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ON THE ROMAN DOMINATION NUMBER OF GRAPHS

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ABSTRACT

In this manuscript we consider the change in the Roman Domination Number of a graph when a vertex is removed from the graph. We prove a necessary and sufficient condition under which the Roman Domination Number of a graph increases. Also we prove a necessary and sufficient condition under which the Roman Domination Number decreases. We deduce that in any graph there are vertices whose removal does not increase the Roman Domination Number.

Keywords: *Roman Dominating Function, Roman Domination Number, Minimal Roman Dominating Function, Minimum Roman Dominating Function, Private Neighbourhood.*

AMS Mathematics Subject Classification (2010): *05C69.*

1. INTRODUCTION

The concept of Roman Domination was introduced in [5] by Ernie J. Cockayne, T.W.Haynes, and others. A Roman Dominating Function gives rise to a Dominating Function and vice-versa. In [5] a necessary and sufficient condition has been proved under which the Domination Number of a graph increases when a vertex is removed from the graph. A necessary and sufficient condition was also proved under which the Domination Number decreases when a vertex is removed from the graph [5]. Here we prove the conditions for the same operations for the Roman Domination Number.

2. PRELIMINARIES AND NOTATIONS

In this paper we consider only those graphs which are simple and finite. If G is a graph, $V(G)$ will denote the vertex set of graph G and $E(G)$ will denote the edge set of graph G. If G is a graph and $v \in V(G)$ then $G - v$ will denote the subgraph obtained by removing the vertex v from G. The Roman Domination Number of the graph G is denoted as $\gamma_R(G)$, whereas the Domination Number of the graph G is denoted as $\gamma(G)$. If $f: V(G) \to \{0,1,2\}$ is a function then we write,

$$\boldsymbol{V_2(f)} = \{v \in V(G) \,/\, f(v) = 2\}$$
$$\boldsymbol{V_1(f)} = \{v \in V(G) \,/\, f(v) = 1\}$$
$$\boldsymbol{V_0(f)} = \{v \in V(G) \,/\, f(v) = 0\}$$

Obviously the above sets are mutually disjoint and their union is the vertex set V(G). The weight of this function $f = \sum_{v \in V(G)} f(v)$. This number is denoted as $w(f)$. We will also use the following notations:

$$\boldsymbol{V^0} = \{v \in V(G) \,/\; \gamma_R(G - v) = \gamma_R(G)\}$$
$$\boldsymbol{V^+} = \{v \in V(G) \,/\; \gamma_R(G - v) > \gamma_R(G)\}$$
$$\boldsymbol{V^-} = \{v \in V(G) \,/\; \gamma_R(G - v) < \gamma_R(G)\}$$

If $S \subset V(G)$ and $v \in S$, then the private neighbourhood of v with respect to the set S = { $v \in V(G) \,/\, N[u] \cap S = \{v\}$}. It is denoted as $pv[v, S]$.

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Definition 2.1 [4]: Let G be a graph. A function $f: V(G) \to \{0, 1, 2\}$ is called a *Roman Dominating Function* if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$.

Definition 2.2 [4]: A Roman Dominating Function with minimum weight is called a *Minimum Roman Dominating Function.*

Definition 2.3 [4]: The weight of a Minimum Roman Dominating Function is called the *Roman Domination Number* of the graph. It is denoted as $\gamma_R(G)$.

Definition 2.4 [4]: Let G be a graph. A function $f: V(G) \to \{0, 1, 2\}$ is called a *Minimal Roman Dominating Function* if (i) f is a Roman Dominating Function. (ii) Whenever $g: V(G) \to \{0, 1, 2\}$ and $g < f$ then g is not a Roman Dominating Function.

Remark 2.5:
i) For any graph $G, \gamma(G) \le \gamma_R(G) \le 2\gamma(G)$.
ii) In [2] it is proved that a Roman Dominating Function f is minimal if and only if each of the following two conditions is satisfied:
   1) If $v \in V(G)$ and $f(v) = 2$ then there is a vertex x such that $f(x) = 0$, x is adjacent to v and x is not adjacent to any other vertex w for which $f(w) = 2$.
   2) If $f(w) = 1$ then w is not adjacent to any vertex x for which $f(x) = 2$.

3. VERTEX REMOVAL AND ROMAN DOMINATION NUMBER

Proposition 3.1: Let G be a graph and $v \in V(G)$. If $\gamma_R(G - v) < \gamma_R(G)$, then $\gamma_R(G - v) = \gamma_R(G) - 1$.

Proof: Let g be a Minimum Roman Dominating Function of $G - v$, then $w(g) < \gamma_R(G)$.

Now define $f: V(G) \to \{0,1,2\}$ as follows:

$f(v) = 1$ and

$f(w) = g(w); if\ w \neq v$

Obviously f is a Roman Dominating Function on G and $w(f) = w(g) + 1$.

Since $\gamma_R(G - v) < \gamma_R(G)$, f must be a Minimum Roman Dominating Function of G.

Thus $\gamma_R(G) = w(f) = w(g) + 1 = \gamma_R(G - v) + 1$.

i.e. $\gamma_R(G - v) = \gamma_R(G) - 1$.

Theorem 3.2: Let G be a graph and $v \in V(G)$ then $\gamma_R(G - v) < \gamma_R(G)$ if and only if there is a Minimum Roman Dominating Function f on G such that $f(v) = 1$.

Proof: First suppose $\gamma_R(G - v) < \gamma_R(G)$. Let g be a Minimum Roman Dominating Function of $G - v$. Now define $f: V(G) \to \{0,1,2\}$ as follows:

$f(v) = 1$ and

$f(w) = g(w); if\ w \neq v$

Obviously f is a Minimum Roman Dominating Function on G and $f(v) = 1$.

Conversely suppose there is a Minimum Roman Dominating Function f such that $f(v) = 1$.

Define $g: V(G - v) \to \{0,1,2\}$ as follows:

$g(w) = f(w); \forall\ w \in V(G - v)$

Then obviously g is a Roman Dominating Function on $G - v$ and $w(g) < w(f) = \gamma_R(G)$.

Therefore $\gamma_R(G - v) < \gamma_R(G)$.

Corollary 3.3: Let G be a graph and $v \in V(G)$ be an isolated vertex then $\gamma_R(G - v) < \gamma_R(G)$.

Proof: For any Minimum Roman Dominating Function f on G, $f(v) = 1$ as v is an isolated vertex [2].

Hence $\gamma_R(G - v) < \gamma_R(G)$.

Consider the following example in which the vertex set is $\{v_1, v_2, v_3, v_4\}$.

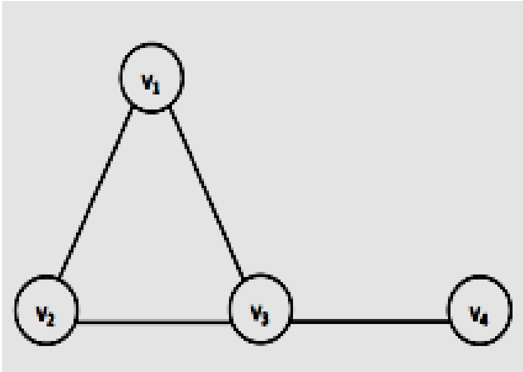

Figure-1: (GRAPH G)

Let $f: V(G) \rightarrow \{0,1,2\}$ be any function such that

$$f(v_1) = 0, f(v_2) = 0, f(v_3) = 2 \quad and \quad f(v_4) = 0$$

The Roman Domination Number of the graph G is **2**;

Whereas the Roman Domination Number of $G - v_3$ is **3**.

Thus we have $\boldsymbol{\gamma_R(G - v_3) > \gamma_R(G)}$.

Now we state and prove the necessary and sufficient conditions under which the Roman Domination Number increases when a vertex is removed from the graph.

Theorem 3.4: Let G be a graph and $v \in V(G)$ then $\gamma_R(G - v) > \gamma_R(G)$ if and only if the following conditions are satisfied:

i) v is not an isolated vertex in G.
ii) $f(v) = 2$ for every Minimum Roman Dominating Function f on G.
iii) There is no Roman Dominating Function g on $G - v$ such that $w(g) \leq \gamma_R(G)$ and $V_2(g)$ is a subset of $V(G) - N[v]$.

Proof: Suppose $\gamma_R(G - v) > \gamma_R(G)$

i) Suppose v is an isolated vertex, then for every minimum function f on $G, f(v) = 1$ and therefore $\gamma_R(G - v) < \gamma_R(G)$ by **Theorem 3.2**; which is a contradiction.
Therefore v is not isolated vertex in G.

ii) Suppose for some minimum function f on $G, f(v) = 0$.
Now define g on $G - v$ as follows:

$$g(w) = f(w); \forall\, w \in V(G - v)$$

Obviously g is a Roman Dominating Function on $G - v$.
Then $\gamma_R(G - v) \leq w(g) \leq w(f) = \gamma_R(G)$; which is a contradiction.
Suppose for some minimum function f on $G, f(v) = 1$, then $\gamma_R(G - v) < \gamma_R(G)$ by theorem 3.2; which is a contradiction.
Hence $f(v) = 2$ for every Minimum Roman Dominating Function f on G

iii) Suppose there is a Roman Dominating Function g on $G - v$ with $w(g) \leq \gamma_R(G)$ and $V_2(g)$ is a subset of $V(G) - N[v]$.
Then obviously $\gamma_R(G - v) \leq w(g) \leq \gamma_R(G)$; which is a contradiction.

Conversely suppose conditions (i), (ii) and (iii) are satisfied. Suppose $\gamma_R(G - v) = \gamma_R(G)$.
Let g be a Minimum Roman Dominating Function on $G - v$. Consider the set $V_2(g)$. First suppose v is adjacent to some vertex of $V_2(g)$. Now define $f: V(G) \rightarrow \{0,1,2\}$ as follows:

$$f(v) = 0 \text{ and } f(w) = g(w); \forall\, w \neq v$$

Then f is a Minimum Roman Dominating Function on G and $w(f) = w(g)$.

But $f(v) = 0$ which contradicts condition (ii).

Suppose v is not adjacent to any vertex of $V_2(g)$, then $V_2(g)$ is a subset of $V(G) - N[v]$, $w(g) \leq \gamma_R(G)$ and g is a Roman Dominating Function on $G - v$; which contradicts condition (iii).

Thus $\gamma_R(G-v)=\gamma_R(G)$ is not possible.

Suppose $\gamma_R(G-v)<\gamma_R(G)$. Let g be a Minimum Roman Dominating Function on $G-v$.

Let h be defined as follows:

$$h(v)=0 \text{ and } h(w)=g(w); \forall\, w\neq v$$

Then $w(h)=w(g)$. Therefore h can not be a Roman Dominating Function on G. Therefore there is a vertex x of G such that $h(x)=0$, but x is not adjacent to any vertex y for which $h(y)=2$. From the definition of h it is clear that $x=v$ and v is not adjacent to any vertex w for which $g(w)=2$.

Now $V_2(g)\subset V(G)-N[v]$ and also $w(g)\leq\gamma_R(G)$ and g is a Roman Dominating Function on $G-v$; this again contradicts condition (iii).

Therefore $\gamma_R(G-v)<\gamma_R(G)$ is also not possible.

Thus $\gamma_R(G-v)>\gamma_R(G)$.

Remark 3.5: We may note that if f is a Minimal Roman Dominating Function and $f(u)=1$ and $f(v)=2$, then u and v can not be adjacent vertices [2].

Corollary 3.6: Let G be a graph and $u,v\in V(G)$ such that $\gamma_R(G-u)>\gamma_R(G)$ and $\gamma_R(G-v)<\gamma_R(G)$ then u and v are non adjacent vertices.

Proof: Since $\gamma_R(G-v)<\gamma_R(G)$ there is a Minimum Roman Dominating Function f such that $f(v)=1$. Also by theorem 3.4 we have $f(u)=2$.

So by the above remark 2.5(2) u and v cannot be adjacent vertices.

Theorem 3.7: Let G be a graph and $v\in V(G)$ then $\gamma_R(G-v)>\gamma_R(G)$. Let f be a Minimum Roman Dominating Function of G, then $f(v)=2$ and there is atleast two distinct vertices u_1 and u_2 such that $f(u_1)=f(u_2)=0$, $u_1,u_2\in pv[v,V_2(f)]$ and also u_1 , u_2 are non adjacent vertices.

Proof: Since $\gamma_R(G-v)>\gamma_R(G), f(v)=2$.Since f is a Minimal Roman Dominating Function by theorem 3.1 of [2] there is at least one vertex u such that $f(u)=0$ and $u\in pv[v,V_2(f)]$.Suppose u is the only such vertex. Now define $g\colon V(G)\to\{0,1,2\}$ as follows:

$$g(v)=0\,, g(u)=2 \text{ and}$$
$$g(w)=f(w); for\ all\ w\in\{u,v\}$$

If x is a vertex different from u and v, then $x\notin pv[v,V_2(f)]$. Therefore there is a vertex y such that $y\neq v, g(y)=f(v)=2$ and x is adjacent to y. Thus g is a Roman Dominating Function. Also $w(g)=w(f)$. Therefore g is a Minimum Roman Dominating Function and $g(v)=0$; which contradicts theorem 3.4 condition (ii).

Therefore there are at least two vertices say u and u' such that $f(u)=f(u')=0$ and $u,u'\in pv[v,V_2(f)]$.

Suppose any two of them are adjacent. Suppose $u_1$and $u_2$are any two vertices such that $f(u_1)=f(u_2)=0$ and $u_1,u_2\in pv[v,V_2(f)]$.

Now define $h\colon V(G)\to\{0,1,2\}$ as follows:

$$h(v)=0\,, h(u_1)=2 \text{ and}$$
$$h(w)=f(w); for\ all\ other\ vertices$$

Let z be a vertex such that $f(z)=0$ and $z\in pv[v,V_2(f)]$. If $z\neq u_1$ then z is adjacent to $u_1$because of our assumption. Therefore h is a Roman Dominating Function with $w(h)=w(f)$ and therefore h is a Minimum Roman Dominating Function with $h(v)=0$; which is a contradiction.

Thus there must be at least two vertices u and u' such that $f(u)=f(u')=0$ and $u,u'\in pv[v,V_2(f)]$ and they are non adjacent.

Remark 3.8: In the above theorem we have proved that if $\gamma_R(G-v)>\gamma_R(G)$ then for any minimum function, $f(v)=2$ and there are two non adjacent vertices u and u' which are adjacent to v and $f(u)=f(u')=0$.

Since u and v are adjacent $\gamma_R(G-u) < \gamma_R(G)$ is not possible by corollary 3.6. Since $f(u) = 0$, $\gamma_R(G-u) > \gamma_R(G)$ is not possible by theorem 3.4 condition (ii). Thus it must be true that $\gamma_R(G-u) = \gamma_R(G)$.

Similarly, $\gamma_R(G-u') = \gamma_R(G)$

Thus every vertex $v \in V^+$ gives rise to at least two vertices in V^0 and therefore $|V^0| \geq 2|V^+|$.

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