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NEW RESULTS ON A SUBCLASS OF ANALYTIC FUNCTIONS

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ABSTRACT

In this work, we define a new subclass of analytic functions C_δ^n *which generalizes some known subclasses of analytic functions studied by many authors. For the class* C_δ^n, *we obtained the coefficient bound and an upper bound for the functional* $\left|a_3 - a_2^2\right|$.

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1. INTRODUCTION

Let A be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disk $U = \{z \in C : |z| < 1\}$. Let S be the class of A consisting of univalent function. Let S^* denote the class of starlike functions, the class of all functions $f(z) \in A$ such that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0, \quad z \in U \tag{2}$$

Given $\delta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $g \in S^*$, let $C_\delta(g)$ denote the class of functions $f \in A$ such that

$$\operatorname{Re}\left\{e^{i\delta}\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in U \tag{3}$$

The above class is called the class of close –to- convex function with argument δ with respect to g. By using specific starlike functions g, inequality (3) defines the related classes of $C_\delta(g)$.

Given $\alpha \in [0,1]$

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Let

$$g_{\delta}(z) = \frac{z}{(1-\alpha z)^2}$$

then (3) is of the form

$$\mathrm{Re}[e^{i\delta}(1-\alpha z)^2 f'(z)] > 0, \quad z \in U$$

and defines the class $C_{\delta}(g_{\alpha})$. These classes of functions were studied in [3].

The qth Hankel determinant of a function $f(z)$ given by (1) is defined for $q \geq 1$ and $n \geq 0$ by [8] as follows

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

For starlike functions, the sharp inequality $H_2(2) \leq 1$ was found in [4]. Babalola [1] found sharp bound for the functional $|a_2 a_3 - a_4|$ in the subclasses $R,\ S^*$, and C of the class S where S^* is the class of starlike univalent functions while R consists of functions such that $\mathrm{Re}[f'(z)] > 0.$ Fekete and Szego further generalized the estimate $|a_3 - \mu a_2^2|$ with real μ and $f \in A.$

Definition 1: A function $f(z) \in A$ is said to be in the class C_{δ}^{n} , if it satisfies the condition

$$\mathrm{Re}\, e^{i\delta}\left[(1-\alpha z)^2 \frac{D^{n+1} f(z)}{z}\right] > 0 \tag{4}$$

where $\delta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), \quad 0 \leq \alpha \leq 1 \quad n \in N_0 = N \bigcup \{0\}$ and $z \in U.$

Remark 1: It is observed that

(i) for $n = 0$, we obtain the class of functions studied by [4]

(ii) when $\delta = 0,\ \alpha = 1\ and\ n = 0$ our class C_{δ}^{n} gives the class CR^{+} investigated by [7]

The aim of this work is to investigate the coefficient bounds and the bounds for the Second Hankel determinant for the class C_{δ}^{n}.

2. PRELIMINARY LEMMAS

Let P be the class of analytic functions $p(z)\ in\ U$ such that $p(0) = 1$, and $\mathrm{Re}[p(z)] > 0.$

The class P is called the class of Caratheodory function. The following result will be required for proving our results.

Lemma 2 [5]: Let the function $p \in P$ given by the series

$$p(z) = 1 + c_1 z + c_2 z^2 + \ldots \qquad (z \in U) \tag{5}$$

then

$$|c_k| \leq 2 \quad (k \in N) \tag{6}$$

Lemma 2.2 [2]: Let $p \in P$, then

$$\left|c_2 - \sigma \frac{c_1^2}{2}\right| = \begin{cases} 2(1-\sigma) & if\ \sigma \leq 0 \\ 2 & if\ 0 \leq \sigma \leq 2 \\ 2(\sigma - 1) & if\ \sigma \geq 2 \end{cases}$$

3. MAIN RESULTS

Theorem 3.1: Let $f(z)=z+\sum_{k=2}^{\infty}a_k z^k \in C_{\delta}^{n}$ then

(i) $|a_2| \le \frac{1+\alpha}{2^n}$

(ii) $|a_3| \le \frac{2+\alpha(4+3\alpha)}{3^{n+1}}$

(iii) $|a_4| \le \frac{1+\alpha(2+3\alpha+2\alpha^2)}{2^{2n+1}}$

Proof: Let $\operatorname{Re} e^{i\delta}\left[(1-\alpha z)^2 \frac{D^{n+1}f(z)}{z}\right] > 0, \quad \delta \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right),\ 0\le\alpha\le 1 \quad n\in N_0 = N\bigcup 0$

then

$$\exists\ q(z)=\cos\delta + i\sin\delta + \sum_{n=1}^{\infty} q_n z^n \text{ such that } \operatorname{Re} q(z) > 0$$

this implies that

$$e^{i\delta}(1-\alpha z)^2 \frac{D^{n+1}f(z)}{z} = q(z)$$

Let

$$p(z)=\frac{q(z)-i\sin\delta}{\cos\delta}$$

then $p(0)=1$

$$\operatorname{Re}\ p(z)=\frac{q(z)}{\cos\delta} > 0 \text{ and } p(z)=1+c_1 z + c_2 z^2 + \ldots$$

this imply that

$$e^{i\delta}(1-\alpha z)^2 \frac{D^{n+1}f(z)}{z} = p(z)\cos\delta + i\sin\delta$$

$$\cos\delta \quad p(z)=q(z)-i\sin\delta$$

$$\cos\delta + c_1\cos\delta\, z + c_2\cos\delta\, z^2 + \ldots = \cos\delta + q_1(z) + q^2 z^2 + \ldots$$

$$q_n = c_n \cos\delta \tag{7}$$

But

$$e^{i\delta}\left[(1-\alpha z)^2 \frac{D^{n+1}f(z)}{z}\right] = e^{i\delta}(1-2\alpha z+\alpha^2 z^2)(1+\sum_{k=2}^{\infty}k^{n+1}a_k z^{k-1})$$

$$= e^{i\delta}\left[1+\sum_{k=2}^{\infty}k^{n+1}a_k z^{k-1} - 2\alpha z - 2\alpha\sum_{k=2}k^{n+1}a_k z^k + \alpha^2 z^2 + \alpha^2\sum_{k=2}^{\infty}k^{n+1}a_k z^{k+1}\right]$$

$$= e^{i\delta}\left[1+(2^{n+1}a_2-2\alpha)z + (3^{n+1}a_3 - 2\alpha 2^{n+1}a_2 + \alpha^2)z^2 + (4^{n+1}a_4 - 2\alpha 3^{n+1}a_3 + \alpha^2 2^{n+1}a_2)z^3 + \ldots\right] \tag{8}$$

Comparing the coefficients of (7) and (8) we obtain

$$2^{n+1}a_2 e^{i\delta} - 2\alpha e^{i\delta} = q_1$$

$$a_2 = \frac{1}{2^{n+1}}(c_1 e^{-i\delta}\cos\delta + 2\alpha) \tag{9}$$

Also,

$$e^{i\delta}\left[3^{n+1}a_3 - 2\alpha 2^{n+1}\left(\frac{c_1 e^{-i\delta}\cos\delta}{2^{n+1}} + \frac{2\alpha}{2^{n+1}}\right) + \alpha^2\right] = q_2$$

$$a_3 = \frac{1}{3^{n+1}}[c_2 e^{-i\delta}\cos\delta + 2\alpha c_1 e^{-i\delta}\cos\delta + 3\alpha^2]$$

Moreover,

$$4^{n+1}a_4 - 2\alpha c_2 e^{-i\delta}\cos\delta - 4\alpha^2 c_1 e^{-i\delta}\cos\delta - 6\alpha^3 + \alpha^2 c_1 e^{-i\delta}\cos\delta + 2\alpha^3 = c_3 e^{-i\delta}\cos\delta$$

$$a_4 = \frac{1}{4^{n+1}}[c_3 e^{-i\delta}\cos\delta + 2\alpha c_2 e^{-i\delta}\cos\delta + 3\alpha^2 c_1 e^{-i\delta}\cos\delta + 4\alpha^3] \tag{11}$$

From equations (9), (10) and (11) we solve for their bounds using lemma (2.1)

$$|a_2| \le \left|\frac{e^{i\delta}c_1\cos\delta}{2^{n+1}}\right| + \left|\frac{2\alpha}{2^{n+1}}\right| \le \frac{|e^{i\delta}||c_1||\cos\delta|}{2^{n+1}} + \frac{2\alpha}{2^{n+1}}$$

$$|a_2| \le \frac{1+\alpha}{2^n}$$

$$|a_3| \le \frac{|c_2||e^{-i\delta}||\cos\delta| + 2\alpha|c_1||e^{-i\delta}||\cos\delta| + 3\alpha^2}{3^{n+1}}$$

Therefore,

$$|a_3| \le \frac{2+\alpha(4+3\alpha)}{3^{n+1}}$$

$$|a_4| \le \frac{|c_3||e^{-i\delta}||\cos\delta| + 2\alpha|c_2||e^{-i\delta}||\cos\delta| + 3\alpha^2|c_1||e^{-i\delta}||\cos\delta| + 4\alpha^3}{4^{n+1}}$$

$$|a_4| \le \frac{1+\alpha(2+3\alpha)+2\alpha^2}{2^{2n+1}}$$

Remark 2: When $\alpha = 0$ *and* $n = 0$, our results give the result stated by [1] for $|a_k| \le \frac{2}{k}$ for the class R where $k = 2,3,4.$

Theorem 3.2: Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in C_\delta^n$, then

$$|a_3 - a_2^2| \le \frac{3\alpha^2 + 4\alpha + 2}{3^{n+1}}$$

Proof: From theorem 3.1 we have that

$$a_2 = \frac{1}{2^{n+1}}(c_1 e^{-i\delta}\cos\delta + 2\alpha)$$

$$a_3 = \frac{1}{3^{n+1}}(c_2 e^{-i\delta}\cos\delta + 2\alpha c_1 e^{-i\delta}\cos\delta + 3\alpha^2)$$

then applying lemmas (2.1) and (2.2) we obtain

$$|a_3 - a_2^2| = \left|\frac{1}{3^{n+1}}c_2 e^{-i\delta}\cos\delta + \frac{2\alpha}{3^{n+1}}c_1 e^{-i\delta}\cos\delta + \frac{\alpha^2}{3^n} - \frac{1}{2^{2n+2}}c_1^2 e^{-2i\delta}\cos^2\delta - \frac{\alpha}{2^n}c_1 e^{-i\delta}\cos\delta - \frac{\alpha^2}{2^{2n}}\right|$$

$$\le \left|\alpha^2\left(\frac{1}{3^n} - \frac{1}{2^{2n}}\right) + \left(\frac{2\alpha}{3^{n+1}} - \frac{\alpha}{2^{2n}}\right)c_1 e^{-i\delta}\cos\delta\right| + \frac{1}{3^{n+1}}e^{-i\delta}\cos\delta\left|c_2 - \frac{3^{n+1}e^{-i\delta}\cos\delta}{2^{2n+1}}\frac{c_1^2}{2}\right|$$

$$= \left|\alpha^2\left(\frac{1}{3^n} - \frac{1}{2^{2n}}\right) + \left(\frac{2\alpha}{3^{n+1}} - \frac{\alpha}{2^{2n}}\right)2\right| + \frac{2}{3^{n+1}}e^{-i\delta}\cos\delta$$

$$\le \left|\alpha^2\left(\frac{1}{3^n} - \frac{1}{2^{2n}}\right) + \left(\frac{4\alpha}{3^{n+1}} - \frac{\alpha}{2^{2n}}\right)\right| + \frac{2}{3^{n+1}}$$

$$= \frac{\alpha^2(3.2^{2n} - 3^{n+1}) + 2\alpha(2.2^{2n} - 3^{n+1}) + 2.2^{2n}}{3^{n+1}.2^{2n}}$$

$$\le \frac{\alpha^2.3.2^{2n} + 2\alpha.2.2^{2n} + 2.2^{2n}}{3^{n+1}.2^{2n}}$$

$$= \frac{3\alpha^2 + 4\alpha + 2}{3^{n+1}}$$

Hence,

$$\left|a_3 - a_2^2\right| \leq \frac{3\alpha^2 + 4\alpha + 2}{3^{n+1}}$$

Remark 3: It is known that if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is analytic and univalent in U, then $\left|a_3 - a_2^2\right| \leq 1$. Our result improves this result for $n = 0$, $\alpha = 0$ *and* $\delta = 0$.

4. CONCLUSION

We have investigated a new subclass of analytic functions and obtained the coefficient bounds and the upper bound of Hankel determinant $H_2(1)$.

REFERENCES

1. Babalola, K.O. *On $H_3(1)$ Hankel determinant for some classes of univalent functions*. Journal of Inequality Theory and Applications. 6, (2007) 1-7.
2. Babalola, K.O. *Some new results on a certain family of analytic functions defined by the Salagean derivatives*. Ph.D Thesis, University of Ilorin, Ilorin (2005). Unpublished dissertation.
3. Babalola, K.O and Opoola, T.O. *On the coefficients of certain analytic and univalent functions*. Advances in Ineq. For Series. Nova Science Publishers. (2006) 5-17.
4. Bogumila, K and Lecko, A. (2014), The *Fekete- Szego inequality for close-to- convex functions with respect to certain starlike function dependent on a real parameter*. Journal of Inequalities and Applications. 1-16.
5. Duren, P.L. *Univalent functions*.Graduate text in mathematics. Springer-Verlag.New York Inc. (1983).
6. Fekete, M. and Szego, G. *Eine Bermerkung ube ungerade schlichte functionen*. Jour. London Math. Soc. 8, (1933) 85-89.
7. Marjono and Thomas, D.K. *The Second hankel determinant of functions convex in one direction*. Int. Jour. Of Math. Analysis, 10 (9), (2016) 423-428.
8. Nooman, J.W and Thomas, D.K. Second *hankel determinant of a really mean p-valent functions*. Trans. Amer. Math. Soc. 223, (1976) 337-346.

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