. The independent domination number $i(G)$ is the minimum cardinality of an independent domination set [2,3].

A set $F$ of edges in a graph $G = (V, E)$ is called an edge dominating set of $G$ if every edge in $E - F$ is adjacent to at least one edge in $F$. The edge domination number $\gamma'(G)$ of a graph $G$ is the minimum cardinality of an edge dominating set of $G$.

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A set $X$ of elements in $G$ is an vertex-edge dominating set of $G$, if every element not in $X$ is either adjacent or incident to at least one element in $X$. The vertex-edge domination number $\gamma_{ve}(G)$ is the order of a smallest vertex-edge dominating set of $G$ [6].

In this paper, we define a new parameter as follows:

An vertex-edge dominating set $X$ of elements in $G$ is an independent vertex-edge dominating set of $G$ if any two elements in $X$ are neither adjacent nor incident. The independent vertex-edge domination number $\gamma_{ve}^i(G)$ of $G$ is the smallest cardinality of an independent vertex-edge dominating set of $G$.

2. MAIN RESULTS

Observation: For any graph $G$, $\gamma_{ve}^i(G) = \gamma_i(T(G))$, where $T(G)$ denote the total graph of a graph $G$.

In next theorem, we compute the independent vertex-edge domination number of some standard class of graphs.

Theorem 2.1:

(i) For any complete graph $K_p$: $p \geq 2$, $\gamma_{ve}(K_p) = \left\lceil \frac{p}{2} \right\rceil$.

(ii) For any cycle $C_p$: $p \geq 4$, $\gamma_{ve}(C_p) = \left\lceil \frac{p}{2} \right\rceil$.

(iii) For any path $P_p$: $p \geq 2$, $\gamma_{ve}(P_p) = \left\lceil \frac{p}{2} \right\rceil$.

(iv) For any complete bipartite graph $K_{p_1,p_2}$, $\gamma_{ve}(K_{p_1,p_2}) = p_1 \leq p_1 \leq p_2$.

(v) For any wheel $W_p$: $p \geq 4$, $\gamma_{ve}(W_p) = \left\lceil \frac{p}{2} \right\rceil$.

In the following theorem, a relation between $\gamma_{ve}(G)$ and $\gamma_{ve}^i(G)$ is obtained.

Theorem 2.2: For any graph $G$, $\gamma_{ve}(G) \leq \gamma_{ve}^i(G)$.

Proof: It is known that, for any graph $G$, $\gamma(G) \leq \gamma_i(G)$. Therefore similarly $\gamma_{ve}(G) \leq \gamma_{ve}^i(G)$.

Next we obtain the relation between $\gamma_i(G)$, $\gamma_i'(G)$ and $\gamma_{ve}^i(G)$.

Theorem 2.3: For any graph $G$

$$\frac{\gamma_i(G) + \gamma_i'(G)}{2} \leq \gamma_{ve}^i(G) \leq \gamma_i(G) + \gamma_i'(G).$$

Proof. First we establish the lower bound. $D$ and $F$ be the minimum independent dominating and independent edge dominating sets of $G$ respectively. Let $X = D \cup F$ be a minimum independent vertex-edge dominating set of $G$.

For each edge $e = uv$ in $F$, choose a vertex $u$ or $v$ but not both which are independent. Let $F'$ be the collection of such vertices. Clearly $D \cup F'$ is an independent dominating set of $G$. Therefore

$$\gamma_i(G) \leq |D \cup F| = |D \cup F| = \gamma_{ve}^i(G). \tag{2.1}$$

Now for each vertex $u$ in $D$, choose exactly one edge incident with $u$ which is independent. Let $D'$ be the collection of such edges. Clearly $D' \cup F$ is an independent edge dominating set of $G$. Therefore

$$\gamma_i(G) \leq |D \cup F| = |D \cup F| = \gamma_{ve}^i(G). \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$\frac{\gamma_i(G) + \gamma_i'(G)}{2} \leq \gamma_{ve}^i(G).$$

Now for the upper bound, let $D$ and $F$ be the minimum independent dominating and independent edge dominating sets of $G$ respectively. Then $D \cup F$ is an independent vertex-edge dominating set. Thus $\gamma_{ve}^i(G) \leq |D \cup F| = \gamma_i(G) + \gamma_i'(G)$.

Hence, $\gamma_{ve}^i(G) \leq \gamma_i(G) + \gamma_i'(G)$.

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Theorem 2.4: If \( G \) is a claw-free graph, then \( y_{ve}(G) \leq \gamma(G) + \gamma(L(G)) \).

Proof: In [1], it is proved that, if \( G \) is a claw free-graph, then \( \gamma(G) = \gamma_{i}(G) \) and \( \gamma(L(G)) = \gamma_{i}(L(G)) \). Therefore using Theorem 2.3, we get the required result.

Theorem A(6): For any connected graph \( G \) of order \( p \)
\[
\frac{p+q}{2\Delta(G)+1} \leq y_{ve}(G).
\]

Now we establish another lower bound for \( y_{ve}(G) \).

Theorem 2.5: For any graph \( G \)
\[
\frac{p+q}{2\Delta(G)+1} \leq y_{ve}^{i}(G).
\]

Proof: The proof follows from Theorem A and the fact that \( y_{ve}(G) \leq y_{ve}^{i}(G) \).

In a graph \( G \) if \( \text{deg}(v) = 1 \), then \( v \) is called a pendant vertex of \( G \).

Lemma 1: Let \( G \) be a connected graph of order \( p \geq 3 \). Then there exist two nonadjacent vertices \( u \) and \( v \) having a common neighbor \( w \) such that \( G - \{u,v\} \) is connected.

Proof: Let \( T \) denote a spanning tree of \( G \). If \( T \) has exactly one nonpendant vertex, then the removal of any two pendant vertices \( u \) and \( v \) of \( T \), results in a connected graph. Suppose \( T \) has at least two nonpendant vertices. Then there exist at least two nonpendant vertices each of which is adjacent to exactly one nonpendant vertex. If \( w \) is adjacent to at least two pendant vertices \( u \) and \( v \), then removal of \( u \) and \( v \) results in a connected graph. If \( w \) is adjacent to exactly one pendant vertex \( u \), then removal of \( w \) and \( u \) results in a connected graph. This completes the proof.

Theorem 2.6: For any connected graph \( G \) of order \( p \geq 2 \) and \( \delta(G) \geq 2 \).
\[
y_{ve}^{i}(G) \leq \left\lfloor \frac{p}{2} \right\rfloor.
\]

Proof: We prove the result by induction on \( p \).

If \( p = 3 \) or \( 4 \), then the result can be verified. Assume the result is true for all connected graphs \( G \) with \( \delta(G) \geq 2 \) and \( p - 1 \) vertices. Let \( G_{1} \) be a connected graph with \( \delta(G_{1}) \geq 2 \) and \( p \) vertices. Let \( u \) and \( v \) denote two nonadjacent vertices having a common neighbor \( w \) such that \( G = G_{1} - \{uv\} \) is connected. Let \( X \) be a minimum independent vertex-edge dominating set of \( G \), then either \( X \cup \{w\} \) or \( X \cup \{uv\} \) is an vertex-edge dominating set of \( G_{1} \).

Thus \( y_{ve}(G_{1}) \leq |X| + 1 \leq \left\lfloor \frac{p-1}{2} \right\rfloor + 1 \leq \left\lfloor \frac{p}{2} \right\rfloor \).

In the following theorem we give characterization of graphs in which \( y_{ve}(G) = y_{ve}^{i}(G) \).

Theorem 2.7: For any graph \( G \) of order \( p \geq 2 \).
\[
y_{ve}(G) = y_{ve}^{i}(G).
\]

Proof: Let \( y_{ve}(G) = y_{ve}^{i}(G) \). If possible, suppose every vertex-edge dominating set of \( G \) is not independent. Then there exists at least one vertex-edge dominating set \( X \) of \( G \) such that at least two elements of \( X \) are either adjacent or incident. Therefore \( y_{ve}(G) < y_{ve}^{i}(G) \), a contradiction. Sufficiency is obvious.

Theorem 2.8: Let \( G \) be a connected graph of order at least 2. Then \( y(G) = y_{ve}(G) \) if and only if any two adjacent vertices form a minimal dominating set.

Proof: Let \( G \) be a connected graph and let a set consisting of any two vertices of \( G \) form a minimal dominating set of \( G \). Also by the fact that a set consisting of any two adjacent vertices of \( G \) forms a minimal dominating set of \( G \) if and only if \( G \) is isomorphic to the complete \( k \)-partite graph \( K_{p_{1},p_{2},\ldots,p_{k}} \); \( p_{1} \geq 2 \) for each \( i \in \{1,2,\ldots,k\} \) with
the partite sets of sizes \( p_1, p_2, \ldots, p_k \).

\[ G = K_{p_1, p_2, \ldots, p_k} \]

Hence \( \gamma(G) = \gamma^i_{ve}(G) \).

Conversely, let \( \gamma(G) = \gamma^i_{ve}(G) \) and any two adjacent do not form a minimal dominating set. Let \( D = \{v_1, v_2, \ldots, v_k\} \) be the set of all maximal independent vertex set of \( G \). Then \( \gamma(G) = |D| \). Let \( D' = \{e_1, e_2, \ldots, e_s\} \) be the maximal independent set of edges of \( G \). Then \( \gamma^i_{ve}(G) \leq |D \cup D'| > \gamma(G) \), a contradiction. Hence any two adjacent vertices of \( G \) form a minimal dominating set of \( G \) or \( G = K_{p_1, p_2, \ldots, p_k} \).

This completes the proof.

We need the following definition for our next results.

**Definition:** A graph \( G \) is \( k \)-partite, \( k \geq 1 \), if it is possible to partition \( V(G) \) into \( k \) subsets \( V_1, V_2, \ldots, V_k \) called partite sets, such that every element of \( E(G) \) joins a vertex of \( V_i \) to a vertex of \( V_j \), \( i \neq j \).

**Theorem 2.9:** For any connected graph \( G \) of order at least 2, \( \gamma_i(G) = \gamma^i_{ve}(G) \), if and only if \( G = K_{p_1, p_2, \ldots, p_k} \).

**Proof:** Since every independent dominating set is a dominating set, therefore the proof of the following theorem is similar to Theorem 2.8.

**Theorem 2.10:** Let \( G \) be any connected graph of order at least 2. Then \( \gamma^i(G) = \gamma^i_{ve}(G) \), if and only if \( G \) is \( k \)-partite graph.

**Proof:** The proof is similar to the proof of Theorem 2.8.

Finally we prove Nordhaus-Gaddum type results for \( \gamma^i_{ve}(G) \).

**Theorem 2.11:** Let a graph \( G \) and its complement \( \overline{G} \) be connected with \( \delta(G) \geq 2 \). Then

(i) \( \gamma^i_{ve}(G) + \gamma^i_{ve}(\overline{G}) \leq 2 \left[ \frac{\rho}{2} \right] \)

(ii) \( \gamma^i_{ve}(G) \cdot \gamma^i_{ve}(\overline{G}) \leq \left[ \frac{\rho^2}{2} \right] \)

**Proof:** The result follows from Theorem 2.6.

**REFERENCES**


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