International Journal of Mathematical Archive-8(2), 2017, 120-123
Available online through www.ijma.info ISSN 2229 – 5046

GEOMETRY AND TOPOLOGY RELATED GROUP MOTION

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(Received On: 22-01-17; Revised & Accepted On: 15-02-17)

ABSTRACT

Geometry and PDEs are intimately related and many schools have worked on this aspect of geometry. There is a clear description to connect the topology of the space and its geometry subject to group actions. As a result the structural stability of the space which is an open problem invites complex situation to understanding the flows on the manifold This exposition discusses some of these issues.

Keywords: *Manifolds, Groups, Group actions.*

1. INTRODUCTION

Group actions on manifolds and vector fields arising from such group actions have interesting implications on the invariants associated with the manifolds. In understanding the geometry and topology of these manifolds and in particular the low dimensional ones, now is an active area of research. Applications to various problems arising in physical sciences are plenty and there are many open problems and conjectures that require extensive study in this regard. The nature of the systems and its dynamics subject to topological and geometric transformations has come under main focus of research. The computational tools and the computing power of the devices have further enabled us to obtain the graphic account of the geometric estimates if not accurately but to its close theoretical limits. We suppose that our manifolds are smooth, finite dimensional and the sub manifolds are smooth immersions.

However, Vector fields are defined on low dimensional manifolds. A physical problem could be flows on one and two dimensional manifolds. Curves on them are simple and closed and in fact smooth. The problem associated with such vector fields are singularities developed over finite time. Indeed this is an open problem for the non linear partial differential equations (PDEs) in particular the navier stokes equation. Such problem arise in fluid mechanics where the flow in 2 and 3 dimensional smooth manifolds demand qualitatively smooth solutions

The intrinsic nature associated with the manifold comes to fore when the vector fields are subject to close scrutiny. It means, we have to take note of large- scale features over the small flows. The characterization of these flows independent of scale invariance is indeed a problem with daunting task for solution.

An extensive study pertaining to the problems of fluid motions in two and three dimensions, their simulation and understanding of singularities developed from such flows on manifolds provide relevant information about them.

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2. LOW DIMENSIONAL MANIFOLDS

The familiar examples of smooth manifolds with standard Euclidean geometric structure are the spaces $\mathbb{R}^n$, $n \geq 1$. They are also called the prototype and describe the manifold locally and globally as well. For $n = 1$ we have $\mathbb{R}^1$ the real line endowed with standard metric. This Euclidean structure and the topology induced by the Euclidean metric makes $\mathbb{R}^1$ a smooth 1- dimensional manifold. $\mathbb{R}^2$ and $\mathbb{R}^3$ are 2 and 3-dimensional examples for smooth manifolds. Non trivial examples come from the quotient topology, as for example the circle group (S^1) in the plane and the sphere S^2 in $\mathbb{R}^3$.

Taking the Cartesian product of S^1 (n times) give us there higher analogs. An Interesting thing about S^1 is its sub groups which are finite. While, T^2 is a two-dimensional smooth closed surface and the two spheres stereographic projection is a compact version of the Riemann sphere.

In the early 40's topologists were engaged in the classification of surfaces and they succeeded in classifying all simple closed surfaces up to homeomorphism. Efforts to classify the higher dimensional spaces, lead to some serious problems with bottle necks. In case of 3 -manifolds it turns out to be notorious. And it remains an open problem till recently and known as Poincare conjecture. Thurston's geometrization program initiated by Thurston himself in early 70's made a beginning in providing rich base for the classification problem of low dimensional manifolds.

3. DIFFERENTIAL EQUATIONS

A typical way of depicting the motion of a fluid is through its velocity field. If $V(z,t)$ denotes the velocity vector field, Z for space coordinate and t the time parameter then to each point of the manifold $V(z,t)$ gives the speed of the motion of the fluid. Not only that the fluid motion also carries with is it vorticity. Vorticity basically represents the rotational speed of the fluid either clockwise or anticlockwise in direction. It was Helmholtz more than 150 years ago realized this fact and observed that the vorticity of the fluid carries important information about the nature of the flow. By this we mean it measures the swirl in a fluid. Thus, vorticity can be defined more precisely as the curl of the fluid velocity and the following expression gives this description

$$w(z,t) = \nabla \times V(z,t) \tag{3.1}$$

We notices that in two dimensional spaces, only one component of the vorticity is Non –zero. But in case of 3-dimention is it is not and hence is basically a scalar field.

For an in compressible fluid the fluid velocity satisfies the Navier-Stokes equations Non linear PDEs. These are a system of coupled Non –linear partial differential equations given by

$$\frac{\partial}{\partial t} V(z,t) + V(z,t).\nabla v(z,t) = v\Delta v(z,t) - \frac{1}{\rho}\nabla \rho \tag{3.2}$$

$$\nabla. V(z,t) = 0 \tag{3.3}$$

Where ν is the kinetic viscosity of the fluid which we will assume it to be constant. ρ is the fluid density. Since the fluid is incompressible, ρ will be constant and P is pressure P(z, t) in the fluid.

Note that (3.2) is simply the Newton's law for fluid with left hand side representing the acceleration and the right hand side the force acting on the fluid. We will assume that the only force acting on the fluid is the internal viscous forces and pressure forces (the first and second term appearing on the right hand side). The external forces acting on the fluid could be incorporated by adding additional terms to the right hand side of the equation. We will ignore the effect of boundaries on the fluid by assuming that the fluid occupies all of $\mathbb{R}^d$, $d = 2{,}3$. For more details on this problem we refer to [C.Engene wayne] and reference there in. (vertices and two dimensional fluid motion, notice of the AMS, volume 58, No.1, January 2011-reference).

4. VORTICITY EVOLUTION AND DYNAMICS

Actually the group action on manifold sets the dynamic motion of the space and the problem is one that of the evolution of the system which develops singularities in finite time. In this section we make a brief mention about vorticity evolution and the associated dynamics. The differential equations for 2 and 3 dimensional flows (Navier-stokes equation) are strikingly different and they are closely related with each other. Further we make an attempt to present the vorticity evolution and associated dynamics of the problem.

$$\frac{\partial}{\partial t} W(z,t) + V.\nabla W(z,t) = \gamma\Delta W(z,t) \tag{4.1}$$

In two dimensions, which is only one scalar equation however in the 3-dimensional case, the equation is given by

$$\frac{\partial}{\partial t} W(z,t) - W.\nabla v(z,t) + V.\nabla w(z,t) = \gamma\Delta W(z,t) \tag{4.2}$$

Here, $W(z,\ t)$ denotes the vorticity vector (bold case). In (4.1) W denotes the single non zero component of the vorticity in *2D* case. In (4.2) $W.\nabla v(z,t)$ is the vortex stretching term that features the significant difference between 2 and 3 dimension cases.

5. GROUP ACTIONS AND RELATED ISSUES

Let *G* be a group and *M* be a smooth manifold. *G* is said to act on *M* (assume that, the group is abelian) otherwise we will have to make specific mention about the group action, acting either on its left or right) if the map $\emptyset: G \times M \to M$ is continuous

i.e $(g,m) \to \emptyset(g,m)$, note that $\emptyset(g,m) \in M$.

For simplicity we write $\emptyset(g,m) = gm$. If no confusion arise, $\emptyset(g,m)$ can be written as $\emptyset g(m)$. Then $\{\emptyset_g : g \in G\}$ is a family of functions that act on *M*.

Now, $for\ g,h \in G, gh \in G, so\ \emptyset(g,h,m) = \emptyset_{gh}(m)$. But $\emptyset g, \emptyset h$ are the maps that act on M. The composition of $\emptyset_g$and $\emptyset_h$ denoted by $\emptyset g \circ \emptyset h$ and can be equated with $\emptyset gh = \emptyset g \circ \emptyset h$. For $g \in G, g^{-1} \in G$ Therefore, $\emptyset_{g\circ g^{-1}} = \emptyset_{g^{-1}\circ g} = \emptyset_e$ and $\emptyset_{g^{-1}} = \emptyset_g^{-1}$, $so\ \{\emptyset_g : g \in G\}$, establishes a linear isomorphism (automorphism) on M under the group action.

In (differential) geometry one often would like to find a subtle relationship with the topology of the underlying set and the non trivial problems arising from the vector fields on them. Basically they are one related to the singularities, developed over finite time. The global and local issues also popup and come to fore. Thus it is not only with the geometric issues but also that of topological ones. Global results and local information about the space is all about the integral. Equations on the other hand rope in algebraic tools and topology arising from this background is totally now trivial, unlike the point set topology of the space.

Understanding topology and geometry of the low dimensional manifolds mean first and for mostly, their classification. For the 2-dimensional ones, we mean surfaces, other than planar, simple and closed are the spheres and spheres with k-handles, where k is some positive integer. This problem that was successfully settled by the topologist in the 40's, also saw that the problems associated with them, as for examples, the singularities developed over finite time to the flows, and was guided by the geometry of the underlying space. In this 2- dimensional case it was the three types of geometries namely, the spherical geometry (of the sphere), planar geometry (of flat surfaces) and hyperbolic geometry (negative curvature)

Topologically, a surface- simple closed is a homeomorphism copy of a sphere S^2. If $\gamma: I \to X$ is such that γ is simple and closed curve in X and X is a simple closed surface (hence compact) then $X \simeq S^2$ it is a possible to deform γ to a great circle.

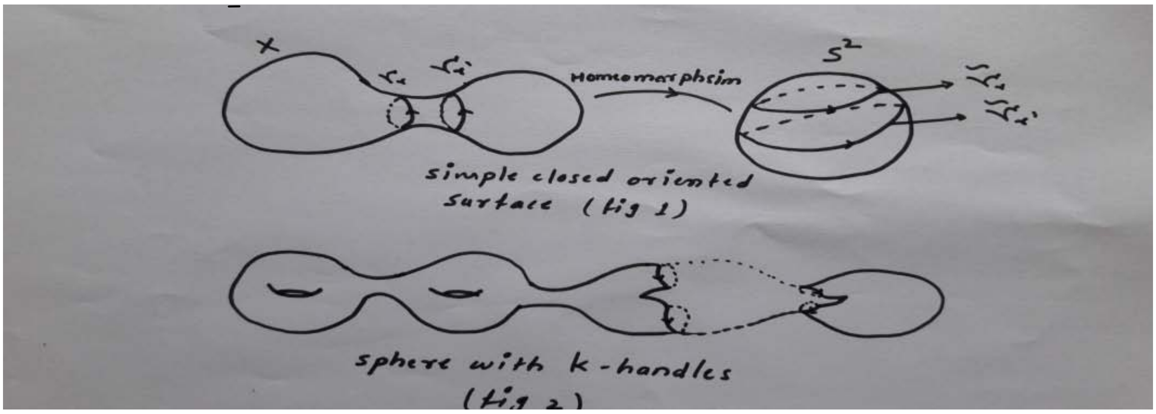
For each $r > 0$ positive real number, we can think of a family of circles, $C_r^{'s}$ where r is the radius of these circle and C_1 obiously the great circle, while $r = 0$ corresponds to the vanishing of a circle to a point.

From this observation we have the following proposition:

Proposition: The family of simple closed curves in *X*, where $X \simeq S^2$ will have at least two fixed points.

Proof: Follows from the group action on S^2 and the group being the circle group.

The following picture gives a heuristic proof of the propositions (see fig.1 and 2)



Since, each $\gamma_\gamma\colon \mathrm{I} \rightarrow X$, on one hand and $\widetilde{\gamma_\gamma}\colon \mathrm{I} \rightarrow S^2$, on the other hand will give us the bijection of these bi continuous maps and our claim that, group action moves the point of the circle exact the anti podal points.

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Source of support: Nil, Conflict of interest: None Declared.