THE FORCING STEINER DOMINATION NUMBER OF A GRAPH

K. RAMALAKSHMI*1, K. PALANI2

1Assistant Professor, Department of Mathematics, Sri Sarada College for Women, Tirunelveli 627 011, Tamil Nadu, India.
2Assistant Professor, Department of Mathematics, A. P. C. Mahalakshmi College, Tuticorin 628 002, Tamil Nadu, India.

(Received On: 17-01-17; Revised & Accepted On: 13-02-17)

ABSTRACT

In this paper, the forcing steiner domination number of a graph is introduced. Also, this number is found for some standard graphs.

Keywords: Domination, Steiner number, Steiner domination number and Forcing Steiner domination number.

1. INTRODUCTION

The concept of domination in graphs was introduced by Ore and Berge [4]. Throughout this paper $G = (V, E)$ denotes a finite undirected simple graph with vertex set $V$ and edge set $E$. A subset $D$ of $V(G)$ is a dominating set of $G$ if every vertex in $V - D$ is adjacent to at least one vertex in $D$. The minimum cardinality of a dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. The concept of Steiner number of a graph was introduced by G.Chatrand and P. Zhang [1]. For a nonempty set $W$ of vertices in a connected graph $G$, the Steiner distance $d(W)$ of $W$ is the minimum size of a connected subgraph of $G$ containing $W$. Necessarily each such subgraph is a tree and is called a Steiner tree with respect to $W$ or a Steiner $W$-tree. The set of all vertices of $G$ that lie in some Steiner $W$-tree is denoted by $S(W)$. If $S(W) = V$, then $W$ is called a Steiner set for $G$. A Steiner set with minimum cardinality is the Steiner number of $G$ and is denoted by $s(G)$.

The concept of Steiner domination number of a graph was introduced by J. John et al., [3]. For a connected graph $G$, a set of vertices $W$ in $G$ is called a Steiner dominating set if $W$ is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of $G$ is its Steiner domination number and is denoted by $\gamma_s(G)$. A steiner dominating set of cardinality $\gamma_s(G)$ is said to be a $\gamma_s$-set.

The concept of Forcing $(G, D)$-number was introduced by K.Palani and A.Nagarajan [5]. For a connected graph $G$, let $S$ be a $\gamma_G$-set of $G$. A subset $T$ of $S$ is called a forcing subset for $S$ if $S$ is the unique $\gamma_G$-set of $G$ containing $T$. A forcing subset $T$ of $S$ with minimum cardinality is called a minimum forcing subset for $S$. The forcing $(G,D)$-number of $S$, denoted by $f_{G,D}(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing $(G, D)$-number of $G$ is the minimum of $f_{G,D}(S)$, where the minimum is taken over all $\gamma_G$-sets $S$ of $G$ and it is denoted by $f_{G,D}(G)$.

Theorem 1.1 [3]: For the complete bipartite graph $G = K_{m,n}$,

$$s(G) = \gamma_s(G) = \begin{cases} 2 & \text{if } m = n = 1 \\ n & \text{if } n \geq 2, m = 1 \\ \min\{m,n\} & \text{if } m, n \geq 2 \end{cases}$$
Theorem 1.2 [8]: For a Wheel graph \( W_{1,n}, n \geq 5 \), \( \gamma_s(W_{1,n}) = n - 2 \).

Theorem 1.3 [7]: For the Wheel \( W_p = K_1 + C_{p-1} \) \((p \geq 5)\), \( s(W_p) = p - 3 \) and \( f_s(W_p) = p - 4 \).

Theorem 1.4 [7]: For the complete bipartite graph \( G = K_{m,n} \) \((m,n \geq 2)\), \( f_s(G) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \)

Theorem 1.5 [3]: Each extreme vertex of a connected graph \( G \) belongs to every minimum Steiner dominating set of \( G \).

Theorem 1.6 [3]: For the complete graph \( K_p \) \((p \geq 2)\), \( \gamma_s(K_p) = p \).

Theorem 1.7 [6]: \( \left\lfloor \frac{n-4}{3} \right\rfloor + 2 \) if \( n \geq 5 \); 
\( 2 \) if \( n = 2,3 \) or 4.

Theorem 1.8 [6]: For \( n > 5 \), \( \gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil \).

2. FORCING STEINER DOMINATION NUMBER

Definition 2.1: Let \( G \) be a connected graph and \( W \) be a minimum steiner dominating set of \( G \). A subset \( T \) of \( W \) is called a forcing subset for \( W \) if \( W \) is the unique minimum steiner dominating set of \( G \) containing \( T \). A forcing subset \( T \) of \( W \) with minimum cardinality is called a minimum forcing subset for \( W \). The forcing steiner domination number of \( W \), denoted by \( \gamma_f(W) \), is the cardinality of a minimum forcing subset \( W \) of \( G \). The forcing steiner domination number of \( G \) is the minimum of \( \gamma_f(W) \) where the minimum is taken over all steiner dominating sets \( W \) of \( G \) and it is denoted by \( \gamma_f(G) \). That is, \( \gamma_f(G) = \min \{ f_s(W) : W \text{ is any steiner dominating set of } G \} \).

Example 2.2: Consider the graph \( G \) in Figure 2.1. \( W_1 = \{ u, w \} \) and \( W_2 = \{ x, v \} \) are the only two minimum steiner dominating sets of \( G \). Forcing subsets of \( W_1 \) are \( \{ u \} \), \( \{ w \} \) and \( \{ u, w \} \). Therefore, \( f_s(W_1) = 1 \). Similarly, Forcing subsets of \( W_2 \) are \( \{ x \} \), \( \{ v \} \) and \( \{ x, v \} \). Therefore, \( f_s(W_2) = 1 \). Hence, \( \gamma_f(G) = \min \{ 1, 1 \} = 1 \).

Remark 2.3: For a connected graph \( G \), \( 0 \leq f_s(G) \leq s(G) \). Here, the lower bound is sharp, since for any complete graph \( W = V(G) \) is the unique steiner dominating set. Therefore, \( T = \emptyset \) is a forcing subset for \( W \) and \( f_s(K_p) = 0 \).

Observation 2.4: Let \( G \) be a connected graph. Then,

i. From the definition of forcing steiner dominating set, \( f_s(G) = 0 \) if and only if \( G \) has a unique steiner dominating set.

ii. \( f_s(G) = 1 \) if and only if \( G \) has at least two steiner dominating sets, one of which has forcing number equal to 1.

iii. \( f_s(G) = \gamma_s(G) \) if and only if every steiner dominating set \( W \) of \( G \) has the property, \( f_s(W) = |W| = \gamma_s(G) \).

iv. For a connected graph \( G \), if every minimum steiner set \( W \) itself is a minimum steiner dominating set then, \( f_s(G) = f_f(G) \).
Theorem 2.5: \[ f_{\gamma_s}(P_n) = \begin{cases} 0 & \text{if } n = 2, 3 \text{ and } n \equiv 1(\text{mod } 3) \\ 1 & \text{otherwise} \end{cases} \]

Proof: Let \( P_n = (v_1, v_2, \ldots, v_n) \).

Case-(i): \( n \equiv 0(\text{mod } 3) \)

When \( n = 3 \), there is a unique steiner dominating set and by observation 2.4(i), \( f_{\gamma_s}(P_3) = 0 \).

If \( n > 3 \), \( P_n \) has more than one steiner dominating set. Further, for every \( n \), there exists only one steiner dominating set containing \( v_3 \). Similar is the case with \( v_{n-2} \) also.

Therefore, \( f_{\gamma_s}(P_n) = 1 \).

Case-(ii): \( n \equiv 1(\text{mod } 3) \)

In this case, \( P_n \) has a unique steiner dominating set.

Therefore, \( f_{\gamma_s}(P_3) = 0 \).

Case-(iii): \( n \equiv 2(\text{mod } 3) \)

When \( n = 2 \), there is a unique steiner dominating set and therefore, \( f_{\gamma_s}(P_2) = 0 \).

When \( n > 2 \), As in case 1, \( P_n \) has more than one steiner dominating set. Further, for every \( n \), there exists only one steiner dominating set containing \( v_2 \). Similar is the case with \( v_{n-1} \) also.

Therefore, \( f_{\gamma_s}(P_n) = 1 \).

Theorem 2.6: Let \( G \) be any graph with atleast two steiner dominating sets. Suppose \( G \) has a steiner dominating set \( W \) satisfying the property, “\( W \) has a vertex \( u \) such that \( u \not\in W' \) for every steiner dominating set \( W' \) different from \( W \)”, then \( f_{\gamma_s}(G) = 1 \).

Proof: As \( G \) has atleast two steiner dominating sets, by Observation 2.4(i), \( f_{\gamma_s}(G) \neq 0 \). If \( G \) satisfies the given condition that is, \( W \) has a vertex \( u \) such that \( u \not\in W' \), for every \( W' \) different from \( W \), then by the definition of forcing steiner domination number, \( f_{\gamma_s}(W) = 1 \). Therefore, by observation 2.4(ii), \( f_{\gamma_s}(G) = 1 \).

Corollary 2.7: Let \( G \) be any graph with at least two steiner dominating sets. Suppose \( G \) has a steiner dominating set \( W \) such that \( W \cap W' = \emptyset \) for every steiner dominating set \( W' \) different from \( W \), then \( f_{\gamma_s}(G) = 1 \).

Corollary 2.8: Let \( G \) be any graph with at least two steiner dominating sets. If pair wise intersection of distinct steiner dominating sets of \( G \) is empty, then \( f_{\gamma_s}(G) = 1 \).

Theorem 2.9: \[ f_{\gamma_s}(C_n) = \begin{cases} 1 & \text{if } n = 4 \text{ and } n \equiv 0(\text{mod } 3)(n > 3) \\ 2 & \text{otherwise} \end{cases} \]

Proof: Let \( C_n = (v_1, v_2, \ldots, v_n) \).

Case-(i): \( n \equiv 0 \text{ (mod } 3) \) and \( n > 3 \).

The steiner dominating sets of \( C_n \) are \( W_1 = \{v_1, v_4, \ldots, v_{3k-1}\} \), \( W_2 = \{v_2, v_5, \ldots, v_{3k-2}\} \) and \( W_3 = \{v_3, v_6, \ldots, v_{3k}\} \).

Here, \( W_i \cap W_j = \emptyset \), for \( 1 \leq i, j \leq 3 \).

Therefore, by Corollary 2.7, \( f_{\gamma_s}(C_n) = 1 \) if \( n \equiv 0 \text{ (mod } 3) \) and \( n > 3 \).
Case-(ii): \( n \equiv 1 \) (mod3).

When \( n = 4 \), the steiner dominating sets of \( C_4 \) are \( W_1 = \{v_1, v_3\} \) and \( W_2 = \{v_2, v_4\} \).

Here, \( W_1 \cap W_2 = \emptyset \). Therefore, as in case 1, \( f_{\gamma}^s(C_4) = 1 \).

When \( n > 4 \), it is easy to observe that any pair of adjacent vertices lie in exactly one steiner dominating set and any single element lie in atleast two steiner dominating sets.

Therefore, \( f_{\gamma}^s(C_n) = 2 \).

Case-(iii): \( n \equiv 2 \) (mod3).

When \( n = 5 \), the cycle is \( C_5 \) and it is easy to observe that any pair of adjacent vertices of \( C_5 \) lie in exactly one steiner dominating set and any single element lie in atleast two steiner dominating sets.

Therefore, \( f_{\gamma}^s(C_5) = 2 \).

When \( n > 5 \), any two vertices \( u \) and \( v \) with \( d(u, v) \) lie in exactly one steiner dominating set and any single element set lie in atleast two steiner dominating sets.

Therefore, \( f_{\gamma}^s(C_n) = 2 \).

Remark 2.10: In the above theorem, if \( n=3 \), then the cycle is \( C_3 \) and it has a unique steiner dominating set. Therefore, by observation 2.4(i), \( f_{\gamma}^s(C_3) = 0 \).

Theorem 2.11: For the wheel \( W_p = K_1 + C_{p-1}(p \geq 5) \), \( f_{\gamma}^s(W_p) = p - 4 \).

Proof: From Theorem 1.2 and 1.3, it is observed that, for a wheel graph \( W_p \), every steiner set is a steiner dominating set and vice versa. Then, by Observation 2.4(iv), \( f_s(W_p) = f_{\gamma}^s(W_p) \). Hence, by Theorem 1.3, \( f_{\gamma}^s(W_p) = p - 4 \).

Theorem 2.12: For the complete bipartite graph \( G = K_{m,n}(m,n \geq 2) \), \( f_{\gamma}^s(G) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases} \)

Proof: From Theorem 1.1, it is observed that, for complete bipartite graph every steiner set is a steiner dominating set and vice versa. Therefore, proceeding as in Theorem 2.11 & by Theorem 1.4, \( f_{\gamma}^s(G) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases} \)

Definition 2.13: A vertex \( v \) of a graph \( G \) is said to be a steiner dominating vertex of \( G \) if \( v \) belongs to every minimum steiner dominating set of \( G \).

Example 2.14: For the graph \( G \) in Figure 2.2, \( W_1 = \{v_2, v_4, v_5\} \) and \( W_2 = \{v_2, v_4, v_6\} \) are the minimum steiner dominating sets of \( G \). \( v_2 \) and \( v_4 \) lie in every steiner dominating set of \( G \). Therefore, \( v_2 \) and \( v_4 \) are the steiner dominating vertices of \( G \).
Remark 2.15:

1. Since all the extreme vertices of a graph $G$ belongs to every minimum steiner dominating set of $G$, all the extreme vertices of $G$ are steiner dominating vertices of $G$.
2. If $G$ has a unique steiner dominating set $W$, then every vertex of $W$ is a steiner dominating vertex of $G$.
3. Let $u \in V(G)$ be a steiner dominating vertex of $G$, Suppose $W$ is a minimum steiner dominating set of $G$ and $T$ is a minimum forcing subset of $W$, then $u \not\in T$.

Let $G$ be a connected graph and let $W$ be a minimum steiner dominating set of $G$. Suppose, $T$ is one of the minimum forcing subset of $W$. Let $E = W - T$ be the relative complement of $T$ in its minimum steiner dominating set $W$. Define $\bar{E} = \{E : E$ is the relative complement of a minimum forcing subset $T$ in its minimum steiner dominating set $W$ of $G\}$.

Theorem 2.16: Let $G$ be a connected graph. Then, $\bigcap_{E \in \bar{E}} E$ is the set of all steiner dominating vertices of $G$.

Proof: Let $S$ be the set of all steiner dominating vertices of $G$.

To Prove: $S = \bigcap_{E \in \bar{E}} E$

Let $v \in S$. By Definition 2.13, $v$ is in every minimum steiner dominating set of $G$. Let $W$ be a minimum steiner dominating set of $G$ and $T$ be a minimum forcing subset of $W$. Then, $v \in W$. Then by Remark 2.15(3), $v \not\in T$. So, $v \in E = W - T$. Hence, $v \in E$ for every $E \in \bar{E}$. That is, $v \in \bigcap_{E \in \bar{E}} E$.

Conversely, let $v \in \bigcap_{E \in \bar{E}} E$. Then, $v \in E = W - T$, where $T$ is a minimum forcing subset of the minimum steiner dominating set $W$. Therefore, $v \in W$ for every steiner dominating set $W$ of $G$. Therefore, $v \in S$.

Hence, $\bigcap_{E \in \bar{E}} E$ is the set of all steiner dominating vertices of $G$.

Remark 2.17:

1. Let $W$ be a minimum steiner dominating set of a graph $G$ and let $T$ be a minimum forcing subset of $W$. If $S$ is the set of all steiner dominating vertices of $G$, then, $S \cap T = \emptyset$. For, by Definition 2.13, $u \in S$ if and only if $u \not\in T$.
2. The above result holds even if $G$ has a unique minimum steiner dominating set. For, $T = \emptyset$ for the unique minimum steiner dominating set.
3. Let $S$ be the set of all steiner dominating vertices of a graph $G$. Then by (1), $f_{\gamma_s} (G) \leq \gamma_s (G) - |S|$.
4. The above inequality is strict. For example, Consider the graph $G$ in Figure 2.3, $W_1 = \{v_1, v_4, v_5\}$, $W_2 = \{v_1, v_3, v_5\}$ and $W_3 = \{v_1, v_4, v_6\}$ are the distinct minimum steiner dominating sets of $G$. Therefore, $\gamma_s (G) = 3$.

$f_{\gamma_s}(W_1) = 2$ and $f_{\gamma_s}(W_2) = f_{\gamma_s}(W_3) = 1$. Therefore, $f_{\gamma_s}(G) = \min \{ f_{\gamma_s}(W) : W$ is a minimum steiner dominating set of $G\} = 1$. Here, $S = \{v_1\}$. Therefore, $\gamma_s (G) - |S| = 3 - 1 = 2$. Hence, $f_{\gamma_s}(G) < \gamma_s (G) - |S|$

5. By Remark 2.15(1), $f_{\gamma_s}(G) \leq \gamma_s (G) - k$ where $k$ is the number of extreme vertices of $G$. 

$G$: Figure-2.3
Observation 2.18: For a complete graph \( G = K_p \), \( f\gamma_s(G) = 0 \) and \( |S| = p \).

For, \( K_p \) has unique minimum steiner dominating set and so \( f\gamma_s(G) = 0 \). By Remark 2.15(1), \( S = V(G) \) and \( |S| = p \).

Remark 2.19: From the above observation, we get that \( f\gamma_s(K_p) = \gamma_s(K_p) - |S| \). Hence the upper bound in Remark 2.17(3) is sharp as well.

REFERENCES


Source of support: Nil, Conflict of interest: None Declared.

[Copy right © 2016. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]