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Essential concepts of pg**- closed sets

Dr. A. PUNITHA THARANI
Associate Professor, St. Mary's College, Thoothukudi – India.

Mrs. G. PRISCILLA PACIFICA*
Assistant Professor, St. Mary's College, Thoothukudi – India.

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ABSTRACT

*In this paper we defined pg**- neighbourhood, pg**closure, pg**interior and pg**-boundary by means of pg**-closed and pg**-open sets and studied their properties. Furtherpg**-multiplicative and pg** - additive are also defined and implemented.*

Key words: *pg**-multiplicative, pg**- additive, pg**- neighbourhood, pg**closure, pg**interior, pg**- boundary.*

1. INTRODUCTION

Levine [3] introduced the class of g-closed sets in 1970. Veerakumar [7] introduced g*-closed sets. P M Helen [5] introduced g**-closed sets. A.S.Mashhour, M.E Abd El. Monsef and S.N.EI. Deeb [4] introduced a new class of pre-open sets in 1982. We have already introduced pg**-closed sets [6] and investigated their properties. The purpose of this paper is to introduce pg**- multiplicative, pg**- additive, pg**- neighbourhood, pg**closure, pg**interior, pg**-boundary and analyse their properties.

2. PRELIMINARIES

Definition 2.1: A subset A of a topological space(X,τ) is called a pre-open set [4] if $A \subseteq int(cl(A)$ and a pre-closed set if $cl(int(A)) \subseteq A$.

Definition 2.2: A subset A of topological space (X,τ) is called

1. generalized closed set (g-closed) [3] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).
2. g*-closed set [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X,τ).
3. g**-closed set [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g*-open in(X,τ).
4. pg**- closed set[6] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is g*-open in(X,τ).

3. Essential concepts of pg**- closed sets

If A and B are pg**- closed subsets of (X,τ), then $A \cup B$ is also a pg**- closed set[6]and hence the finite union of pg**- closed sets is pg**- closed. Equivalently finite intersection of pg**- open sets is open. But arbitrary union of pg**- open sets need not be pg**- open. Hence $PG^{**}O(X,\tau)$ is not a topology. To make it a topology, we need the following definition.

Definition 3.1: A topological space (X,τ) is said to be *pg**-multiplicative* (resp. pg**-finitely multiplicative, pg**-countably multiplicative) if arbitrary (resp. finite, countable) intersection of pg**- closed sets is pg**- closed. Equivalently arbitrary (resp. finite, countable) union of pg**- open sets is pg**- open.

Remark 3.2: In a pg**-multiplicative space $PG^{**}O(X,\tau)$ is a topology.For,

1. φ and X are pg**- open sets.
2. Arbitrary union of pg**- open sets is pg**- open.
3. Finite intersection of pg**- open sets is pg**-open.

Corresponding Author: Mrs. G. Priscilla Pacifica*
Assistant Professor, St. Mary's College, Thoothukudi – India.

Example 3.3: An infinite set with cofinite topology is pg**-multiplicative.

Consider ℝ with infinite cofinite topology. In this space, Let $\{F_\alpha\}$ be an arbitrary collection of pg**- closed sets. Therefore each F_α is either finite or φ or is all of X. Then $\cap F_\alpha$ finite or φ or X and hence arbitrary intersection of pg**- closed sets is pg**- closed. Therefore ℝ with infinite cofinite topology is a pg**-multiplicative space.

Definition 3.4: A topological space (X,τ) is said to be *pg** - additive* (resp. pg**-countably additive) if arbitrary (resp. countable) union of pg**- closed sets is pg**- closed. Equivalently arbitrary (resp. countable) intersection of pg**- open sets is pg**- open.

Example 3.5: Consider ℝ with cofinite topology is not pg**-countably additive and not pg**-additive. Let $A_n = \{-n, -(n-1), \dots, (n-1), n\}$ then A_n's are pg**- closed but $\cup A_n = Z$ is not pg**- closed. Therefore ℝ with infinite cofinite topology is not pg**-additive.

Definition 3.6: A topological space (X,τ) is said to be *pg**-discrete* if every subset of X is pg**-open. Equivalently every subset is pg**-closed.

Example 3.7: All the discrete and indiscrete topological spaces are pg**-discrete.

Example 3.8: ℝ with infinite cofinite topology is not pg**-discrete.

Definition 3.9: Let(X,τ)be a topological space and $x \in X$. Every pg**- open set containing x is said to be a *pg**-neighbourhood* of x. Differently a set U in X is said to be a*pg**-neighbourhood* of x if $x \in G \subseteq U$ for some pg**- open set G in X.The collection N_x of allpg**- neighbourhoods of x is called the pg**- neighbourhood system of x.

Theorem 3.10: Let A be a subset of a pg**-multiplicative space(X,τ). Then A is pg**- open if and only if A contains a pg**-neighbourhood of each of its points.

Proof: Let A be a pg**- open set in (X,τ) and $x \in A$. Then A is a pg**- open set containing x and hence A is a pg**- neighbourhood of x, contained in A.Conversely suppose A contains apg**-neighbourhood of each of its points. For every $x \in A$, there exists a pg**-neighbourhood G_x of x such that $x \in G_x \subseteq A$ and hence $\bigcup_x G_x \subseteq A$. Let $x \in A$, then there exists apg**-neighbourhood G_x such that $x \in G_x$. Therefore $x \in \bigcup_x G_x$. Hence $A = \bigcup_x G_x$. Since (X,τ) is a pg**- multiplicative space $\bigcup_x G_x$ is pg**- open, and hence A is pg**- open.

Theorem 3.11: Let (X,τ)be a pg**-multiplicative space. If F is a pg**-closed subset of X and $x \in F^c$, then there exists a pg**- neighbourhood U of x such that $U \cap F = \varphi$.

Proof: Let F be pg**-closed subset of X and $x \in F^c$. Then F^c is pg**- open set of X. Then by theorem (3.7) F^c contains a pg**-neighbourhood of each of its points. Hence there exists apg**- neighbourhood U of x such that $U \subset F^c$. Therefore $U \cap F = \varphi$.

Theorem 3.12: Every neighbourhood U of $x \in X$ is pg**-neighbourhood of x.

Proof: Follows from every open set is pg**- open.

Remark 3.13: In general a pg**-neighbourhood U of $x \in X$ need not be a neighbourhood of x, as seen from the following example.

Example 3.14: Let (X,τ), where $X = \{a,b,c\}, \tau = \{\varphi, X, \{a\}, \{a,c\}\}$ be a topological space.

Here $** O(X,\tau) = \{\varphi, X, \{b\}, \{c\}, \{a,b\}, \{b,c\}\}$. The set $\{a,b\}$ is a pg**-neighbourhood of the point $b \in X$. However, the set $\{a,b\}$ is not a neighbourhood of the point b.

Definition 3.15: Let A be a subset of (X,τ). A point $x \in X$ is said to be *pg**-limit point* or pg**-cluster point or pg**- accumulation point of A if every pg**-neighborhood of x contains a point of A other than x. Said differently, x is a pg**-limit point of A if it belongs to the pg**-closure of $A - \{x\}$. The set of all pg**-limit point of $A$is called pg**- derived set of A and is denoted by the symbol A'.

Example 3.16: Consider ℝ with infinite cofinite topology and the subset ℚ.
$PG ** O(\mathbb{R}) = \{\varphi, \mathbb{R}, all\ infinite\ subsets\}$. Let $x \in \mathbb{R}$ be arbitrary and U, apg**-neighbourhood of x. Then U is infinite and U contains a point of ℚ other than x. Therefore x is a pg**-limit point of ℚ.

Example 3.17: Consider $\mathbb{R}$ with discrete topology. $PG ** O(\mathbb{R}) = \{ all\ subsets\}$.

The set of all rationals $\mathbb{Q}$ has no pg**-limit point. Since for any $\in \mathbb{R}$, $\{x\}$ is pg**-neighbourhood of x which contains no point of $\mathbb{Q}$ other than x. In fact, in any set with discrete topology, no subset has a pg**-limit point.

Theorem 3.18: If A and B are subsets of a space (X, τ), then $A \subset B \Longrightarrow A^{'} \subset B^{'}$.

Proof: Let $x \in A^{'}$. Then every pg**-neighbourhood U of x contains a point y of A with $y \neq x$. Since $A \subset B$, $y \in B$. Hence every pg**-neighbourhood U of x contains a point y of B with $y \neq x$. Hence $x \in B^{'}$. Therefore, $A^{'} \subset B^{'}$.

Definition 3.19: Let A be a subset of a topological space(X, τ). A is said to be *pg**-perfect* if A is pg**-closed and every point of A is a pg**-limit point of A.

Definition 3.20: Let A be a subset of a topological space(X, τ). $pg ** cl(A)$ is defined to be the intersection of all pg**-closed sets containing A.

Note:

(i) Since intersection of pg**-closed sets need not be pg**-closed, $pg ** cl(A)$ need not be pg**-closed. If A is pg**-closed then $pg ** cl(A) = A$. But $pg ** cl(A) = A$ need not imply A is pg**-closed.

(ii) If (X, τ) is pg**-multiplicative then $pg ** cl(A) = A$ if and only if A is pg**-closed.

Theorem 3.21: If A is a subset of a topological space(X, τ), then $pg ** cl(A) \subset cl(A)$.

Proof: Let A be a subset of a topological space(X, τ). $cl(A) = \cap \{F \subset X : A \subset F \in C(X)\}$. Since every closed set is pg**-closed$A \subset F \in C(X)$, implies$A \subset F \in PG ** C(X)$. That is $pg ** cl(A) \subset F$. Therefore $pg ** cl(A) \subset \cap \{F \subset X : A \subset F \in C(X)\} = cl(A)$. Hence $pg ** cl(A) \subset cl(A)$.The converse of the above Theorem need not be true in general as seen in the following example.

Example 3.22: Let $X = \{a, b, c\}$ with topology $\tau = \{\varphi, X, \{a\}, \{a, b\}\}$. Let $A = \{a\}$ where $pg ** cl(A) = \{a, c\}$ and $cl(A) = X$. Hence $pg ** cl(A) \neq cl(A)$.

Theorem 3.23: For any $x \in X$, $x \in pg ** cl(A)$ if and only if $A \cap U \neq \varphi$ for every pg**-open set U containing x.

Proof: Let $x \in pg ** cl(A)$. Suppose there exists a pg**-open set U containing x such that$A \cap U = \varphi$. Then $A \subseteq X - U$. Since $X - U$ is pg**-closed, $pg ** cl(A) \subseteq X - U$. This implies $x \notin pg ** cl(A)$ which is a contradiction. Hence $A \cap U \neq \varphi$ for every pg**-open set U containing x. Conversely, let $A \cap U \neq \varphi$ for every pg**-open set U containing x. Suppose $x \notin pg ** cl(A)$, then there exists a pg**-closed set F containing A such that $x \notin F$. Then $x \in X - F$ and $X - F$ ispg**-open. Also $(X - F) \cap A = \varphi$this is a contradiction to the hypothesis. Hence $x \in pg ** cl(A)$.

Theorem 3.24: $Let\ A$ be a subset of a topological space(X, τ). Then $pg ** cl(A) = A \cup A^{'}$.

Proof: Clearly $A \subseteq pg ** cl(A)$. Let $x \in A^{'}$ and suppose $x \notin pg ** cl(A)$, then there exists a pg**-closed set F containing A such that $x \notin F$. Then $x \in X - F$ and $X - F$ is pg**-open. Also $(X - F) \cap (A - \{x\}) = \varphi$ which is not true. Therefore $x \in pg ** cl(A)$. Therefore$A \cup A^{'} \subseteq pg ** cl(A)$. Let $x \in pg ** cl(A)$ and $x \notin A$. Suppose $x \notin A^{'}$ then there exists apg**-neighbourhood U of x such that $A \cap U = \varphi$. Therefore $A \subseteq X - U$ which is pg**-closed containing A and $x \notin X - U$. which is a contradiction. Therefore $pg ** cl(A) \subseteq A \cup A^{'}$. Hence $pg ** cl(A) = A \cup A^{'}$.

Theorem 3.25: The subset A of pg**-multiplicative space(X, τ) is pg**-closed if and only if $A^{'} \subseteq A$.

Proof: By theorem (3.21) A is pg**-closed if and only if $A = A \cup A^{'} \Leftrightarrow A^{'} \subseteq A$.

Definition 3.26: Let A be a subset of a topological space(X, τ). Then A is *pg**-dense* in X if every point of X is a pg**-limit point of A or a point of A.

Definition 3.27: Atopological space having countable pg**-dense subset is said to be *pg**-separable*.

Example 3.28: In $\mathbb{R}$ with cofinite topology $\mathbb{Q}$ is pg**-dense in $\mathbb{R}$. Also $\mathbb{R}$ is pg**-separable.

Definition 3.29: Let A be a subset of a topological space (X, τ). A point $x \in A$ is said to be*pg**-interior point* of A if there exists a pg**-open set U such that $x \in U \subset A$.

Definition 3.30: Let A be a subset of a topological space(X, τ). *pg**int(A)* is defined to be the union of all pg**-open sets contained in A.

Equivalently $pg ** int(A) = \cup \{ U: U \subseteq A, U \in PG ** O(X)\}$.

Example 3.31:
(1) Consider $\mathbb{R}$ with discrete topology. Then $\mathbb{Q}$ is pg**-open and hence every point in $\mathbb{Q}$ is a pg**-interior point.
(2) Consider $\mathbb{R}$ with cofinite topology, the subset $\mathbb{Q}$ and $x \in \mathbb{Q}$ be arbitrary. Suppose x is a pg**-interior point of $\mathbb{Q}$, then there exists a pg**-neighbourhood U of x such that $x \in U \subset \mathbb{Q}$. This implies $\mathbb{Q}^c$ must be finite which is not true. Therefore x is not apg**-interior point of $\mathbb{Q}$. Since x is arbitrary $\mathbb{Q}$ has no pg**-interior point.

Note Any subset of $\mathbb{R}$ with cofinite topology whose complement is not finite has no pg**-interior point.

Note:
(1) Obviously $pg ** int(A)$is the set of all pg**-interior point of A.
(2) $pg ** int(A)$need not be pg**-open but if A is pg**-open then $pg ** int(A) = A$.
(3) If (X, τ) is pg**-multiplicative space then $pg ** int(A) = A$ if and only if A is pg**-open.

Theorem 3.32: For any two subsets A and B of (X, τ). Then,
1. $int(A) \subseteq pg ** int(A) \subseteq A$.
2. If $A \subseteq B$, then $pg ** int(A) \subseteq pg ** int(B)$.
3. $pg ** int(A \cup B) \supseteq pg ** int(A) \cup pg ** int(B)$.
4. $pg ** int(A \cap B) = pg ** int(A) \cap pg ** int(B)$.

Proof: follows from the definition.

Remark 3.33: For a subset A of $Xpg ** int(A) \neq int(A)$ as seen from the following example.

Example 3.34: Let $X = \{a, b, c\}, \tau = \{\varphi, X, \{a\}, \{a, b\}\}$ Let $A = \{a, c\}$ where $pg ** int(A) = \{a, c\}$ and $int(A) = \{a\}$. Hence $pg ** int(A) \neq int(A)$.

Remark 3.35: $pg ** int(A) = pg ** int(B)$ does not imply that $A = B$. This is revealed by the following example.

Example 3.36: Let (X, τ), where $X = \{a, b, c\}, \tau = \{\varphi, X, \{a\}, \{c\}, \{a, c\}\}$ be a topological space. Here $PG^{**}O(X, \tau) = \{\varphi, X, \{a\}, \{c\}, \{a, c\}\}$. Let $A = \{a, b\}$ and $B = \{a\}$, then $pg ** int(A) = pg ** int(B)$ but $A \neq B$.

Theorem 3.37: Let A be a subset of (X, τ), then the following are true.
(1) $(pg ** int(A))^c = pg ** cl(A^c)$.
(2) $pg ** int(A) = (pg ** cl(A^c))^c$.
(3) $pg ** cl(A) = (pg ** int(A^c))^c$.

Proof:
(1) Let $x \in (pg ** int(A))^c$. Then $x \notin pg ** int(A)$. That is every pg**-open set U containing x is such that U is not a proper subset of A. Thus $U \cap A^c \neq \varphi$ for every pg**-open set U containing x. Thus $x \in pg ** cl(A^c)$. Conversely, suppose $x \in pg ** cl(A^c)$, then for every pg**-open set U containing x, $U \cap A^c \neq \varphi$. Then by the definition of pg**int(A), $x \notin pg ** int(A)$, hence $x \in (pg ** int(A))^c$.
Therefore $(pg ** int(A))^c = pg ** cl(A^c)$.
(2) Follows by taking complements in (1).
(3) Follows by replacing A by A^c in (1).

Theorem 3.38: For any $A \subseteq X$, $(X - pg ** int(A)) = pg ** cl(X - A)$.

Proof: Let $x \in X - pg ** int(A)$. Then $x \notin pg ** int(A)$, that is every pg**-open set G containing x is such that $G \nsubseteq A$. Therefore every pg**-open set G containing x intersects $X - A$. That is $G \cap X - A \neq \varphi$ and hence $x \in pg ** cl(X - A)$. Conversely let $x \in pg ** cl(X - A)$. Then every pg**-open set G containing x intersects $X - A$, that is $G \cap X - A \neq \varphi$. To be precise every pg**-open set G containing x is such that $G \nsubseteq A$. This implies $x \notin pg ** int(A)$. Therefore $x \in X - pg ** int(A)$ and hence $(X - pg ** int(A)) = pg ** cl(X - A)$.

Remark 3.39: For any $A \subseteq X$, we have
(i) $(X - pg ** cl(X - A)) = pg ** int(A)$.
(ii) $(X - pg ** int(X - A)) = pg ** cl(A)$. Taking complement in the above theorem and by replacing A by $X - A$ in theorem (3.38) the results (i) and (ii) follow.

Definition 3.40: A subset A of a topological space(X,τ) is called *pg**-clopen* if it is both pg******- open and pg******- closed in X.

Example 3.41: Consider $\mathbb{R}$ with usual topology $\mathbb{Q}$ and $\mathbb{Q}^c$ are pg******-clopen.

Definition 3.42: A point $x \in X$ is said to be a *pg**-boundary point* of A if every pg******- open set containing x intersects both A and $X - A$.

Definition 3.43: Let A be any subset of a topological space (X,τ). Then the *pg**-boundary* of A is defined as pg******Bd(A) = pg $**$ cl(A) $\cap$ pg $**$ cl(A^c).

Example 3.44: Consider $\mathbb{R}$ with discrete topology and $\mathbb{Q}$, the set of rationals. Let $r \in \mathbb{R}$ be arbitrary, then $\{r\}$ is a pg******- open set containing r which cannot intersect both $\mathbb{Q}$ and $\mathbb{Q}^c$. Therefore $\mathbb{Q}$ has no pg******-boundary point.

Example 3.45: Consider $\mathbb{R}$ with finite complement topology and $\mathbb{Q}$, the set of rationals. Let $r \in \mathbb{R}$ be arbitrary and U be a pg******-neighbourhood of r, then U is infinite and hence contains atleast one point of $\mathbb{Q}$. Therefore U intersects both both $\mathbb{Q}$ and $\mathbb{Q}^c$. Therefore every real number is a pg******-boundary point of $\mathbb{Q}$.

Infact, any infinite subset A of $\mathbb{R}$ whose complement is also infinite has every real number as its pg******-boundary point.

Definition 3.46: If (X,τ) is a topological space, a point $x \in$ X is said to be a *pg**- isolated point* of X if the one-point set $\{x\}$ is pg******- open in X.

Definition 3.47: Let (X,τ) be a topological space and A be a subset of X. A point x in A is called a *pg**- isolated point* of A if it has a pg******- neighborhood of x which contains no other point of A.

Definition 3.48: Let (X,τ) be a topological space and $A \subseteq X$. Then the *pg**-border* of A is defined as
$b\mathrm{pg}**(\mathrm{A}) = \mathrm{A} - \mathrm{pg}**\mathrm{int}(\mathrm{A})$.

Definition 3.49: Let A be any subset of a topological space (X,τ). Then the *pg**-exterior* of A is defined as
$pg ** Ext(A) = pg ** int(A^c)$.

Theorem 3.50: Let A and B be any two sets of a topological space (X,τ), then the following conditions hold:

(i) $\mathrm{pg}** Bd(A) = \mathrm{pg}** \mathrm{Bd}(\mathrm{A^c})$.
(ii) $\mathrm{pg}** Bd(A) \subseteq \mathrm{pg}** \mathrm{cl}(\mathrm{A^c})$.
(iii) If A is pg******-closed, then $\mathrm{pg}** Bd(A) \subseteq A$.
(iv) If A is pg******-open, then $\mathrm{pg}** Bd(A) \subseteq \mathrm{A^c}$.
(v) Let $A \subseteq B$ and $B \in \mathrm{pg}** \mathrm{Cl}(X,\tau)$(resp. $B \in \mathrm{pg}** \mathrm{O}(X,\tau)$). Then,
(vi) $\mathrm{pg}** Bd(A) \subseteq B$ (resp. $\mathrm{pg}** Bd(A) \subseteq \mathrm{B^c}$) where $\mathrm{pg}** \mathrm{Cl}(X,\tau)$ denotes the class of pg******-closed (resp. $\mathrm{pg}** \mathrm{O}(X,\tau)$ denotes the class of pg******-open) sets in X.
(vii) $(\mathrm{pg}** Bd(A))^c = \mathrm{pg}** \mathrm{int}(\mathrm{A}) \cup \mathrm{pg}** \mathrm{int}(\mathrm{A^c})$.

Proof: (i) pg****** $Bd(A)$ = $\mathrm{pg}** \mathrm{cl}(\mathrm{A}) \cap \mathrm{pg}** \mathrm{cl}(\mathrm{A^c}) = \mathrm{pg}** \mathrm{cl}(A^c)^c \cap \mathrm{pg}** \mathrm{cl}(\mathrm{A^c}) = \mathrm{pg}** \mathrm{Bd}(\mathrm{A^c})$.
(ii) and (iii) Follows from Definition of pg******$Bd(A)$.
(iv) $\mathrm{pg}** \mathrm{Bd}(\mathrm{A}) \subseteq \mathrm{pg}** \mathrm{cl}(\mathrm{A}) = \mathrm{A}$. Hence $\mathrm{pg}** \mathrm{Bd}(\mathrm{A}) \subseteq \mathrm{A}$.
(v) Suppose A is pg******-open then A^{c}is pg******-closed, also pg******$Bd(\mathrm{A^c}) \subseteq \mathrm{A^c}$. Hence by (i) $\mathrm{pg}** \mathrm{Bd}(\mathrm{A}) \subseteq \mathrm{A}$.
(vi) Since $\mathrm{A} \subseteq B, \mathrm{pg}** \mathrm{cl}(\mathrm{A}) \subseteq \mathrm{pg}** \mathrm{cl}(\mathrm{B})$.
Now pg******$\mathrm{Bd(A)} \subseteq \mathrm{pg}** \mathrm{cl}(\mathrm{A})\ ) \subseteq \mathrm{pg}** \mathrm{cl}(\mathrm{B}) = B$. Hence $\mathrm{pg}** \mathrm{Bd}(\mathrm{A}) \subseteq B$.
(vii) $(\mathrm{pg}** \mathrm{Bd}(\mathrm{A}))^c = (\mathrm{pg}** \mathrm{cl}(\mathrm{A}) \cap \mathrm{pg}** \mathrm{cl}(\mathrm{A^c}))^c = (\mathrm{pg}** \mathrm{cl}(\mathrm{A}))^c \cup (\mathrm{pg}** \mathrm{cl}(\mathrm{A^c}))^c$
$= \mathrm{pg}** \mathrm{int}(\mathrm{A^c}) \cup \mathrm{pg}** \mathrm{int}(A^c)^c = \mathrm{pg}** \mathrm{int}(\mathrm{A^c}) \cup \mathrm{pg}** \mathrm{int}(A)$.

Theorem 3.51: Let A be a subset of a topological space (X,τ), then the following conditions hold:

(i) $\mathrm{pg}** Bd(A) \subseteq Bd(A)$, where $Bd(A)$ denotes the boundary of A.
(ii) $\mathrm{pg}** \mathrm{cl}(\mathrm{A}) = \mathrm{pg}** \mathrm{int}(\mathrm{A}) \cup \mathrm{pg}** Bd(A)$
(iii) $\mathrm{pg}** \mathrm{int}(\mathrm{A}) \cap \mathrm{pg}** \mathrm{Bd}(\mathrm{A}) = \varphi$.
(iv) $\mathrm{pg}** \mathrm{Bd}\big(\mathrm{int}(\mathrm{A})\big) \subseteq \mathrm{pg}** Bd(A)$.
(v) $\mathrm{pg}** \mathrm{Bd}\big(\mathrm{cl}(\mathrm{A})\big) \subseteq \mathrm{pg}** Bd(A)$.
(vi) $b\mathrm{pg}** (\mathrm{A}) \subseteq \mathrm{pg}** Bd(A)$.

Proof: (i) pg******$Bd(A)$ = $\mathrm{pg}** \mathrm{cl}(\mathrm{A}) \cap \mathrm{pg}** \mathrm{cl}(\mathrm{A^c}) \subseteq \mathrm{cl}(\mathrm{A}) \cap \mathrm{cl}(\mathrm{A^c}) = Bd(A)$.
(ii) $\mathrm{pg}** \mathrm{int}(\mathrm{A}) \cup \mathrm{pg}** \mathrm{Bd}(\mathrm{A}) = \mathrm{pg}** \mathrm{int}(\mathrm{A}) \cup (\mathrm{pg}** \mathrm{cl}(\mathrm{A}) \cap \mathrm{pg}** \mathrm{cl}(\mathrm{A^c})) = \mathrm{pg}** \mathrm{cl}(\mathrm{A})$.
(ii) $\mathrm{pg}** \mathrm{int}(\mathrm{A}) \cap \mathrm{pg}** \mathrm{Bd}(\mathrm{A}) = \mathrm{pg}** \mathrm{int}(\mathrm{A}) \cap (\ \mathrm{pg}** \mathrm{cl}(\mathrm{A}) \cap \mathrm{pg}** \mathrm{cl}(\mathrm{A^c})) = \varphi$.

(iv) $pg ** Bd\big(int(A)\big) = (pg ** cl(int(A)) \cap pg ** cl(int(A))^c)$
$= (pg ** cl(int(A)) \cap (pg ** int(int(A)))^c \subseteq pg ** cl(A) \cap (pg ** int(A))^c = pg ** Bd(A)$.
(v) $pg ** Bd\big(cl(A)\big) = (pg ** cl(cl(A)) \cap pg ** cl(cl(A))^c) \subseteq pg ** cl(A) \cap (pg ** int(A))^c = pg ** Bd(A)$.
(vi) $bpg ** (A) = A - pg ** int(A) \subseteq pg ** cl(A) \cap (pg ** int(A))^c = pg ** Bd(A)$.

Theorem 3.52: Let A be a subset of a topological space (X, τ), then the following conditions hold:
(i) $bpg ** (A) \subseteq b(A)$, where $b(A)$ denotes the border of A.
(ii) $A = pg ** int(A) \cup bpg ** (A)$.
(iii) $pg ** int(A) \cap bpg ** (A) = \varphi$.
(iv) If A is pg $**$-open, then $bpg ** (A) = \varphi$.
(v) $bpg ** (A) = A \cap pg ** cl(A^c)$.

Proof: (i) follows from definition ofpg******-$border\ of A$ and $A - pg ** int(A) \subseteq A - int(A)$.
(ii) and (iii) follows from the definition of pg******-$border\ of A$.
(iv) If A is pg $**$-open, then $pg ** int(A) = A$. Thus $bpg ** (A) = \varphi$.
(v) $bpg ** (A) = A - pg ** int(A) = A - (pg ** cl(A^c))^c = A \cap pg ** cl(A^c)$.

Theorem 3.53: Let A be a subset of a topological space (X, τ), then the following conditions hold:
(i) $Ext(A) \subseteq pg ** Ext(A)$, where $Ext(A)$ denotes the exterior of A.
(ii) $pg ** Ext(X) = \varphi$.
(iii) $pg ** Ext(\varphi) = X$.
(iv) $pg ** Ext(A) = (pg ** cl(A))^c$.
(v) $pg ** Ext\big(pg ** Ext(A)\big) = pg ** int\big(pg ** cl(A)\big)$.
(vi) If $A \subseteq B$ then $pg ** Ext(A) \supseteq pg ** Ext(B)$.
(vii) $pg ** Ext(A \cup B) \subseteq pg ** Ext(A) \cup pg ** Ext(B)$.
(viii) $pg ** Ext(A \cap B) \supseteq pg ** Ext(A) \cap pg ** Ext(B)$.
(ix) $pg ** int(A) \subseteq pg ** Ext\big(pg ** Ext(A)\big)$.

Proof: (i) (ii) (iii) and (iv) follows from the definition of $pg ** Ext(A)$.
(v) $pg ** Ext\big(pg ** Ext(A)\big) = pg ** Ext(pg ** cl(A))^c = pg ** int\big(pg ** cl(A)\big)$.
(vi) If $A \subseteq B$ then $A^c \supseteq B^c \Rightarrow pg ** int(A^c) \supseteq pg ** int(B^c) \Rightarrow pg ** Ext(A) \supseteq pg ** Ext(B)$.
(vii) and (viii) follows from (vi).
(ix) $pg ** int(A) \subseteq pg ** int\big(pg ** cl(A)\big) = pg ** int\big(pg ** Ext(A)\big)^c = pg ** Ext\big(pg ** Ext(A)\big)$.

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