Essential concepts of pg**- closed sets

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(Received On: 11-02-17; Revised & Accepted On: 10-03-17)

ABSTRACT

In this paper we define pg**- neighbourhood, pg**-closure, pg**-interior and pg**-boundary by means of pg**- closed and pg**-open sets and studied their properties. Further pg**-multiplicative and pg**-additive are also defined and implemented.

Key words: pg**-multiplicative, pg**-additive, pg**-neighbourhood, pg**-closure, pg**-interior, pg**-boundary.

1. INTRODUCTION


2. PRELIMINARIES

Definition 2.1: A subset 𝐴𝐴 of a topological space (𝑋, 𝜏) is called a pre-open set [4] if 𝐴𝐴 ⊆ int(cl(A)) and a pre-closed set if cl(int(A)) ⊆ 𝐴𝐴.

Definition 2.2: A subset 𝐴𝐴 of topological space (𝑋, 𝜏) is called
1. generalized closed set (g-closed) [3] if cl(A) ⊆ 𝑈𝑈 whenever 𝐴𝐴 ⊆ 𝑈𝑈 and 𝑈𝑈 is open in (𝑋, 𝜏).
2. g*-closed set [7] if cl(A) ⊆ 𝑈𝑈 whenever 𝐴𝐴 ⊆ 𝑈𝑈 and 𝑈𝑈 is g-open in (𝑋, 𝜏).
3. g**-closed set [5] if cl(A) ⊆ 𝑈𝑈 whenever 𝐴𝐴 ⊆ 𝑈𝑈 and 𝑈𝑈 is g*-open in (𝑋, 𝜏).
4. pg**-closed set [6] if 𝑝𝑝𝑐𝑐𝑐𝑐(A) ⊆ 𝑈𝑈 whenever 𝐴𝐴 ⊆ 𝑈𝑈 and 𝑈𝑈 is g*-open in (𝑋, 𝜏).

3. Essential concepts of pg**- closed sets

If 𝐴𝐴 and 𝐵𝐵 are pg**-closed subsets of (𝑋, 𝜏), then 𝐴𝐴∪𝐵𝐵 is also a pg**-closed set[6] and hence the finite union of pg**-closed sets is pg**-closed. Equivalently finite intersection of pg**-open sets is open. But arbitrary union of pg**-open sets need not be pg**-open. Hence PG**O(X, 𝜏) is not a topology. To make it a topology, we need the following definition.

Definition 3.1: A topological space (𝑋, 𝜏) is said to be pg**-multiplicative (resp. pg**-finitely multiplicative, pg**-countably multiplicative) if arbitrary (resp. finite, countable) intersection of pg**- closed sets is pg**-closed. Equivalently arbitrary (resp. finite, countable) union of pg**-open sets is pg**-open.

Remark 3.2: In a pg**-multiplicative space PG**O(X, 𝜏) is a topology. For,
1. 𝜙 and 𝑋 are pg**-open sets.
2. Arbitrary union of pg**-open sets is pg**-open.
3. Finite intersection of pg**-open sets is pg**-open.

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**Example 3.3:** An infinite set with cofinite topology is $\text{pg}^\ast$-multiplicative.

Consider $\mathbb{R}$ with infinite cofinite topology. In this space, let $\{F_n\}$ be an arbitrary collection of $\text{pg}^\ast$-closed sets. Therefore each $F_n$ is either finite or $\varnothing$ or is all of $\mathbb{R}$. Then $\bigcap F_n$ is finite or $\varnothing$ or $\mathbb{R}$ and hence arbitrary intersection of $\text{pg}^\ast$-closed sets is $\text{pg}^\ast$-closed. Therefore $\mathbb{R}$ with infinite cofinite topology is a $\text{pg}^\ast$-multiplicative space.

**Definition 3.4:** A topological space $(X, \tau)$ is said to be $\text{pg}^\ast$-additive (resp. $\text{pg}^\ast$-countably additive) if arbitrary (resp. countable) union of $\text{pg}^\ast$-closed sets is $\text{pg}^\ast$-closed. Equivalently arbitrary (resp. countable) intersection of $\text{pg}^\ast$-open sets is $\text{pg}^\ast$-open.

**Example 3.5:** Consider $\mathbb{R}$ with cofinite topology is not $\text{pg}^\ast$-countably additive and not $\text{pg}^\ast$-additive. Let $A_n = \{-n, -(n-1), \ldots, (n-1), n\}$ then $A_n$'s are $\text{pg}^\ast$-closed but $\bigcup A_n = \mathbb{Z}$ is not $\text{pg}^\ast$-closed. Therefore $\mathbb{R}$ with infinite cofinite topology is not $\text{pg}^\ast$-additive.

**Definition 3.6:** A topological space $(X, \tau)$ is said to be $\text{pg}^\ast$-discrete if every subset of $X$ is $\text{pg}^\ast$-open. Equivalently every subset is $\text{pg}^\ast$-closed.

**Example 3.7:** All the discrete and indiscrete topological spaces are $\text{pg}^\ast$-discrete.

**Example 3.8:** $\mathbb{R}$ with infinite cofinite topology is not $\text{pg}^\ast$-discrete.

**Definition 3.9:** Let $(X, \tau)$ be a topological space and $x \in X$. Every $\text{pg}^\ast$-open set containing $x$ is said to be a $\text{pg}^\ast$-neighbourhood of $x$. Differently a set $U$ in $X$ is said to be an $\text{pg}^\ast$-neighbourhood of $x$ if $x \in G \subseteq U$ for some $\text{pg}^\ast$-open set $G$ in $X$. The collection $\mathcal{V}_x$ of all $\text{pg}^\ast$-neighbourhoods of $x$ is called the $\text{pg}^\ast$-neighbourhood system of $x$.

**Theorem 3.10:** Let $A$ be a subset of a $\text{pg}^\ast$-multiplicative space $(X, \tau)$. Then $A$ is $\text{pg}^\ast$-open if and only if $A$ contains a $\text{pg}^\ast$-neighbourhood of each of its points.

**Proof:** Let $A$ be a $\text{pg}^\ast$-open set in $(X, \tau)$ and $x \in A$. Then $A$ is a $\text{pg}^\ast$-neighbourhood of $x$, contained in $A$. Conversely suppose $A$ contains $\text{pg}^\ast$-neighbourhood of each of its points. For every $x \in A$, there exists a $\text{pg}^\ast$-neighbourhood $G_x$ of $x$ such that $x \in G_x \subseteq A$ and hence $\bigcup G_x \subseteq A$. Let $x \in A$, then there exists $\text{pg}^\ast$-neighbourhood $G_x$ such that $x \in G_x$. Therefore $x \in \bigcup G_x$. Hence $A = \bigcup G_x$. Since $(X, \tau)$ is a $\text{pg}^\ast$-multiplicative space $\bigcup G_x$ is $\text{pg}^\ast$-open, and hence $A$ is $\text{pg}^\ast$-open.

**Theorem 3.11:** Let $(X, \tau)$ be a $\text{pg}^\ast$-multiplicative space. If $F$ is a $\text{pg}^\ast$-closed subset of $X$ and $x \in F^c$, then there exists a $\text{pg}^\ast$-neighbourhood $U$ of $x$ such that $U \cap F = \varnothing$.

**Proof:** Let $F$ be $\text{pg}^\ast$-closed subset of $X$ and $x \in F^c$. Then $F^c$ is $\text{pg}^\ast$-open set of $X$. Then by theorem (3.7) $F^c$ contains a $\text{pg}^\ast$-neighbourhood of each of its points. Hence there exists $\text{pg}^\ast$-neighbourhood $U$ of $x$ such that $U \subseteq F^c$. Therefore $U \cap F = \varnothing$.

**Theorem 3.12:** Every neighbourhood $U$ of $x \in X$ is $\text{pg}^\ast$-neighbourhood of $x$.

**Proof:** Follows from every open set is $\text{pg}^\ast$-open.

**Remark 3.13:** In general a $\text{pg}^\ast$-neighbourhood $U$ of $x \in X$ need not be a neighbourhood of $x$, as seen from the following example.

**Example 3.14:** Let $(X, \tau)$, where $X = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{a, c\}\}$ be a topological space.

Here $\ast O(X, \tau) = \{\varnothing, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$. The set $\{a, b\}$ is a $\text{pg}^\ast$-neighbourhood of the point $b \in X$. However, the set $\{a, b\}$ is not a neighbourhood of the point $b$.

**Definition 3.15:** Let $A$ be a subset of $(X, \tau)$. A point $x \in X$ is said to be $\text{pg}^\ast$-limit point or $\text{pg}^\ast$-cluster point of $A$ if every $\text{pg}^\ast$-neighbourhood of $x$ contains a point of $A$ other than $x$. Said differently, $x$ is a $\text{pg}^\ast$-limit point of $A$ if it belongs to the $\text{pg}^\ast$-closure of $A - \{x\}$. The set of all $\text{pg}^\ast$-limit point of $A$ is called $\text{pg}^\ast$-derived set of $A$ and is denoted by the symbol $A'$.

**Example 3.16:** Consider $\mathbb{R}$ with infinite cofinite topology and the subset $\mathbb{Q}$.

$\text{PG} \ast O(\mathbb{R}) = \{\varnothing, \mathbb{R}, \text{all infinite subsets}\}$. Let $x \in \mathbb{R}$ be arbitrary and $U$, $\text{pg}^\ast$-neighbourhood of $x$. Then $U$ is infinite and $U$ contains a point of $\mathbb{Q}$ other than $x$. Therefore $x$ is a $\text{pg}^\ast$-limit point of $\mathbb{Q}$.
Example 3.17: Consider \( \mathbb{R} \) with discrete topology. \( PG \ast\ast O(\mathbb{R}) = \{ \text{all subsets} \} \).

The set of all rationals \( \mathbb{Q} \) has no \( pg\ast\ast \)-limit point. Since for any \( x \in \mathbb{R} \), \( \{ x \} \) is \( pg\ast\ast \)-neighbourhood of \( x \) which contains no point of \( \mathbb{Q} \) other than \( x \). In fact, in any set with discrete topology, no subset has a \( pg\ast\ast \)-limit point.

Theorem 3.18: If \( A \) and \( B \) are subsets of a space \((X, \tau)\), then \( A \subset B \implies A' \subset B' \).

Proof: Let \( x \in A' \). Then every \( PG\ast\ast \)-neighbourhood \( U \) of \( x \) contains a point \( y \) of \( A \) with \( y \neq x \). Since \( A \subset B \), \( y \in B \). Hence every \( pg\ast\ast \)-neighbourhood \( U \) of \( x \) contains a point \( y \) of \( B \) with \( y \neq x \). Hence \( x \in B' \). Therefore, \( A' \subset B' \).

Definition 3.19: Let \( A \) be a subset of a topological space \((X, \tau)\). A is said to be \( pg\ast\ast \)-perfect if \( A \) is \( pg\ast\ast \)-closed and every point of \( A \) is a \( pg\ast\ast \)-limit point of \( A \).

Definition 3.20: Let \( A \) be a subset of a topological space \((X, \tau)\). \( pg \ast cl(A) \) is defined to be the intersection of all \( pg\ast\ast \)-closed sets containing \( A \).

Note:
1. Since intersection of \( pg\ast\ast \)-closed sets need not be \( pg\ast\ast \)-closed, \( pg \ast cl(A) \) need not be \( pg\ast\ast \)-closed. If \( A \) is \( pg\ast\ast \)-closed then \( pg \ast cl(A) = A \). But \( pg \ast cl(A) = A \) need not imply \( A \) is \( pg\ast\ast \)-closed.
2. If \((X, \tau)\) is \( pg\ast\ast \)-multiplicative then \( pg \ast cl(A) = A \) if and only if \( A \) is \( pg\ast\ast \)-closed.

Theorem 3.21: If \( A \) is a subset of a topological space \((X, \tau)\), then \( pg \ast cl(A) \subset cl(A) \).

Proof: Let \( A \) be a subset of a topological space \((X, \tau)\). \( cl(A) = \cap \{ F \subset X : A \subset F \subset C(X) \} \). Since every closed set is \( pg\ast\ast \)-closed \( A \subset F \in C(X) \), implies \( A \subset F \in PG \ast cl(A) \). That is \( pg \ast cl(A) \subset F \). Therefore \( pg \ast cl(A) \subset \cap \{ F \subset X : A \subset F \subset C(X) \} = cl(A) \). Hence \( pg \ast cl(A) \subset cl(A) \).

The converse of the above Theorem need not be true in general as seen in the following example.

Example 3.22: Let \( X = \{ a, b, c \} \) with topology \( \tau = \{ \varnothing, X, [a], [a, b] \} \). Let \( A = \{ a \} \) where \( pg \ast cl(A) = \{ a, c \} \) and \( cl(A) = X \). Hence \( pg \ast cl(A) \neq cl(A) \).

Theorem 3.23: For any \( x \in X \), \( x \in pg \ast cl(A) \) if and only if \( A \cap U \neq \varnothing \) for every \( pg\ast\ast \)-open set \( U \) containing \( x \).

Proof: Let \( x \in pg \ast cl(A) \). Suppose there exists a \( pg\ast\ast \)-open set \( U \) containing \( x \) such that \( A \cap U = \varnothing \). Then \( A \subset X - U \). Since \( X - U \) is \( pg\ast\ast \)-closed, \( pg \ast cl(A) \subset X - U \). This implies \( x \notin pg \ast cl(A) \) which is a contradiction. Hence \( A \cap U \neq \varnothing \) for every \( pg\ast\ast \)-open set \( U \) containing \( x \). Conversely, let \( A \cap U \neq \varnothing \) for every \( pg\ast\ast \)-open set \( U \) containing \( x \). Suppose \( x \notin pg \ast cl(A) \), then there exists a \( pg\ast\ast \)-closed set \( F \) containing \( A \) such that \( x \notin F \). Then \( x \in X - F \) and \( X - F \) is \( pg\ast\ast \)-open. Also \( (X - F) \cap A = \varnothing \) this is a contradiction to the hypothesis. Hence \( x \in pg \ast cl(A) \).

Theorem 3.24: Let \( A \) be a subset of a topological space \((X, \tau)\). Then \( pg \ast cl(A) = A \cup A' \).

Proof: Clearly \( A \subset pg \ast cl(A) \). Let \( x \in A' \) and suppose \( x \notin pg \ast cl(A) \), then there exists a \( pg\ast\ast \)-closed set \( F \) containing \( A \) such that \( x \notin F \). Then \( x \in X - F \) and \( X - F \) is \( pg\ast\ast \)-open. Also \( (X - F) \cap (A - \{ x \}) = \varnothing \) which is not true. Therefore \( x \in pg \ast cl(A) \). Therefore \( A \cup A' \subset pg \ast cl(A) \). Let \( x \in pg \ast cl(A) \) and \( x \notin A \). Suppose \( x \in A' \) then there exists an \( pg\ast\ast \)-neighbourhood \( U \) of \( x \) such that \( A \cap U = \varnothing \). Therefore \( A \subset X - U \) which is \( pg\ast\ast \)-closed containing \( A \) and \( x \notin X - U \). which is a contradiction. Therefore \( pg \ast cl(A) \subset A \cup A' \). Hence \( pg \ast cl(A) = A \cup A' \).

Theorem 3.25: The subset \( A \) of \( pg\ast\ast \)-multiplicative space \((X, \tau)\) is \( pg\ast\ast \)-closed if and only if \( A' \subseteq A \).

Proof: By theorem (3.21) \( A \) is \( pg\ast\ast \)-closed if and only if \( A = A \cup A' \iff A' \subseteq A \).

Definition 3.26: Let \( A \) be a subset of a topological space \((X, \tau)\). Then \( A \) is \( pg\ast\ast \)-dense in \( X \) if every point of \( X \) is a \( pg\ast\ast \)-limit point of \( A \) or a point of \( A \).

Definition 3.27: A topological space having countable \( pg\ast\ast \)-dense subset is said to be \( pg\ast\ast \)-separable.

Example 3.28: In \( \mathbb{R} \) with cofinite topology \( \mathbb{Q} \) is \( pg\ast\ast \)-dense in \( \mathbb{R} \). Also \( \mathbb{R} \) is \( pg\ast\ast \)-separable.

Definition 3.29: Let \( A \) be a subset of a topological space \((X, \tau)\). A point \( x \in A \) is said to be \( pg\ast\ast \)-interior point of \( A \) if there exists a \( pg\ast\ast \)-open set \( U \) such that \( x \in U \subset A \).
**Definition 3.30:** Let $A$ be a subset of a topological space $(X, \tau)$. $pg**\text{int}(A)$ is defined to be the union of all $pg**$-open sets contained in $A$.

Equivalently $pg**\text{int}(A) = \bigcup \{ U: U \subseteq A, U \in PG**O(X) \}$.

**Example 3.31:**

1. Consider $\mathbb{R}$ with discrete topology. Then $\mathbb{Q}$ is $pg**$-open and hence every point in $\mathbb{Q}$ is a $pg**$-interior point.
2. Consider $\mathbb{R}$ with cofinite topology, the subset $\mathbb{Q}$ and $x \in \mathbb{Q}$ be arbitrary. Suppose $x$ is a $pg**$-interior point of $\mathbb{Q}$, then there exists a $pg**$-neighbourhood $U$ of $x$ such that $x \in U \subset \mathbb{Q}$. This implies $\mathbb{Q}^c$ must be finite which is not true. Therefore $x$ is not $apg**$-interior point of $\mathbb{Q}$. Since $x$ is arbitrary $\mathbb{Q}$ has no $pg**$-interior point.

**Note:** Any subset of $\mathbb{R}$ with cofinite topology whose complement is not finite has no $pg**$-interior point.

**Note:**

1. Obviously $pg**\text{int}(A)$ is the set of all $pg**$-interior point of $A$.
2. $pg**\text{int}(A)$ need not be $pg**$-open but if $A$ is $pg**$-open then $pg**\text{int}(A) = A$.
3. If $(X, \tau)$ is $pg**$-multiplicative space then $pg**\text{int}(A) = A$ if and only if $A$ is $pg**$-open.

**Theorem 3.32:** For any two subsets $A$ and $B$ of $(X, \tau)$. Then,

1. $\text{int}(A) \subseteq pg**\text{int}(A) \subseteq A$.
2. If $A \subseteq B$, then $pg**\text{int}(A) \subseteq pg**\text{int}(B)$.
3. $pg**\text{int}(A \cup B) \supseteq pg**\text{int}(A) \cup pg**\text{int}(B)$.
4. $pg**\text{int}(A \cap B) = pg**\text{int}(A) \cap pg**\text{int}(B)$.

**Proof:** follows from the definition.

**Remark 3.33:** For a subset $A$ of $X$ $pg**\text{int}(A) \neq \text{int}(A)$ as seen from the following example.

**Example 3.34:** Let $X = \{a,b,c\}, \tau = \{\varnothing, X, \{a\}, \{a,b\}\}$ Let $A = \{a,c\}$ where $pg**\text{int}(A) = \{a,c\}$ and $\text{int}(A) = \{a\}$. Hence $pg**\text{int}(A) \neq \text{int}(A)$.

**Remark 3.35:** $pg**\text{int}(A) = pg**\text{int}(B)$ does not imply that $A = B$. This is revealed by the following example.

**Example 3.36:** Let $(X, \tau)$, where $X = \{a,b,c\}, \tau = \{\varnothing, X, \{a\}, \{c\}, \{a,c\}\}$ be a topological space. Here $PG**O(X, \tau) = \{\varnothing, X, \{a\}, \{c\}, \{a,c\}\}$. Let $A = \{a, b\}$ and $B = \{a\}$, then $pg**\text{int}(A) = pg**\text{int}(B)$ but $A \neq B$.

**Theorem 3.37:** Let $A$ be a subset of $(X, \tau)$, then the following are true.

1. $(pg**\text{int}(A))^c = pg**\text{cl}(A^c)$.
2. $pg**\text{int}(A) = (pg**\text{cl}(A^c))^c$.
3. $pg**\text{cl}(A) = (pg**\text{int}(A))^c$.

**Proof:**

1. Let $x \in (pg**\text{int}(A))^c$. Then $x \notin pg**\text{int}(A)$. That is every $pg**$-open set $U$ containing $x$ is such that $U$ is not a proper subset of $A$. Thus $U \cap A^c \neq \varnothing$ for every $pg**$-open set $U$ containing $x$. Thus $x \in pg**\text{cl}(A^c)$. Conversely, suppose $x \in pg**\text{cl}(A^c)$, then for every $pg**$-open set $U$ containing $x$, $U \cap A^c \neq \varnothing$. Then by the definition of $pg**\text{int}(A)$, $x \notin pg**\text{int}(A)$, hence $x \in (pg**\text{int}(A))^c$. Therefore $(pg**\text{int}(A))^c = pg**\text{cl}(A^c)$.
2. Follows by taking complements in (1).
3. Follows by replacing $A$ by $A^c$ in (1).

**Theorem 3.38:** For any $A \subseteq X$, $(X - pg**\text{int}(A)) = pg**\text{cl}(X - A)$.

**Proof:** Let $x \in X - pg**\text{int}(A)$. Then $x \notin pg**\text{int}(A)$, that is every $pg**$-open set $G$ containing $x$ is such that $G \not\subseteq A$. Therefore every $pg**$-open set $G$ containing $x$ intersects $X - A$. That is $G \cap X - A \neq \varnothing$ and hence $x \in pg**\text{cl}(X - A)$. Conversely let $x \in pg**\text{cl}(X - A)$. Then every $pg**$-open set $G$ containing $x$ intersects $X - A$, that is $G \cap X - A \neq \varnothing$. To be precise every $pg**$-open set $G$ containing $x$ is such that $G \not\subseteq A$. This implies $x \notin pg**\text{int}(A)$. Therefore $x \in X - pg**\text{int}(A)$ and hence $(X - pg**\text{int}(A)) = pg**\text{cl}(X - A)$.

**Remark 3.39:** For any $A \subseteq X$, we have

1. $(X - pg**\text{cl}(X - A)) = pg**\text{int}(A)$.
2. $(X - pg**\text{int}(X - A)) = pg**\text{cl}(A)$. Taking complement in the above theorem and by replacing $A$ by $X - A$ in theorem (3.38) the results (i) and (ii) follow.
Definition 3.40: A subset $A$ of a topological space $(X, \tau)$ is called \textit{pg**-clopen} if it is both pg**- open and pg**- closed in $X$.

Example 3.41: Consider $\mathbb{R}$ with usual topology $\mathbb{Q}$ and $\mathbb{Q^c}$ are pg**-clopen.

Definition 3.42: A point $x \in X$ is said to be a \textit{pg**-boundary point} of $A$ if every pg**- open set containing $x$ intersects both $A$ and $X - A$.

Definition 3.43: Let $A$ be any subset of a topological space $(X, \tau)$. Then the \textit{pg**-boundary} of $A$ is defined as $\text{pg**Bd}(A) = \text{pg** cl}(A) \cap \text{pg** cl}(A^c)$.

Example 3.44: Consider $\mathbb{R}$ with discrete topology and $\mathbb{Q}$, the set of rationals. Let $r \in \mathbb{R}$ be arbitrary, then $\{r\}$ is a pg**- open set containing $r$ which cannot intersect both $\mathbb{Q}$ and $\mathbb{Q^c}$. Therefore $\mathbb{Q}$ has no pg**-boundary point.

Example 3.45: Consider $\mathbb{R}$ with finite complement topology and $\mathbb{Q}$, the set of rationals. Let $r \in \mathbb{R}$ be arbitrary and $U$ be a pg**-neighbourhood of $r$, then $U$ is infinite and hence contains atleast one point of $\mathbb{Q}$. Therefore $U$ intersects both $\mathbb{Q}$ and $\mathbb{Q^c}$. Therefore every real number is a pg**-boundary point of $\mathbb{Q}$.

Infact, any infinite subset $A$ of $\mathbb{R}$ whose complement is also infinite has every real number as its pg**-boundary point.

Definition 3.46: If $(X, \tau)$ is a topological space, a point $x \in X$ is said to be a \textit{pg**- isolated point} of $X$ if the one-point set $\{x\}$ is pg**- open in $X$.

Definition 3.47: Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. A point $x \in A$ is called a \textit{pg**- isolated point} of $A$ if it has a pg**- neighborhood of $x$ which contains no other point of $A$.

Definition 3.48: Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then the \textit{pg**-border} of $A$ is defined as $\text{bp}_{\text{pg}}(A) = A - \text{pg** int}(A)$.

Definition 3.49: Let $A$ be any subset of a topological space $(X, \tau)$. Then the \textit{pg**-exterior} of $A$ is defined as $\text{pg** Ext}(A) = \text{pg** int}(A^c)$.

Theorem 3.50: Let $A$ and $B$ be any two sets of a topological space $(X, \tau)$, then the following conditions hold:

(i) $\text{pg** Bd}(A) = \text{pg** cl}(A) \cap \text{pg** cl}(A^c) = \text{pg** cl}(A^c)^c \cap \text{pg** cl}(A^c) = \text{pg** Bd}(A^c)$.

(ii) and (iii) Follows from Definition of $\text{pg** Bd}(A)$.

(iv) $\text{pg** Bd}(A) \subseteq \text{pg** cl}(A) \subseteq \text{A}$. Hence $\text{pg** Bd}(A) \subseteq \text{A}$.

(v) Suppose $A$ is pg**-open then $A$ is pg**-closed, also $\text{pg** Bd}(A^c) \subseteq A^c$. Hence by (i) $\text{pg** Bd}(A) \subseteq A$.

(vi) Since $A \subseteq B$, $\text{pg** cl}(A) \subseteq \text{pg** cl}(B)$. Now $\text{pg** Bd}(A) \subseteq \text{pg** cl}(A) \cup \text{pg** cl}(B) = B$. Hence $\text{pg** Bd}(A) \subseteq B$.

(vii) $(\text{pg** Bd}(A))^c = (\text{pg** cl}(A) \cap \text{pg** cl}(A^c))^c = (\text{pg** cl}(A))^c \cup (\text{pg** cl}(A^c))^c = \text{pg** int}(A)^c \cup \text{pg** int}(A^c)^c = \text{pg** int}(A^c) \cup \text{pg** int}(A)$.

Proof: (i) $\text{pg** Bd}(A) = \text{pg** cl}(A) \cap \text{pg** cl}(A^c) = \text{pg** cl}(A^c)^c \cap \text{pg** cl}(A^c) = \text{pg** Bd}(A^c)$.

and (ii) (iii) Follows from Definition of $\text{pg** Bd}(A)$.

(iv) $\text{pg** Bd}(A) \subseteq \text{pg** cl}(A) \subseteq \text{A}$. Hence $\text{pg** Bd}(A) \subseteq \text{A}$.

(v) Suppose $A$ is pg**-open then $A$ is pg**-closed, also $\text{pg** Bd}(A^c) \subseteq A^c$. Hence by (i) $\text{pg** Bd}(A) \subseteq A$.

(vi) Since $A \subseteq B$, $\text{pg** cl}(A) \subseteq \text{pg** cl}(B)$. Now $\text{pg** Bd}(A) \subseteq \text{pg** cl}(A) \cup \text{pg** cl}(B) = B$. Hence $\text{pg** Bd}(A) \subseteq B$.

(vii) $(\text{pg** Bd}(A))^c = (\text{pg** cl}(A) \cap \text{pg** cl}(A^c))^c = (\text{pg** cl}(A))^c \cup (\text{pg** cl}(A^c))^c = \text{pg** int}(A)^c \cup \text{pg** int}(A^c)^c = \text{pg** int}(A^c) \cup \text{pg** int}(A)$.

Theorem 3.51: Let $A$ be a subset of a topological space $(X, \tau)$, then the following conditions hold:

(i) $\text{pg** Bd}(A) \subseteq \text{Bd}(A)$, where $\text{Bd}(A)$ denotes the boundary of $A$.

(ii) $\text{pg** cl}(A) = \text{pg** int}(A) \cup \text{pg** Bd}(A)$

(iii) $\text{pg** int}(A) \cap \text{pg** Bd}(A) = \emptyset$.

(iv) $\text{pg** Bd}(\text{int}(A)) \subseteq \text{pg** Bd}(A)$.

(v) $\text{pg** Bd}(\text{cl}(A)) \subseteq \text{pg** Bd}(A)$.

(vi) $\text{bpg}(A) \subseteq \text{pg** Bd}(A)$.

Proof: (i) $\text{pg** Bd}(A) = \text{pg** cl}(A) \cap \text{pg** cl}(A^c) \subseteq \text{cl}(A) \cap \text{cl}(A^c) = \text{Bd}(A)$.

(ii) $\text{pg** int}(A) \cup \text{pg** Bd}(A) = \text{pg** int}(A) \cup (\text{pg** cl}(A) \cap \text{pg** cl}(A^c)) = \text{pg** cl}(A)$.

(iii) $\text{pg** int}(A) \cap \text{pg** Bd}(A) = \text{pg** int}(A) \cap (\text{pg** cl}(A) \cap \text{pg** cl}(A^c)) = \emptyset$.
Theorem 3.52: Let $A$ be a subset of a topological space $(X, τ)$, then the following conditions hold:

(i) $bp\mathbb{g}** (A) \subseteq b(A)$, where $b(A)$ denotes the border of $A$.

(ii) $A = pg** int(A) \cup bp\mathbb{g}** (A)$.

(iii) $pg** int(A) \cap bp\mathbb{g}** (A) = \varnothing$.

(iv) If $A$ is $pg**$-open, then $bp\mathbb{g}** (A) = \varnothing$.

(v) $bp\mathbb{g}** (A) = A = pg** int(A) \subseteq pg** cl(A) \cap (pg** int(A))^c = pg** Bd(A)$.

Proof: (i) follows from the definition of $pg**$-border of $A$ and $A = pg** int(A) \subseteq A - int(A)$.

(ii) and (iii) follows from the definition of $pg**$-border of $A$.

(iv) If $A$ is $pg**$-open, then $A = pg** int(A)$. Thus $bp\mathbb{g}** (A) = \varnothing$.

Theorem 3.53: Let $A$ be a subset of a topological space $(X, τ)$, then the following conditions hold:

(i) $Ext(A) \subseteq pg** Ext(A)$, where $Ext(A)$ denotes the exterior of $A$.

(ii) $pg** Ext(X) = \varnothing$.

(iii) $pg** Ext(\varnothing) = X$.

(iv) $pg** Ext(A) = (pg** cl(A))^c$.

(v) $pg** Ext(pg** Ext(A)) = pg** int(pg** cl(A))$.

(vi) If $A \subseteq B$ then $pg** Ext(A) \supseteq pg** Ext(B)$.

(vii) $pg** Ext(A \cup B) \subseteq pg** Ext(A) \cup pg** Ext(B)$.

(viii) $pg** Ext(A \cap B) \supseteq pg** Ext(A) \cap pg** Ext(B)$.

(ix) $pg** int(A) \subseteq pg** Ext(pg** Ext(A))$.

Proof: (i) (ii) (iii) and (iv) follows from the definition of $pg** Ext(A)$.

(v) $pg** Ext(pg** Ext(A)) = pg** Ext(pg** cl(A))^c = pg** int(pg** cl(A))$.

(vi) If $A \subseteq B$ then $A^c \supseteq B^c \Rightarrow pg** int(A^c) \supseteq pg** int(B^c) \Rightarrow pg** Ext(A) \supseteq pg** Ext(B)$.

(vii) and (viii) follows from (vi).

(ix) $pg** int(A) \subseteq pg** int(pg** cl(A)) = pg** int(pg** Ext(A))^c = pg** Ext(pg** Ext(A))$.

REFERENCES


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