ON THE SUBSET GRAPH OF A NEAR-RING

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ABSTRACT

Let \( N \) be a near-ring. Let \( F \) be the collection of all non-empty subsets of \( N \). We define a new graph, the subset graph \( F_R \) as the graph with all the members of \( F \) as vertices and any two distinct vertices \( A, B \) are adjacent if and only if \( A + B = \{ a + b : a \in A, b \in B \} \) is a right \( N \)-subset of \( N \). In this paper we discuss about the connectivity, diameter and girth of the graph \( F_R \). We also discuss about some induced subgraphs of \( F_R \) and some graphical parameters of these subgraphs viz. diameter, girth etc.

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1. INTRODUCTION

One of the generalized structures of rings are the near-rings. A near-ring is actually what is required to describe the formation of the endomorphism of various mathematical structures adequately. Let \( M(G) \) be the set of all maps of an additive (not necessarily abelian) group \( G \) into itself. The concept of near-ring arises when we define addition and multiplication on the set \( M(G) \) as \((f + g)(a) = f(a) + g(a)\) and \((fg)(a) = f(g(a))\), for all \( a \in G\); \( f, g \in M(G) \). An algebraic system \((N, +, \cdot)\) is called a near-ring if \((N, +)\) is a group (not necessarily abelian), \((N, \cdot)\) is a semigroup and \((a + b) \cdot c = a \cdot c + b \cdot c\) for all \( a, b, c \in N \). This near-ring is termed as right near-ring. The additive identity of the group \((N, +)\) of a near-ring \( N \) is called the zero element and it is denoted by \( 0 \). In a near-ring \( N \), \( 0a = 0, \forall a \in N \). A near-ring \( N \) is called zero symmetric near-ring if \( a0 = 0, \forall a \in N \).

A normal subgroup \( I \) of \((N, +)\) is called a right ideal of \( N \) if \( IN \subseteq I.A \) non empty subset \( A \) of \( N \) is known as (i) a right \( N \)-subset of \( N \) if \( AN \subseteq A \), (ii) a left \( N \)-subset of \( N \) if \( NA \subseteq A \) and (iii) an invariant subset of \( N \) if \( AN \subseteq A, NA \subseteq A \). It is clear that an invariant subset of a near-ring \( N \) is a left as well as right \( N \)-subset of \( N \). Moreover, every right (left) \( N \)-subset contains the zero element of \( N \). Throughout this paper by a near-ring \( N \) we mean a zero symmetric right abelian near-ring unless otherwise stated.

Let us consider a near-ring \( N \) where \((N, +)\) is an abelian group. Also let \( F \) be the set of all non-empty subsets of \( N \). We define the subset graph \( F_R \) as the graph with all the members of \( F \) as vertices and any two distinct vertices \( A, B \) are adjacent if and only if \( A + B = \{ a + b : a \in A, b \in B \} \) is a right \( N \)-subset of \( N \).

Let \( G \) be a graph. The graph \( G \) is said to be connected if there is a path between any two distinct vertices of \( G \). On the other side, the graph \( G \) is called totally disconnected if no two vertices of \( G \) are adjacent. For vertices \( x \) and \( y \) of \( G \), the distance between \( x \) and \( y \) denoted by \( d(x, y) \) is defined as the length of the shortest path from \( x \) to \( y \); \( d(x, y) = \infty \), if there is no such path. The diameter of \( G \) is \( \text{diam}(G) = \sup \{ d(x, y) : x, y \text{ are vertices of } G \} \). The girth of \( G \), denoted by \( g(G) \), is the length of a shortest cycle in \( G \); \( g(G) = \infty \) if \( G \) contains no cycle.

For usual graph-theoretic terms and definitions, one can look at [1]. General references for the algebraic part of this paper are [2],[3],[4],[5],[6].
Example 1.1: Let us consider the near-ring \( N = \{0, a, b\} \) under the operations defined by the following tables.

\[
\begin{array}{ccc}
+ & 0 & a & b \\
0 & 0 & a & b \\
a & a & 0 & a \\
b & b & a & 0 \\
\end{array}
\quad \begin{array}{ccc}
. & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & 0 & a & a \\
b & b & 0 & b \\
\end{array}
\]

Here we can see that \( N, \{0, a\}, \{0, b\} \) are right \( N \)-subsets of \( N \). The graph \( F_R \) is given below:

![Graph FR](image)

Figure-1.1: The subset graph \( F_R \)

2. MAIN RESULTS

Let \( N \) be a right-near ring and we denote cardinality of \( N \) by \( \alpha \), i.e. \( |N| = \alpha \). \( \alpha \) may be of infinite cardinality. We start our section with the following lemma.

Lemma 2.1 [Lemma 1.1.17[3]]: The sum \( A + B = \{a + b | a \in A \text{ and } b \in B\} \) of two right \( N \)-subsets of \( N \) is also a right \( N \)-subset of \( N \).

Theorem 2.2: For any right near-ring \( N \) with \( |N| \leq 2 \), the graph \( F_R \) is always connected.

**Proof:** If \( |N| = 1 \), then the proof is clear. Let \( |N| = 2 \) and \( N = \{0, a\} \). The vertices \( \{0\} \) and \( N \) are always connected. The vertices \( \{a\} \) and \( \{0\} \) are never adjacent, since \( \{a\} + \{0\} = \{a\} \) which cannot be a right \( N \)-subset of \( N \). Now \( \{a\} + N = N \) and clearly this is a right \( N \)-subset of \( N \). Hence the graph \( F_R \) is connected.

Remark 2.3: In theorem 2.2, we have considered the near-ring as a zero-symmetric near-ring. It is interesting to note that if \( N \) is not zero-symmetric then with the same adjacency relation for \( |N| = 2 \), the graph \( F_R \) is a complete graph \( K_3 \) with diameter 1 and girth 3, e.g. consider the near-ring \( Z_2 = \{0, 1\} \) under addition modulo 2 and \( ' \cdot ' \) on \( Z_2 \) is defined as \( a \cdot b = a, \forall a, b \in Z_2 \).

For \( |N| > 2 \), the graph \( F_R \) is never totally disconnected. However for any near-ring \( N \), it is a tough job to check whether there is an isolated vertex or not in the graph \( F_R \). In the following theorem we develop some conditions under which the graph \( F_R \) never contains an isolated vertex.

Theorem 2.4: For \( |N| > 2 \), the graph \( F_R \) is always connected if one of the following holds:

(i) for any proper subset \( A \), \( A + N = N \).
(ii) for any two subsets \( A, B \), \( A + B = N \).
(iii) every subset of \( N \) is a right \( N \)-subset of \( N \).

**Proof:**

(i) Let \( A \subseteq N \). Now \( A + N \subseteq N \) and \( (A + N)N \subseteq NN \subseteq N \). It is clear that \( (A + N) \) is a right \( N \)-subset of \( N \) only if \( N \subseteq (A + N) \), i.e. \( A + N = N \). Thus \( N \) is adjacent to every other vertex and hence the graph \( F_R \) is connected.
(ii) Let \( A, B \) be any two proper subsets of \( N \) such that \( A + B = N \). Now \( (A + B)N = N \subseteq N = A + B \) and thus \( A \) and \( B \) are adjacent. If either \( A = N \) or \( B = N \), then from above (i) clearly \( A \) and \( B \) are adjacent and thus \( FR \) is connected.

(iii) The proof is clear from lemma 2.1.

Remark 2.5: In the theorem 2.4, if \( N \) satisfies either condition (ii) or (iii), then the resulting graph \( FR \) will be a complete graph \( K_{2\alpha-1} \). However if \( N \) satisfies condition (i) in theorem 2.5, then the graph \( FR \) contains a spanning subgraph isomorphic to the star graph \( K_{1,2\alpha-2} \).

Next, let us find the girth of the graph \( FR \).

Theorem 2.6: For \( |N| \leq 2 \), \( gr(FR) = \infty \).

Proof: The proof is clear from the proof of theorem 2.2.

Theorem 2.7: For any right near-ring \( N \) with \( |N| \geq 3 \), we have \( gr(FR) = 3 \) if one of the following conditions holds.

(i) If for some \( n1(= 0) \in N \), there exists a nonzero element \( n2 (= n_1) \in N \) such that \( n1 + n2 = 0 \).

(ii) If for any two subsets \( A, B, A + B = N \).

(iii) If every subset of \( N \) is a right \( N \)-subset of \( N \).

(iv) If there exists an element \( n1 \in N \) such that \( n1N = 0, \forall n \in N \).

Proof: The proofs of (i), (ii) and (iii) are clear from theorem 2.4.

(iv) Let \( n1 \in N \) such that \( n1n = 0, \forall n \in N \). There may be two cases. In the first case let \( n1 + n1 = 0 \). Then clearly there exists a 3-cycle in \( \{N\} - \{n1\} - \{0, n1\} - \{N\} \) in the graph \( FR \). Next, let \( -n1 (= n_1) \) be the additive inverse of \( n1 \) in the near-ring \( N \). In this case also we can construct a 3-cycle as \( \{N\} - \{n1\} - \{-n1\} - \{N\} \) and hence \( gr(FR) = 3 \).

3. ON SOME SUBGRAPHS OF THE GRAPH \( FR \)

In this section, we discuss about some induced subgraphs of the graph \( FR \). We also try to find some graphical parameters like diameter, girth, chromatic number etc. of these subgraphs.

I. Let us consider the subclass of the family of subsets of \( N \) which consists of all the right \( N \)-subsets of \( N \). Let us denote this induced subgraph of \( FR \) by \( RR \). We have the following results:

Theorem 3.1: The subgraph \( RR \) is a complete subgraph of \( FR \).

Proof: We know that for any two right \( N \)-subsets of \( N \) say \( A, B \), their sum \( A + B = \{a + b : a \in A \text{ and } b \in B\} \) is also a right \( N \)-subset of \( N \). Hence the statement is clear.

Corollary 3.2: For any near-ring \( N \) we have \( diam(RR) \leq 1 \).

Corollary 3.3: \( gr(RR) = 3 \) or \( \infty \).

II. For any near-ring \( N \), an element \( x \in N \) is said to be nilpotent if \( x^t = 0 \), for some \( t \in \mathbb{Z}^+ \). A subset \( S \) of \( N \) is called a nilpotent subset of \( N \) if there exists a \( k \in \mathbb{Z}^+ \) such that \( S^k = 0 \) which means \( s_1s_2s_3...s_k = 0 \) for each \( s_i \in S, i = 1, 2, 3, ..., k \). A near-ring \( N \) is called a strongly semi-prime near-ring if \( N \) has no non-zero nilpotent invariant subset. Let us consider an induced subgraph of \( FR \) whose vertex set consists of all the nilpotent subsets of \( N \). Let us denote this subgraph by \( NilR \).

Lemma 3.4[Lemma 2.1.32,[3]]: A strongly semi-prime near-ring \( N \) has no non-zero nilpotent left(right) \( N \)-subsets of \( N \).

Theorem 3.5: If \( N \) is strongly semi-prime then the subgraph \( NilR \) is the disjoint union of \( K_2 \) and \( K_1 \) ‘s.

Proof: Let \( N \) be a strongly semi-prime near-ring. By lemma 3.4, it is clear that \( N \) has no non-zero nilpotent right \( N \)-subsets of \( N \). Thus the subgraph contains a line \( \{0\} - N \) and other vertices (if any) are isolated. Hence the result.
Remark 3.6: A near-ring $N$ is called regular if $\forall n \in N$, there exists $x \in N$ such that $nxn = n$. A near-ring $N$ is called weakly regular if for any ideal $I$ of $N$, each left $I$- subgroup $A$ of $I$, $A^2 = A$. It can be seen that if $N$ is a weakly regular near-ring, then also theorem 3.5 holds [Lemma 5.2.5,[4]]. As all regular near-rings are weakly regular [Lemma 5.2.7,[4]] so we can finally state that the subgraph $N \cap R$ is disjoint union of $K_2$ and $K_1$’s for a regular near-ring $N$.

REFERENCES

1. Frank Harary, Graph Theory, 1969 by Addison-Wesley Publishing Company, Inc.
3. Khanindra Chowdhury, Near-rings and near-ring groups with finiteness conditions, VDM Verlag Dr.Muller Aktiengesellschaft and Co. KG, Germany, 2009.
4. Khanindra Chowdhury, AMasum Near-rings and goldie characters, VDM Verlag Dr. Muller Aktiengesellschaft and Co. KG, Germany, 2011.

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