International Journal of Mathematical Archive-8(2), 2017, 135-140
Available online through www.ijma.info ISSN 2229 – 5046

COMMON FIXED POINT THEOREMS FOR PAIRS OF COMPATIBLE MAPPINGS OF TYPE (A)

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(Received On: 11-01-17; Revised & Accepted On: 01-02-17)

ABSTRACT.

In this paper, we prove some common fixed point theorems for pairs of compatible mappings of type (A), A-compatible mappings and S-compatible mappings satisfying contractive condition of integral type in a complete metric space. The result of this paper extends and generalized the results of Aage and Salunke [1], Branciari[2], Murthy[5], Pathak and Khan[6], Shahidur Rahman et al. [9],Sharma and Sahu [10], for generalized contraction of integral type.

***Keywords**: Common fixed point, compatible mappings, compatible mappings of type (A), A- compatible, S- compatible.*

***AMS Subject Classification:** 47H10, 54H25.*

1. INTRODUCTION AND PRELIMINARIES

Sessa [8] introduced the concept of weakly commuting mappings and obtained some common fixed point theorems in complete metric space. Jungck [3] defined compatible mappings and discussed few common fixed point theorems in complete metric space. Also he showed weak commuting mappings are compatible mappings but converse need not hold. Further, Jungck *et al.* [4] introduced the new concept i.e. compatible mappings of type (A) and proved some common fixed point theorems in complete metric spaces. Compatible mappings of type (A) is more general than weakly commuting mappings and converse is not true. Pathak and Khan [6] introduced the concept of A-compatible and S-compatible by splitting the definition of compatible mapping of type (A).

Pathak *et al.* [7] proved some common fixed point theorem for compatible mappings of type (P), as application they prove the existence and uniqueness problem of common solution for a class of functional equations arising in dynamic programming. Recently, Shahidur Rahman *et al.* [9] proved generalized common fixed point theorems of A-compatible and S-compatible mappings, and generalized the result of Murthy [5], Sharma and Sahu [10]. Aage and Salunke [1] proved some common fixed point theorem for compatible mappings of type (A) for four self mappings of a complete metric space. Recently, Branciari [2] proved Banach contraction principle for integral type contraction.

Following are the definitions of different types of compatible mappings.

Definition 1.1[8]: Self-mappings S and T of a metric space (X, d) are said to be weakly commuting pair iff $d(STx, TSx) \leq d(Sx, Tx)$ for all $x \in X$.

Clearly, commuting mappings are weakly commuting but converse is not true.

Definition 1.2[3]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} A\,x_n = \lim_{n\to\infty} S\,x_n = t \text{ for some } t \in X.$$

Definition 1.3[4]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be compatible of type (A) if

$$\lim_{n\to\infty} d(ASx_n, SSx_n) = 0 \text{ and } \lim_{n\to\infty} d(SAx_n, AAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

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Definition 1.5[6]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be A-compatible if

$$\lim_{n\to\infty} d(ASx_n, SSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} A\,x_n = \lim_{n\to\infty} S\,x_n = t \text{ for some } t \in X.$$

Definition 1.6[6]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be S-compatible if $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} A\,x_n = \lim_{n\to\infty} S\,x_n = t \text{ for some } t \in X.$$

Definition 1.7[7]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be compatible of type (P) if

$$\lim_{n\to\infty} d(AAx_n, SSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Proposition 1.1[6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is A-compatible on X and St = At for some t ∈X, then ASt= SSt.

Proposition 1.2[6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is S-compatible on X and St = At for some t ∈ X, then SAt= AAt.

Proposition 1.3[6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is A-compatible on X and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$, then $SSx_n \to At$ if A is continuous at t.

Proposition 1.4[6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is S-compatible on X and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$, then $AAx_n \to St$ if S is continuous at t.

Branciari [2] proved the following result.

Theorem 1.1[2]: Let (X, d) be a complete metric space c∈]0,1[and let f :X→X be a mapping such that for each x, y∈X, $\int_0^{d(f_x, f_y)} \Phi(t)\,dt \le c \int_0^{d(x,y)} \Phi(t)dt$, where Φ :[0,+∞]→ [0,+∞] is a Lebesgue integrable mapping which is summable *i.e.* with finite integral on each compact subset of [0,+∞] nonnegative and such that for each $\varepsilon>0$, $\int_0^{\varepsilon} \Phi(t)dt > 0$; then f has a unique fixed point a ∈X such that for each x∈X, $\lim_{n\to+\infty} f^n x = a$.

2. MAIN RESULT

In this section, we prove some common fixed point theorem for compatible mapping of type (A), A-compatible and S-compatible mappings satisfying contractive condition of integral type in a complete metric space. The result of this paper extends and generalized the results of Aage and Salunke [1], Branciari [2], Murthy [5], Pathak and Khan [6], Shahidur Rahman *et.al.*[9],Sharma and Sahu[10]for generalized contraction of integral type.

Theorem 2.1: Let A, B, S and T be self-maps of a complete metric space (X, d) satisfying the following conditions:

(i) S(X) ⊆ B(X) and T(X) ⊆ A(X).

(ii) $\int_0^{d(Sx,Ty)} \Phi(t)dt \le \psi\{\int_0^{M(x,y)} \Phi(t)dt\}$, where $M(x,y)$=d(Ax,Sx).$\{\frac{1+d(By,Ty)}{1+d(Ax,By)}\}$ for all x, y∈X and Φ :[0,+∞]→ [0,+∞] is a Lebesgue integrable mapping which is summable *i.e.* with finite integral on each compact subset of [0,+∞] nonnegative and such that for each $\varepsilon>0$, $\int_0^{\varepsilon} \Phi(t)dt > 0$;also ψ: $R^+ \to R^+$ be a right continuous mapping satisfying the condition ψ(0)=0 and ψ(t) < t for each t >0 .

(iii) One of A, B, S and T is continuous.

(iv) Pairs (A, S) and (B, T) are compatible mappings of type (A).

Then A, B, S and T have unique common fixed point in X.

Proof: Let x_0∈X be arbitrary. Choose a point x_1 in X such that $Sx_0 = Bx_1$. This can done since S(X) ⊆B(X). Let x_2 be another point in X such that $Tx_1 = Ax_2$. This can done since T(X) ⊆A(X). In general, we can choose x_{2n}, x_{2n+1}, x_{2n+2},**...,** such that $Sx_{2n} = Bx_{2n+1}$ and $Tx_{2n+1} = Ax_{2n+2}$. so that we obtain a sequence Sx_0, Tx_1, Sx_2, Tx_3,**... .** Hence in general, we define a sequence $\{y_{2n}\}$ in X as $y_{2n+1} = Sx_{2n} = Bx_{2n+1}$;$y_{2n+2} = Tx_{2n+1} = Ax_{2n+2}$, where n = 0, 1, 2, 3,… .

Now we will show that the sequence $\{y_{2n}\}$ is Cauchy.

For this put $x = x_{2n}$, $y = x_{2n+1}$ in (ii), we have

$$\int_0^{d(Sx_{2n},Tx_{2n+1})} \Phi(t)dt \le \psi \int_0^{M(x,y)} \Phi(t)dt$$

where $M(x, y) = d(Ax_{2n}, Sx_{2n}) \left\{\frac{1+d(Bx_{2n+1},Tx_{2n+1})}{1+d(Ax_{2n},Bx_{2n+1})}\right\}$

$$\int_0^{d(y_{2n+1},y_{2n+2})} \Phi(t)dt \le \psi \int_0^{d(y_{2n},y_{2n+1})\left\{\frac{1+d(y_{2n+1},y_{2n+2})}{1+d(y_{2n},y_{2n+1})}\right\}} \Phi(t)dt$$

This gives

$$\int_0^{d(y_{2n+1},y_{2n+2})[1+(y_{2n},y_{2n+1})]} \Phi(t)dt \le \psi \left[\int_0^{d(y_{2n},y_{2n+1})\,[1+d(y_{2n+1},y_{2n+2})]} \Phi(t)dt\right]$$

This implies
$$\int_0^{d(y_{2n+1},y_{2n+2})} \Phi(t)dt + \int_0^{d(y_{2n+1},y_{2n+2})d(y_{2n},y_{2n+1})} \Phi(t)dt\,.$$
$$\le \psi \int_0^{d(y_{2n},y_{2n+1})} \Phi(t)dt + \psi \int_0^{d(y_{2n},y_{2n+1})d(y_{2n+1},y_{2n+2})} \Phi(t)dt$$

This gives

$$\int_0^{d(y_{2n+1},y_{2n+2})} \Phi(t)dt < \int_0^{d(y_{2n},y_{2n+1})} \Phi(t)dt$$

Hence as a consequence; we have $\int_0^{d(y_{2n+1},y_{2n+2})} \Phi(t)dt \to 0 \ as\ n \to \infty$.

Hence the sequence $\{y_{2n}\}$ is Cauchy sequence in X.

Since $\{y_{2n}\}$ is Cauchy sequence and (X, d) is complete, so the sequence $\{y_{2n}\}$ has a limit point say z in X. Hence the sub sequences $\{Sx_{2n}\} = \{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\} = \{Ax_{2n+2}\}$ also converges to the point z in X.

Suppose that the mapping A is continuous. Then $A^2x_{2n} \to Az$ and $ASx_{2n} \to Az$ as $n\to\infty$.

Since the pair (A, S) is compatible of type (A). We get $SAx_{2n} \to Az$ as $n\to\infty$.

Now by (ii), if we put $x = Ax_{2n}$, $y = x_{2n+1}$, we get

$$\int_0^{d(SAx_{2n},Tx_{2n+1})} \Phi(t)dt \le \psi \int_0^{M(x,y)} \Phi(t)dt$$

Where $M(x, y) = d(AAx_{2n},SAx_{2n})\{\frac{1+d(Bx_{2n+1},Tx_{2n+1})}{1+d(AAx_{2n},\ Bx_{2n+1})}\}$

Letting $n\to\infty$, we get

$$\int_0^{d(Az,z)} \Phi(t)dt \le \psi \int_0^{d(Az,Az)\{\frac{1+d(z,\ z)}{1+d(Az,z)}\}} \Phi(t)dt$$

This gives $\int_0^{d(Az,z)} \Phi(t)dt \le 0$.

Hence Az = z.

Further, if we put $x = z$, $y = x_{2n+1}$ in (ii), we get

$$\int_0^{d(Sz,Tx_{2n+1})} \Phi(t)dt \le \psi \int_0^{d(Az,Sz).\{\frac{1+d(Bx_{2n+1},Tx_{2n+1})}{1+d(Az,\ Bx_{2n+1})}\}} \Phi(t)dt$$

Letting $n\to\infty$, we get

$$\int_0^{d(Sz,z)} \Phi(t)dt \le \psi \int_0^{d(z,Sz).\ \{\frac{1+d(z,z)}{1+d(z,z)}\}} \Phi(t)dt$$

This gives $\int_0^{d(Sz,z)} \Phi(t)dt \le \psi \int_0^{d(z,Sz)} \Phi(t)dt < \int_0^{d(z,Sz)} \Phi(t)dt$

Hence Sz = z .Thus Sz= Az= z.

Since S(X) ⊆ B (X), there is a point u ∈X such that z = Sz= Bu.

Now by (ii),

$$\int_0^{d(z,Tu)} \Phi(t)dt = \int_0^{d(Sz,Tu)} \Phi(t)dt \leq \psi \int_0^{d(Az,Sz)\,.\{\frac{1+d(Bu,\ Tu)}{1+d(Az,\ Bu)}\}} \Phi(t)dt$$

$$\int_0^{d(z,Tu)} \Phi(t)dt \leq \psi \int_0^{d(Az,Az).\{\frac{1+d(Bu,\ Tu)}{1+d(Az,\ Bu)}\}} \Phi(t)dt$$

Hence $\int_0^{d(z,Tu)} \Phi(t)dt \leq 0$

This gives Tu = z = Bu.

Take y_n= u for n ≥ 1.

Then Tyn→ Tu= z and By_n→ Bu= z as n → ∞.

Since the pair (B, T) is compatible of type (A), we get $\lim_{n\to\infty} d(TBy_n,\ BBy_n) = 0$

Implies (Tz, Bz) = 0, since By_n= z for all n ≥ 1. Hence Tz= Bz.

Finally, by (ii), we have

$$\int_0^{d(z,Tz)} \Phi(t)dt = \int_0^{d(Sz,Tz)} \Phi(t)dt \leq \psi \int_0^{d(Az,Sz)\{\frac{1+d(Bz,\ Tz)}{1+d(Az,\ Bz)}\}} \Phi(t)dt$$

We get $\int_0^{d(z,Tz)} \Phi(t)dt \leq 0$.

Hence z = Tz = Bz. Thus z = Sz= Az=Tz = Bz.

Therefore z is common fixed point of S, A, T and B, when the continuity of A is assumed.

Now suppose that S is continuous. Then S^2x_{2n}→ Sz, SAx_{2n}→ Sz as n → ∞.

Since the pair (A, S) is compatible of type (A) therefore ASx_{2n}→ Sz as n → ∞.

By condition (ii), we have

$$\int_0^{d(S^2x_{2n},Tx_{2n+1})} \Phi(t)dt = \psi \int_0^{d(SSx_{2n},Tx_{2n+1})} \Phi(t)dt$$

$$\leq \psi \int_0^{d(ASx_{2n},SSx_{2n}).\{\frac{1+d(Bx_{2n+1},\ Tx_{2n+1})}{1+d(ASx_{2n},\ Bx_{2n+1})}\}} \Phi(t)dt$$

Letting n→∞, we get

$$\int_0^{d(Sz,z)} \Phi(t)dt \leq \psi \int_0^{d(Sz,Sz)\{\frac{1+d(z,\ z)}{1+d(Sz,\ z)}\}} \Phi(t)dt$$

We get $\int_0^{d(Sz,z)} \Phi(t)dt \leq 0$. Hence Sz= z.

But S(X) ⊆ B(X), there is a point v ∈X such that z = Sz= Bv.

Now by (ii), we have

$$\int_0^{d(S^2x_{2n},Tv)} \Phi(t)dt = \int_0^{d(SSx_{2n},Tv)} \Phi(t)dt \leq \psi \int_0^{d(ASx_{2n},SSx_{2n})\{\frac{1+d(Bv,\ Tv)}{1+d(ASx_{2n},Bv)}\}} \Phi(t)dt$$

Letting n→∞ and using Sz = z, we get

$$\int_0^{d(z,Tv)} \Phi(t)dt \leq \psi \int_0^{d(z,z)\{\frac{1+d(Bv,\ Tv)}{1+d(z,\ Bv)}\}} \Phi(t)dt$$

This gives $\int_0^{d(z,Tv)} \Phi(t)dt \leq 0$. Hence Tv = z.

Thus z = Bv = Tv for v ∈X.

Let y_n = v. Then $Ty_n \to$ Tv = z and $By_n \to$ Tv = z.

Since (B, T) is compatible of type (A), we have $\lim_{n\to\infty} d(TBy_n, BBy_n) = 0$, this gives TBv = BTv or Tz = Bz. Further by (ii), we have

$$\int_0^{d(Sx_{2n},Tz)} \Phi(t)dt \leq \psi \int_0^{d(Ax_{2n},Sx_{2n})\{\frac{1+d(Bz,\ Tz)}{1+d(Ax_{2n},Bz)}\}} \Phi(t)dt$$

Letting n→∞ and using the results above, we get

$$\int_0^{d(z,Tz)} \Phi(t)dt \leq \psi \int_0^{d(z,z)\{\frac{1+d(Bz,\ Tz)}{1+d(z,\ Bz)}\}} \Phi(t)dt$$

This gives $\int_0^{d(z,Tz)} \Phi(t)dt \leq 0$. Thus z = Tz. Hence z = Tz = Bz.

Since T(X) ⊆ A(X), there is a point w ∈X such that z = Tz = Aw.

Thus by (ii), we have

$$\int_0^{d(Sw,z)} \Phi(t)dt = \int_0^{d(Sw,Tz)} \Phi(t)dt \leq \psi \int_0^{d(Aw,Sw)\{\frac{1+d(Bz,\ Tz)}{1+d(Aw,\ Bz)}\}} \Phi(t)dt$$

$$\int_0^{d(Sw,z)} \Phi(t)dt \leq \psi \int_0^{d(z,Sw)\{\frac{1+d(z,\ z)}{1+d(z,\ z)}\}} \Phi(t)dt$$

$$\int_0^{d(Sw,z)} \Phi(t)dt \leq \psi \int_0^{d(z,Sw)} \Phi(t)dt < \int_0^{d(z,Sw)} \Phi(t)dt$$

This gives Sw = z.

Take y_n = w then $Sy_n \to$ Sw = z, $Ay_n \to$ Aw = z. Since (A, S) is compatible of type (A), we get

$$\lim_{n\to\infty} d(SAy_n, AAy_n) = 0.$$

This implies that SAw = ASw or Sz = Az. Thus we have z = Sz = Az = Bz = Tz.

Hence z is a common fixed point of A, B, S and T, when S is continuous.

The proof is similar that z is common fixed point of A, B, S and T, when the continuity of B or T is assumed.

Uniqueness

Let z and t be two common fixed point of A, B, S and T.
i.e. z = Sz = Az = Tz = Bz and t = St = At = Tt = Bt.

By condition (ii), we have

$$\int_0^{d(z,t)} \Phi(t)dt = \int_0^{d(Sz,Tt)} \Phi(t)dt \leq \psi \int_0^{d(Az,Sz)\left\{\frac{1+d(Bt,\ Tt)}{1+d(Az,\ Bt)}\right\}} \Phi(t)dt$$

This gives $\int_0^{d(z,t)} \Phi(t)dt \leq 0$.

Thus we have z = t. Hence z is a unique common fixed point of mappings A, B, S and T.

Theorem 2.2: Let A, B, S and T be self maps of a complete metric space (X, d) satisfying the conditions (i), (ii), (iv) and ψ be as in theorem 2.1 satisfying the inequality

$$d(Sx, Ty) \leq \psi d(Ax, Sx)\{\frac{1+d(By,Ty)}{1+d(Ax,By)}\},$$

Then A, B, S and T have unique common fixed point in X

Proof: The proof of the theorem 2.2 is follows from theorem 2.1 by putting $\Phi(t) = 1$ in (ii).

Corollary 2.1: Let A, B, S and T be self-maps of a complete metric space (X, d) satisfying the following conditions (i),(ii),(iii) of theorem 2.1and if pairs (A, S) and (B, T) are A- compatible or S- compatible. Then A, B, S and T have unique common fixed point in X.

Proof: The proof of the corollary directly follows by splitting the definition of compatible mappings of type (A) into A-compatible or S- compatible mappings and using the Proposition 1.1 to 1.4.

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Source of support: Nil, Conflict of interest: None Declared.

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