CERTAIN UNIFIED INTEGRALS INVOLVING I- FUNCTION AND M-SERIES WITH GENERALIZED MULTIVARIABLE POLYNOMIALS

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(Received On: 22-12-16; Revised & Accepted On: 07-03-17)

ABSTRACT

In the present paper, we consider new unified integrals associated with the I-function and M-series with generalized polynomial. Some interesting special cases of main results are considered in form of many corollaries. The obtained results of this paper provide an extension of the results given by the literature.

Key Words: H-function, I-function, M-series, Generalized Mittag-Leffler function, and Generalized polynomials.

MSC: 33C45, 33C60, 33C70, 33E12.

1. INTRODUCTION

The I-function, which is more general the fox’s H-function [3], introduced by V.P. Saxena [8], by means of the following Mellin-Barnes type contour integral

\[ I[z] = \frac{1}{2\pi i} \int_L \theta(s) z^s ds \]

\[ \theta(s) = \frac{\prod_{i=1}^{m} \Gamma(b_i - s) \prod_{j=1}^{n} \Gamma(1 - a_i + 2s)}{\sum_{k=1}^{m} \prod_{j=1}^{n} \Gamma(1 - a_i + b_k - s)} \]

Where \( p_i, q_i \) (\( i = 1, 2, \ldots, r \)), \( m, n \) are integers satisfying \( 0 \leq n \leq p_i \), \( 0 \leq m \leq q_i \), \( \alpha, \beta, \gamma \) are real and positive and \( a_i, b_i, \alpha, \beta, \gamma \) are complex numbers, \( L \) is a suitable contour of the Mellin-Barnes type running from \( \gamma - i\alpha \) to \( \gamma + i\alpha \) (\( \gamma \) is real) in the complex s-plane. Detail regarding existence conditions and various parametric restriction of I-function, we may refer [8].

The generalized polynomials introduced by Srivastava [10] is as follows:

\[ s_{\alpha_1, \ldots, \alpha_t}^{m_1, \ldots, m_t} [z_1, \ldots, z_t] = \sum_{\alpha_1=0}^{m_1} \cdots \sum_{\alpha_t=0}^{m_t} \frac{(-1)^{m_1} \alpha_1! \cdots (-1)^{m_t} \alpha_t!}{\alpha_1! \cdots \alpha_t!} A[\alpha_1, \alpha_1; \ldots; \alpha_t, \alpha_t] z_1^{\alpha_1} \cdots z_t^{\alpha_t} \]  

Where \( n_i = 0, 1, 2, \ldots \) \( \forall i = (1, \ldots, t) \), \( m_1, \ldots, m_t \) arbitrary positive integers and the coefficients \( A[\alpha_1, \alpha_1; \ldots; \alpha_t, \alpha_t] \) are arbitrary constants, real or complex.

The generalized M-series is defined by Sharma and Jain [9]

\[ a_n^M [a_1, a_2; \ldots; a_p; b_1, b_2; \ldots; b_q; z] = a_n^M [a_1; b_1] \cdots [a_p; b_p] z^n \]

\[ = \sum_{a_1=0}^{a_1} \cdots \sum_{a_p=0}^{a_p} \frac{(a_1)! \cdots (a_p)!}{(a_1)! \cdots (a_p)!} z^n, \quad a_1, \ldots, a_p, b_1, \ldots, b_p \in \mathbb{C}, R(\alpha) > 0 \]  

Where \( (a_1), \ldots, (b_p) \) are the know pochhammer symbols. The series (1) is defined when none of the parameters \( b_j, j = 1, \ldots, q \) is a negative integer or zero; if any numerator parameter \( a_j \) is a negative integer or zero, then the series terminates to a polynomial in \( z \).
The three-parameter Mittag-Leffler function, also called generalized Mittag-Leffler function was introduced by Prabhakar [7] by means of the following series representation

\[ E_{\lambda}^{\mu} \left( 2 \mu \right) = \sum_{m=0}^{\infty} \frac{(2 \mu)^m}{m!} \]

A three-parameter Mittag-Leffler function is given by the generalized series representation

\[ E_{\lambda}^{\mu} \left( 2 \mu \right) = \sum_{m=0}^{\infty} \frac{(2 \mu)^m}{m!} \]

Let the condition of theorem 1 be satisfied and set \( r = 1 \), the equation (8) reduces to
Corollary 1.2: If we take \(m_1, ..., m_r \to m\) and \(n_2, ..., n_r \to n\) i.e. \((1,..., t) \to 1\) in the integral (8) and let the condition of theorem 1 be satisfied, we arrive at the following result which is obtained by Misra & Pandey [5].

\[
\int_0^\infty x^{\nu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} H_{1,1}^{m,n}(x) \left[ \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})^k} \right] \left[ \begin{array}{c} (a_j, a_{ij})_1 \vfill_n \end{array} \right] \left[ \begin{array}{c} (b_j, b_{ij})_1 \vfill_q \end{array} \right] dx
\]

\[
= 2^{1-\mu} a^{\mu-\lambda-\rho \alpha} \Gamma(2\mu) \sum_{a=0}^{[m/n]} \left( \begin{array}{c} \alpha \vfill_{m,n} \end{array} \right) \left[ \begin{array}{c} \beta \vfill_{m,n} \end{array} \right] \left( \begin{array}{c} \gamma \vfill_{m,n} \end{array} \right) \left( \begin{array}{c} \delta \vfill_{m,n} \end{array} \right) A[n, \alpha] z^\alpha \left( -\rho \alpha, k \right) ; (1 - \lambda + \mu - \rho \alpha, k) ; (a_j, a_{ij})_1 ; (b_j, b_{ij})_1 \right] \]

Corollary 1.3: If we take \(r = 1\) and \(m_1, ..., m_r \to m\) and \(n_2, ..., n_r \to n\) i.e. \((1,..., t) \to 1\) in the integral (8) and let the condition of theorem 1 be satisfied, we arrive at the following result which is obtained by Garg and Mittal [4].

\[
\int_0^\infty x^{\nu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} H_{1,1}^{m,n}(x) \left[ \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})^k} \right] \left[ \begin{array}{c} (a_j, a_{ij})_1 \vfill_n \end{array} \right] \left[ \begin{array}{c} (b_j, b_{ij})_1 \vfill_q \end{array} \right] dx
\]

\[
= 2^{1-\mu} a^{\mu-\lambda-\rho \alpha} \Gamma(2\mu) \sum_{a=0}^{[m/n]} \left( \begin{array}{c} \alpha \vfill_{m,n} \end{array} \right) \left( \begin{array}{c} \beta \vfill_{m,n} \end{array} \right) \left( \begin{array}{c} \gamma \vfill_{m,n} \end{array} \right) \left( \begin{array}{c} \delta \vfill_{m,n} \end{array} \right) A[n, \alpha] z^\alpha a^{\rho \alpha} H_{1,1}^{m,n+2}(x) \left[ \begin{array}{c} \gamma \vfill_{m,n+2} \end{array} \right] \left( \begin{array}{c} \delta \vfill_{m,n+2} \end{array} \right) \left( \begin{array}{c} \beta \vfill_{m,n+2} \end{array} \right) \left( \begin{array}{c} \alpha \vfill_{m,n+2} \end{array} \right) \left( 1 - \lambda - \rho \alpha, k \right) ; (1 - \lambda + \mu - \rho \alpha, k) ; (a_j, a_{ij})_1 \right] \]

Corollary 1.4: If we take \(r = 1\) and \(m_1, ..., m_r \to 1\) in the integral (8) and let the condition of theorem 1 be satisfied, we arrive at the following result which is obtained by Chouhan and Khan [2].

\[
\int_0^\infty x^{\nu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} H_{1,1}^{m,n}(x) \left[ \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})^k} \right] \left[ \begin{array}{c} (a_j, a_{ij})_1 \vfill_n \end{array} \right] \left[ \begin{array}{c} (b_j, b_{ij})_1 \vfill_q \end{array} \right] dx
\]

\[
= 2^{1-\mu} a^{\mu-\lambda-\rho \alpha} \Gamma(2\mu) \left[ \begin{array}{c} \beta \vfill_{m,n+2} \end{array} \right] \left( \begin{array}{c} \alpha \vfill_{m,n+2} \end{array} \right) \left( \begin{array}{c} \gamma \vfill_{m,n+2} \end{array} \right) \left( \begin{array}{c} \delta \vfill_{m,n+2} \end{array} \right) A[n, \alpha] z^\alpha a^{\rho \alpha} \left( -\rho \alpha, k \right) ; (1 - \lambda + \mu - \rho \alpha, k) ; (a_j, a_{ij})_1 \right] \]

Corollary 1.5: If we take \(r = 1\), \(\alpha_i = \beta_i = 1\) and \(S_{m_1, ..., m_r} \to 1\) in the integral (8) and let the condition of theorem 1 be satisfied, we arrive at the following result which is obtained by Choi and Agarwal [1].

\[
\int_0^\infty x^{\nu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} G_{1,1}^{m,n}(x) \left[ \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})^k} \right] \left[ \begin{array}{c} (a_i) \vfill_n \end{array} \right] \left[ \begin{array}{c} (b_i) \vfill_q \end{array} \right] dx
\]

\[
= 2^{1-\mu} a^{\mu-\lambda-\rho \alpha} \Gamma(2\mu) \left[ \begin{array}{c} \beta \vfill_{m,n+2} \end{array} \right] \left( \begin{array}{c} \alpha \vfill_{m,n+2} \end{array} \right) \left( \begin{array}{c} \gamma \vfill_{m,n+2} \end{array} \right) \left( \begin{array}{c} \delta \vfill_{m,n+2} \end{array} \right) A[n, \alpha] z^\alpha a^{\rho \alpha} \left( -\rho \alpha, k \right) ; (1 - \lambda + \mu - \rho \alpha, k) ; (a_j, a_{ij})_1 \right] \]

Where \(G_{1,1}^{m,n}[z]\) is well known meijer G-function (fox [3]).

Corollary 1.6: If we take \(r = 1\), \(S_{m_1, ..., m_r} \to 1\), \(m = 1, n = 0, p = 0, q = 2, k = 2, w = 2, b_1 = 0, \beta_1 = 1, b_2 = -\nu, \beta_2 = 1, \lambda = \lambda + \nu\) in the integral (8) and let the condition of theorem 1 be satisfied, we arrive at the following result which is obtained by Choi and Agarwal [1].

\[
\int_0^\infty x^{\nu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} F_1 \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx
\]

\[
= 2^{1-\nu} a^{\nu-\lambda-\rho \alpha} \Gamma(2\mu) \left[ \begin{array}{c} (1 + \lambda + \nu, 2) \vfill_2 \end{array} \right] \left[ \begin{array}{c} (1 + \lambda = \lambda + \nu + \mu, 2) \vfill_2 \end{array} \right] \left( \begin{array}{c} (1 + \nu, 2) \vfill_2 \end{array} \right) \left( \begin{array}{c} (1 + \lambda + \nu + \mu, 2) \vfill_2 \end{array} \right) \left( \begin{array}{c} (1 + \nu, 1) \vfill_2 \end{array} \right) \]

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Where $I_n(z)$ is the ordinary Bessel function of first kind [1].

**Corollary 1.7:** If we take $r = 1$, $S^{m_1...m_t}_{n_1...n_t} \to 1$, $m = 1$, $n = 1$, $p = 1$, $q = 2$, $k = 1$, $w = 1$, $b_1 = 0$, $\beta_1 = 1$, $b_2 = 1 - c$, $\beta_2 = 1$, $\lambda = \nu$, $a_1 = 1 - a$, $a_1 = 1$, $\mu = \lambda$, $\gamma = -Y$, in the integral (5) and let the condition of theorem 1 be satisfied, we arrive at the following result which is obtained by Choi and Agarwal [1].

\[
\int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\frac{1}{2}\nu} F_1 \left[ \begin{array}{c} \nu 1, \nu 2, \nu 3  \\ a c  \\ \frac{Y}{x + a + \sqrt{x^2 + 2ax}} \end{array} \right] dx = 2^{1-\lambda} \frac{\Gamma(\nu - r(2\lambda + \nu - 1))}{\Gamma(\nu + \lambda + 1)} \frac{Y}{c, \nu + \lambda + 1, \nu; a} F_3 \left[ \begin{array}{c} a, v - \lambda, \nu + 1; \nu  \\ b 1, c, \nu + \lambda + 1; a  \\ \end{array} \right] \quad (15)
\]

### 3. Unified Integral Involving M-Series with the Generalized Polynomial

We derive the following theorem

**Theorem 2:**

\[
\int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\frac{1}{2}\nu} M_\nu \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})^k} \left[ \begin{array}{c} \nu 1, \nu 2, \nu 3  \\ n_1 \nu 1, n_2 \nu 2, n_3 \nu 3  \\ \end{array} \right] \frac{\left(\nu - \sum_{i=1}^{m} \rho_i a_i, k\right)}{1 + \nu - \sum_{i=1}^{m} \rho_i a_i} \left[ \begin{array}{c} \nu 1, \nu 2, \nu 3  \\ n_1 \nu 1, n_2 \nu 2, n_3 \nu 3  \\ \end{array} \right] \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{q} \Gamma(a_j)} dx = 2^{1-\lambda} \frac{\Gamma(\nu - r(2\lambda + \nu - 1))}{\Gamma(\nu + \lambda + 1)} \frac{Y}{c, \nu + \lambda + 1, \nu; a} F_3 \left[ \begin{array}{c} a, v - \lambda, \nu + 1; \nu  \\ b 1, c, \nu + \lambda + 1; a  \\ \end{array} \right] \quad (16)
\]

where $\{a_i\}_n$, $\{b_i\}_m$ are known pacham symbols and parameters $b_j; s, j = 1, 2, ..., q$ is a negative integer or zero; if any numerator parameter $a_i$ is a negative integer or zero, then the series terminates to a polynomial in $z$.

1. $n_1 = 0, 1, 2, ..., \forall i = (1, \ldots, r), n_1, \ldots, n_r$ arbitrary positive integers and the coefficient $A[n_1, a_1; \ldots; n_r, a_r]$ are arbitrary constants, real or complex. 
2. $k > 0$, $Re(\mu, \lambda, \rho) > 0$.
3. $Re(\mu) - Re(\lambda) - k \min_{1 \leq j \leq m} Re \left[ \frac{b_j}{\rho_j} \right] < 0$.

**Proof:** Using the generalized M-series as the function defined by means of the power series in given by (4) and the definition of a generalized Polynomials given by (3), then interchanging the order of summation and integration, we obtain L.H.S. of (16) as

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{\nu} n!\right)^k}{\left(\nu - \sum_{i=1}^{m} \rho_i a_i\right)_k} \int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\frac{1}{2}\nu} M_\nu \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})^k} \left[ \begin{array}{c} \nu 1, \nu 2, \nu 3  \\ n_1 \nu 1, n_2 \nu 2, n_3 \nu 3  \\ \end{array} \right] \frac{\left(\nu - \sum_{i=1}^{m} \rho_i a_i, k\right)}{1 + \nu - \sum_{i=1}^{m} \rho_i a_i} \left[ \begin{array}{c} \nu 1, \nu 2, \nu 3  \\ n_1 \nu 1, n_2 \nu 2, n_3 \nu 3  \\ \end{array} \right] \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{q} \Gamma(a_j)} dx
\]

And evaluating the inner integral by using an integral formula (7) given by F.Oberhettinger [6] and we get

\[
= 2^{1-\lambda} \frac{\Gamma(\nu - r(2\lambda + \nu - 1))}{\Gamma(\nu + \lambda + 1)} \frac{Y}{c, \nu + \lambda + 1, \nu; a} F_3 \left[ \begin{array}{c} a, v - \lambda, \nu + 1; \nu  \\ b 1, c, \nu + \lambda + 1; a  \\ \end{array} \right] \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{q} \Gamma(a_j)} \Gamma(\lambda + k n + \sum_{i=1}^{m} \rho_i a_i) \Gamma(\lambda + k n + \sum_{i=1}^{m} \rho_i a_i - \mu) \Gamma(1 + \lambda + k n + \sum_{i=1}^{m} \rho_i a_i - \mu)
\]

Now using definition of $H$-function (5) we get the desired result.

**Special Cases**

**Corollary 2.1:** Let the condition of theorem 2 be satisfied and set $p = 1, q = 1$, $b_j = 1$ the generalized M-series convert in generalized Mittag-leffler function (6) introduced by Prabhakar [7] the equation (16) reduces to
\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\rho} E_{\alpha,\beta}^\gamma \left[ \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})} \right] dx
\]

\[
= 2^{1-\mu} a^{\mu-\alpha} \Gamma(2\mu) \sum_{n=0}^{[n/\alpha]!} \frac{(-n)_{(\mu-\alpha)}}{\alpha!} A[n, \alpha] z^{\alpha} \frac{1}{\Gamma(\gamma)} H_{1,5}^{1,4} \left[ -\left( \frac{y}{w} \right)^k \right]
\]

\[
\int_0^\infty x^{\mu-1} \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})} S_n \left[ x(x + a + \sqrt{x^2 + 2ax})^{-\rho} \right] dx
\]

\[
= 2^{1-\mu} a^{\mu-\alpha} \Gamma(2\mu) \sum_{n=0}^{[n/\alpha]!} \frac{(-n)_{(\mu-\alpha)}}{\alpha!} A[n, \alpha] z^{\alpha} \frac{1}{\Gamma(\gamma)} H_{1,5}^{1,4} \left[ -\left( \frac{y}{w} \right)^k \right]
\]

\[
\int_0^\infty \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})} \left( x(x + a + \sqrt{x^2 + 2ax})^{-\rho} \right) dx
\]

\[
= 2^{1-\mu} a^{\mu-\alpha} \Gamma(2\mu) \sum_{n=0}^{[n/\alpha]!} \frac{(-n)_{(\mu-\alpha)}}{\alpha!} A[n, \alpha] z^{\alpha} \frac{1}{\Gamma(\gamma)} H_{1,5}^{1,4} \left[ -\left( \frac{y}{w} \right)^k \right]
\]

\[
\int_0^\infty \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})} \left( x(x + a + \sqrt{x^2 + 2ax})^{-\rho} \right) dx
\]

\[
= 2^{1-\mu} a^{\mu-\alpha} \Gamma(2\mu) \sum_{n=0}^{[n/\alpha]!} \frac{(-n)_{(\mu-\alpha)}}{\alpha!} A[n, \alpha] z^{\alpha} \frac{1}{\Gamma(\gamma)} H_{1,5}^{1,4} \left[ -\left( \frac{y}{w} \right)^k \right]
\]

\[
\int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\rho} E_{\alpha,\beta}^\gamma \left[ \frac{y^k}{w^k(x + a + \sqrt{x^2 + 2ax})} \right] dx
\]

\[
= 2^{1-\mu} a^{\mu-\alpha} \Gamma(2\mu) \sum_{n=0}^{[n/\alpha]!} \frac{(-n)_{(\mu-\alpha)}}{\alpha!} A[n, \alpha] z^{\alpha} \frac{1}{\Gamma(\gamma)} H_{1,5}^{1,4} \left[ -\left( \frac{y}{w} \right)^k \right]
\]

Corollary 2.2: If we take \( m_1, m_2 \rightarrow m \) and \( n_1, n_2 \rightarrow n \) i.e. \( 1, \ldots, t \rightarrow 1 \) in the integral (16) and let the condition of theorem 2 be satisfied, we obtained

Corollary 2.3: If we take \( m_1, m_2 \rightarrow m \) and \( n_1, n_2 \rightarrow n \) i.e. \( 1, \ldots, t \rightarrow 1 \) and set \( p = 1, q = 1 \), \( a_j = \gamma, b_j = 1 \) the generalized M-series convert in generalized Mittag-Leffler function (6) introduced by Prabhakar [7] in the integral (16) and let the condition of theorem 2 be satisfied, we obtained

Corollary 2.4: If we take \( S_{m_1, m_2} \rightarrow 1 \) in the integral (6) and let the condition of theorem 1 be satisfied, we arrive at the following result which is obtained by Chouhan and Khan [2].

Corollary 2.5: If we take \( S_{m_1, m_2} \rightarrow 1 \) and set \( p = 1, q = 1 \), \( a_j = \gamma, b_j = 1 \) the generalized M-series convert in generalized Mittag-Leffler function (6) introduced by Prabhakar [7] in the integral (16) and let the condition of theorem 2 be satisfied, we arrive at the following result

4. CONCLUSION

In this paper we have presented generalized integral formulas involving Sexana I- function and M-series with generalized polynomials. The result so established may be found useful in several interesting situation appearing in the literature on mathematical analysis. Further many known and unknown results have been established in terms of special cases. The results presented in this paper are easily converted in terms of the Fox H-function, generalized Mittag-Leffler function, G-function, Bessel function and hyper geometric function after some suitable parametric replacement. We are also trying to find certain possible applications of those results presented here to some other research areas.
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Source of support: Nil, Conflict of interest: None Declared.

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