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# $pg^{**}$Separation axioms

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***ABSTRACT***

*In this paper the separation axioms via pg**-open sets are analysed in topological and ideal topological spaces.*

***Key words:*** *$pg^{**}T_0$ space, $pg^{**}T_0$ modulo I space, $pg^{**}T_1$ space, $pg^{**}T_1$ modulo I space, $pg^{**}T_2$ space, $pg^{**}T_2$ modulo I space, $pg^{**}$regular space, $pg^{**}$normal space.*

## 1. INTRODUCTION

Levine [3] introduced the class of g-closed sets in 1970. Veerakumar[7] introduced g*****-closed sets. A.S.Mashhour, M.E Abd El. Monsef [4] introduced a new class of pre-open sets in 1982. Ideal topological spaces have been first introduced by K.Kuratowski [2] in 1930. In this paper we generalize the conventional separation axioms through pg******-open sets.

## 2. PRELIMINARIES

**Definition 2.1:** A subset $A$ of a topological space$(X,\tau)$ is called a pre-open set [4] if $A \subseteq int(cl(A)$ and a pre-closed set if $cl(int(A)) \subseteq A$.

**Definition 2.2:** A subset $A$ of topological space $(X,\tau)$ is called

1. generalized closed set (g-closed) [3] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X,\tau)$.
2. g*****-closed set [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g-open in $(X,\tau)$.
3. pg******- closed set[6] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g*****-open in$(X,\tau)$.

**Definition 2.3:** A function $f: (X,\tau) \rightarrow (Y,\sigma)$ is called

1. pg******-irresolute[6] if $f^{-1}(V)$ is a pg******-closed set of $(X,\tau)$ for every pg******-closed set $V$ of $(Y,\sigma)$.
2. pg******-continuous[6] if $f^{-1}(V)$ is a pg******-closed set of $(X,\tau)$for every closed set $V$of $(Y,\sigma)$.
3. pg******-resolute[6] if $f(U)$ is pg******- open in $Y$ whenever $U$ is pg******- open in $X$.

**Definition 2.4:** An ideal [2] $I$ on a nonempty set $X$ is a collection of subsets of $X$ which satisfies the following properties. $(i) A \in I$ , B $\in I \Longrightarrow A \cup B \in I (ii) A \in I, B \subset A \Longrightarrow B \in I$ . A topological space $(X,\tau)$ with an ideal $I$on $X$ is called an ideal topological space and is denoted by $(X,\tau,I)$.

## 3. $pg^{**}T_0$ Space

**Definition 3.1:** The points $x, y \in X$ is said to be *pg******- indistinguishable* if $x \in pg^{**}cl(y)$ and $y \in pg^{**}cl(x)$

**Note:** pg******-indistinguishability is an equivalence relation.

**Definition 3.2:** A topological space $(X,\tau)$ is said to be $pg^{**}T_0$ *space* if no two distinct points are pg******-indistinguishable. Equivalently a topological space $X$ is called $pg^{**}T_0$ space if given any two distinct points $x$ and $y$ there is either a pg******- open set $U$ such that $x \in U, y \notin U$ or $y \in U, x \notin U$.

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**Example 3.3:** Let $(X,\tau)$be an indiscrete topological space has more than one point. Then $X$ is $pg^{**}T_0$ space, since every subset of $X$ is pg******-open.

**Theorem 3.4:** Every $T_0$ space is $pg^{**}T_0$ space but not conversely

**Proof:** Obvious since every open set is pg******- open.

**Example 3.5:** The space in example (3.3) is $pg^{**}T_0$ but not $T_0$. Consider ℝ with trivial topology, take two arbitrary points $x, y \in \mathbb{R}$ such that $x \neq y$. Here $U = \{x\}$ and $V = \{y\}$ are pg******- open sets, therefore ℝ with trivial topology is $pg^{**}T_0$ space. But this space is not $T_0$, since the only open sets are $\varphi$ and ℝ.

**Theorem 3.6:** Let $(X,\tau)$ be a pg******- multiplicative space, then $X$ is $pg^{**}T_0$ space if and only if pg******-closures of distinct points are distinct. (i.e) if $x \neq y \in X, pg^{**}cl(\{x\}) \neq pg^{**}cl(\{y\})$.

**Proof:** Let $(X,\tau)$ be a $pg^{**}T_0$ space and $x$ and $y$ be two distinct points of $X$. Then there is a pg******-open set $U$ such that $x \in U, y \notin U$ and $y \in U^c, x \notin U^c$. $pg^{**}cl(\{y\}) \subseteq U^c$ since $U^c$ is pg******-closed in $X$. Thus $pg^{**}cl(\{x\}) \neq pg^{**}cl(\{y\})$.

Conversely suppose for any pair of distinct points $x$ and $y$ in $pg^{**}cl(\{x\}) \neq pg^{**}cl(\{y\})$. Then we can choose $z \in X$ such that $z \in pg^{**}cl(\{x\})$ but $z \notin pg^{**}cl(\{y\})$. If $x \in pg^{**}cl(\{y\})$, then $pg^{**}cl(\{x\}) \subseteq pg^{**}cl(\{y\})$, this implies $z \in pg^{**}cl(\{y\})$ which is a contradiction. Hence$x \notin pg^{**}cl(\{y\})$ this implies$x \in (pg^{**}cl(\{y\}))^c$which is pg******-open in $X$ containing $x$ but not$y$. Hence $X$ is $pg^{**}T_0$ space.

**Theorem 3.7:** Let $(X,\tau)$ and $(Y,\sigma)$ be two topological spaces and $f : (X,\tau) \to (Y,\sigma)$ be a bijection. Then,

1. $f$ is pg******- continuous and $Y$ is a $T_0$ space$\Longrightarrow X$ is a $pg^{**}T_0$ space.
2. $f$ is continuous and $Y$ is a $T_0$ space $\Longrightarrow$ $X$ is a $pg^{**}T_0$ space.
3. $f$ is pg******-irresolute and $Y$ is $pg^{**}T_0$ space$\Longrightarrow X$ is $pg^{**}T_0$ space.
4. $f$ is pg******-resolute and $X$ is $pg^{**}T_0$ space$\Longrightarrow Y$ is $pg^{**}T_0$ space.
5. $f$ is pg******- open and$X$ is a $T_0$ space $\Longrightarrow$ $Y$ is $pg^{**}T_0$ space.
6. $f$ is strongly pg******- continuous and $Y$ is $pg^{**}T_0$ space$\Longrightarrow X$ is a $T_0$ space.

**Proof:** (1) Let $x$ and $y$ be two distinct points of $X$, then $f(x)$ and $f(y)$ are distinct points of $Y$. Then there is a pg******- open set $U$ in $Y$ such that $f(x) \in U, f(y) \notin U$or $f(y) \in U, f(x) \notin U$. Then $f^{-1}(U)$ is a pg******-open set in $X$ such that $x \in f^{-1}(U)$ , $y \notin f^{-1}(U)$ or $y \in f^{-1}(U), x \notin f^{-1}(U)$. Therefore $X$ is a $pg^{**}T_0$ space.

Proofs for (2) to (6) are similar to the above.

**Remark 3.8:** The property of being $pg^{**}T_0$ space, is a pg******-topological property. This follows from (3) and (4) of the above theorem.

**Theorem 3.9:** Let $f : (X,\tau) \to (Y,\sigma)$ be an injective map and $Y$ is $pg^{**}T_0$ space. If $f$ is pg******- totally continuous then $X$ is ultra-Hausdorff.

**Proof:** Let $x$ and $y$ be any two distinct points in $X$. Since $f$ is injective,$f(x)$ and $f(y)$ are distinct points in $Y$. Since $Y$ is $pg^{**}T_0$ space there exists an pg******- open set $U$ in $Y$containing $f(x)$ but not $f(y)$. Then $\in f^{-1}(U), y \notin f^{-1}(U)$ and $x \in f^{-1}(U), y \in (f^{-1}(U))^c$also $f^{-1}(U)$ is clopen in $X$. This implies every pair of distinct points of $X$ can be separated by disjoint clopen sets. Therefore $X$ is ultra-Hausdorff.

## 4. $pg^{**}T_0$ modulo *I* space

**Definition 4.1:** An ideal topological space $(X,\tau,I)$ is said to be $pg^{**}T_0$ *modulo I*if for every pair of points $x, y \in X$ and $x \neq y$ there exists pg******- open set $U$ such that $x \in U, U \cap \{y\} \in I$ or $y \in U, U \cap \{x\} \in I$.

**Example 4.2:** An ideal topological space $(X,\tau,I)$where $I = \wp(X)$ is a $pg^{**}T_0$ modulo *I* space.

For, if $x, y \in X$ and $x \neq y$, for any pg******- open sets $U_x, U_y$ containing $x, y$ respectively, then $U_x \cap \{y\}, U_y \cap \{x\} \in I$.

**Theorem 4.3:** Every pg******- $T_0$ space is $pg^{**}T_0$ modulo *I* space for every ideal *I*.

**Proof:** Let $x$ and $y$ be any two distinct points in $X$. Since $X$ is $pg^{**}T_0$ spacethere exists disjoint pg******- open sets $U_x, U_y$ containing $x, y$ respectively, then $U_x \cap U_y = \varphi \in I$. Hence $X$ is $pg^{**}T_0$ modulo *I* space.

**Remark 4.4:** If $I = \{\varphi\}$ then both $pg^{**}T_0$ space and $pg^{**}T_0$ modulo *I* space coincide.

 

**Theorem 4.5:** Let $I, J$ be ideals of $X$ and if $I \subseteq J$, then $(X, \tau, I)$ is $pg^{**}T_0$ modulo $I$ implies $(X, \tau, J)$ is $pg^{**}T_0$ modulo $J$.

If $x, y \in X$ and $x \neq y$, then there exists disjoint pg**-open sets $U_x, U_y$ containing $x, y$ respectively such that $U_x \cap U_y = \varphi \in I \subseteq J$. Therefore $(X, \tau, J)$ is a $pg^{**}T_0$modulo $J$ space.

**Theorem 4.6:** Let $(X, \tau, I)$ and $(Y, \sigma, J)$ be two ideal topological spaces and $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ be a bijection where $J = f(I)$ is an ideal in $Y$ then,

1. $f$ is pg**-resolute and $X$ is $pg^{**}T_0$modulo $I$ space$\Longrightarrow$ $Y$ is $pg^{**}T_0$ modulo $J$ space.
2. $f$ is pg**-continuous and $Y$ is a $T_0$modulo $J$ space$\Longrightarrow$ $X$ is a $pg^{**}T_0$ modulo $I$ space.
3. $f$ is continuous and $Y$ is a$T_0$modulo $J$space$\Longrightarrow$ $X$ is a $pg^{**}T_0$ modulo $I$ space.
4. $f$ is pg**-irresolute and $Y$ is $T_0$modulo $J$ space $\Longrightarrow$ $X$ is $pg^{**}T_0$ modulo $I$ space.
5. $f$ is pg**-open and $X$ is a $T_0$ space$\Longrightarrow$ $Y$ is $pg^{**}T_0$ modulo $J$ space.
6. $f$ is open and $X$ is a $T_0$ space$\Longrightarrow$ $Y$ is $pg^{**}T_0$ modulo $J$ space.

**Proof:** (1) Let $y_1 \neq y_2 \in Y$. Since $f$ is a bijection there exists $x_1 \neq x_2 \in X$ such that $f(x_1) = y_1$and $f(x_2) = y_2$. Also there exists pg**-open set $U$ in $X$such that $x_1 \in U, U \cap \{x_2\} \in I$ or $x_2 \in U, U \cap \{x_1\} \in I$ since $X$ is $pg^{**}T_0$modulo $I$ space, which implies $y_1 \in f(U), f(U) \cap \{y_2\} \in J$ or $y_2 \in f(U), f(U) \cap \{y_1\} \in J$ where $f(U)$ is pg**-open in $Y$. Therefore $(Y, \sigma, J)$ is a $pg^{**}T_0$ modulo $J$ space.

Proofs for (2) to (6) are similar to (1).

## 5. $pg^{**}T_1$ Space

**Definition 5.1:** A topological space $(X, \tau)$ is said to be $pg^{**}T_1$ *space* if $x, y \in X$ and $x \neq y$, there exists pg**- open sets $U_x, U_y$ containing $x, y$ respectively, such that $y \notin U_x$ and $x \notin U_y$.

**Example 5.2:** An indiscrete topological space $(X, \tau)$ has more than one point is $pg^{**}T_1$ space, since all the subsets of $X$ is pg**- open.

**Example 5.3:** Consider an infinite set $X$ with cofinite topology, if $x \neq y \in X$, then $U_x = X - \{y\}$ and $U_y = X - \{x\}$ are pg**- open sets such that $y \notin U_x$ and $x \notin U_y$. Therefore $X$ is $pg^{**}T_1$ space.

**Example 5.4:** The one point space is $pg^{**}T_1$, because the definition of $pg^{**}T_1$ space is vacuously satisfied.

**Example 5.5:** Let $X = \{a, b, c\}, \tau = \{\varphi, X, \{a\}, \{c\}, \{a, c\}\}$. Then $PG^{**}O(X) = \{\varphi, X, \{a\}, \{c\}, \{a, c\}\}$. This space is not $pg^{**}T_1$ space.

**Theorem 5.6:** Every $T_1$ space is $pg^{**}T_1$ space.

Proof follows from the fact that every open set is pg**-open.

**Remark 5.7:** The converse of the above theorem is not true from the following example.

**Example 5.8:** An indiscrete topological space $(X, \tau)$ has more than one point is $pg^{**}T_1$ but not $T_1$ space.

**Theorem 5.9:** Every $pg^{**}T_1$ space is $pg^{**}T_0$ space but not conversely.

Proof follows from the definitions.

**Example 5.10:** The space in example (5.5) is $pg^{**}T_0$ but not $pg^{**}T_1$ spaces.

Hence the set of $pg^{**}T_1$topological spaces is a proper subset of all $pg^{**}T_0$topological spaces.

**Theorem 5.11:** A topological space $(X, \tau)$ is a $pg^{**}T_1$ space if and only if every singleton set is pg**-closed.

**Proof:** Let $(X, \tau)$be $pg^{**}T_1$ space and $x \in X$. Let $x \neq y$ be an arbitrary element in $X$. Subsequently there exists pg**-open sets $U_x, U_y$ containing $x, y$ respectively, such that $y \notin U_x$ and $x \notin U_y$.

Now $U_x$ is a pg**- open set containing $x$ not intersecting$\{y\}$. Therefore $x$ is not a pg**- limit point of $\{y\}$. Thus $\{y\}$ is pg**- closed. Conversely let every singleton set is pg**- closed in $X$. If $x$and$y$ are distinct points of $X$, then $U_x = X - \{y\}$ and $U_y = X - \{x\}$ are pg**- open sets such that $y \notin U_x$ and $x \notin U_y$. Therefore $X$ is $pg^{**}T_1$ space.

 

**Theorem 5.12:** If $(X,\tau)$ is a $pg^{**}T_1$ space then every finite subset of $X$ is pg**- closed.

**Proof:** Let$A$ be a finite subset of $X$, then $A = \cup_{x\in A} \{x\}$ is pg**- closed being finite union of pg**- closed sets.

**Theorem 5.13:** In a topological space$(X,\tau)$the following statements are equivalent:

1. $(X,\tau)$ is a $pg^{**}T_1$ space.
2. Every singleton set of $(X,\tau)$is pg**- closed.
3. Every finite subset of $X$ is pg**- closed.
4. Every point $x \in X$ equals the intersection of all pg**-neighbourhoods of $x$.

**Proof:** The proof for $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ follows from theorem (5.11).

$(\mathbf{1}) \Rightarrow (\mathbf{4})$: Let $N_x$ be the intersection of all pg**-neighbourhoods of $x$ in $X$. Let $x \neq y$ be an arbitrary element in $X$. Since $X$ is $pg^{**}T_1$there exists pg**- open set $U_x$ containing $x$, such that $x \in U_x$ and $y \notin U_x$. Therefore $y \notin N_x$ and hence $N_x = \{x\}$.

$(\mathbf{4}) \Rightarrow (\mathbf{1})$: Let$x, y$ be two distinct points in $X$and $N_x$ be the intersection of all pg**-neighborhoods of $x$, then $N_x = \{x\}$. Therefore $y \notin N_x$. Therefore there is atleast one pg**- open set $U_x$ containing $x$ and notcontaining $y$. Correspondingly we can get a pg**- open set $U_y$containing $y$and notcontaining $x$. Thus$(X,\tau)$is a $pg^{**}T_1$ space.

**Theorem 5.14:** A topological space $(X,\tau)$ is a $pg^{**}T_1$ space if and only if $PG^{**}O(X,\tau)$is finer than co finite topology on $X$.

**Proof:** Let $X$ be a $pg^{**}T_1$ space. Let $\tau^*$ denote the co finite topology on $X$. To prove that$\tau^* \subseteq PG^{**}O(X,\tau)$.Let $U \in \tau^*$, then $X - U$ is a finite set. Since $X$ is a $pg^{**}T_1$ space$X - U$ is pg**-closed in $X$. Hence $U$ is pg**-open. Therefore $\tau^* \subseteq PG^{**}O(X,\tau)$ . Conversely presume $\tau^* \subseteq PG^{**}O(X,\tau)$ . Choose $x \in X$ . Then $X - \{x\} \in \tau^* \Rightarrow X - \{x\} \in PG^{**}O(X,\tau)$. This implies$\{x\}$is pg**-closed in $X$. Then by theorem (5.11)$(X,\tau)$ is a $pg^{**}T_1$ space.

**Theorem 5.15:** Every finite $pg^{**}T_1$ space is a pg**-discrete space.

**Proof:** Let $(X,\tau)$ be a finite $pg^{**}T_1$ space, then all the subsets of $X$ is finite and hence pg**-closed. Therefore $X$ is pg**-discrete.

**Theorem 5.16:** In a $pg^{**}T_1$ space $(X,\tau)$ every pg**-connected set containing more than one point is infinite.

**Proof:** Let $A$ be a pg**-connected subset of $X$ has more than one point. Presume that $A$ is finite and let $A = \{x_1, x_2, \dots, x_m\}$, then $A$ is pg**-discrete. Therefore $\{x_1\}$ and $A - \{x_1\}$ are both pg**-clopen. Thus $A$ can be written as the union of two non-empty disjoint pg**-open sets. Which is a contradiction to $A$ is pg**-connected. Therefore $A$ must be infinite.

**Theorem 5.17:** Let $(X,\tau)$ be pg**-additive and $pg^{**}T_1$ space. Then $X$ is a pg**-discrete space.

**Proof:** Let $A$ be a subset of $X$. Then $A = \cup_{x\in A} \{x\}$ and each $\{x\}$ is pg**- closed. Since $X$ is pg**-additive $A$ is pg**-closed. Therefore $X$ is pg**-discrete.

**Theorem 5.18:** Let $(X,\tau)$ be a $pg^{**}T_1$ space and $A$ be a subset of $X$. Then a point $x \in X$ is a pg**-limit point of $A$ if and only if every pg**-open set containing $x$ contains infinitely many points of $A$. Consequently in a $pg^{**}T_1$ space no finite set has a pg**-limit point.

**Proof:** Let$x$ be a pg**-limit point of $A$ and $U$ be a pg**-open set containing $x$. Suppose $U$ intersects $A$ in only finitely many points. Then $U$ also intersects $A - \{x\}$ in finitely many points. Let $E = U \cap A - \{x\} = \{x_1, x_2, \dots, x_m\}$. Then $E$ is pg**-closed, since $X$ is pg**-$T_1$ space. Therefore $E^c \cap U$ is pg**-open set containing $x$. $(E^c \cap U) \cap (A - \{x\}) = E^c \cap E = \varphi$, which is a contradiction to $x$ is a pg**-limit point of $A$. Therefore $U$ intersects $A$ ininfinitely many points of $A$. Conversely if every pg**-open set containing $x$ contains infinitely many points of $A$, it certainly intersects $A$ in some point other than $x$ itself, so that $x$ is a pg**-limit point of $A$.

**Corollary 5.19:** Any finite subset of $pg^{**}T_1$ space has no pg**-limit point.

Proof follows from theorem (5.18).

**Theorem 5.20:** In a $pg^{**}T_1$space $X$, if every infinite subset has a pg******-limit point then $X$ is pg******-countably compact.

**Proof:** Let every infinite subset has apg******-limit point. We need to prove $X$is pg******-countably compact. Suppose not, then there exists a countable pg******-open cover $\{U_n\}$ has no finite subcover.

In view of the fact that $U_1 \neq X$, then there exists $x_1 \notin U_1$ also $X \neq U_1 \cup U_2$, then there exists $x_2 \notin U_1 \cup U_2$. Proceeding like this there exists $x_n \notin U_1 \cup U_2 \cup \dots \cup U_n$for all $n$. Now $A = \{x_n\}$ is an infinite set. If $x \in X$ then $x \in U_n$ for some $n$. But $x_m \notin U_n, \ \forall m \geq n$. Since $X$ is $pg^{**}T_1$ space $U_n - \{x_1, x_2, \dots, x_{n-1}\}$ is a pg******-open set containing $x$ which does not have a point of $A$ other than $x$. Contradicting the fact that every infinite subset of $X$ has a pg******-limit point. Therefore $X$ is pg******-countably compact.

**Remark 5.21:** A sequence in a $pg^{**}T_1$ spaceis pg******-congregates to more than one pg******-limit. In fact a sequence can pg******-congregates to every point of the space. Consider the following example.

Let $(X, \tau)$ be an infinite topological space with co finite topology, $\langle x_n \rangle$ be any sequence in X and $x \in X$. To prove $\langle x_n \rangle \xrightarrow{pg^{**}} x$. Let $U \in \tau$ such that $x \in U$. $U \in \tau$implies $U \in PG^{**}O(X, \tau)$and $X - U$ is a finite. Find the largest $n_0 \in \mathbb{N}$ such that $x_{n_0} \in X - U$. Therefore $x_n \in U \ \ \forall\, n \geq n_0$. This shows that $\langle x_n \rangle \xrightarrow{pg^{**}} x$ in $X$. Since $x \in X$ is arbitrary, we get any sequence in $(X, \tau)$ pg******-congregates to every point of the space.

**Theorem 5.22:** If $X$ is infinite pg******-additive $pg^{**}T_1$ space then it is not pg******-compact.

**Proof:** In a $pg^{**}T_1$ space$\{x\}$ is pg******-closed for all $x \in X$. Therefore every subset of $X$ is pg******-clopen. Therefore $\{\{x\}/x \in X\}$ is a pg******-open cover for $X$ which has no finite subcover.

**Theorem 5.23:** Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a bijection. Then,

1. $f$ is pg******-continuous and $Y$ is a $T_1$ space$\Longrightarrow$ $X$ is a $pg^{**}T_1$ space.
2. $f$ is continuous and $Y$is a$T_1$ space $\Longrightarrow$ $X$ is a $pg^{**}T_1$ space.
3. $f$ is pg******-irresolute and $Y$ is $pg^{**}T_1$ space $\Longrightarrow$ $X$ is $pg^{**}T_1$ space.
4. $f$ is pg******- resolute and $X$ is $pg^{**}T_1$ space$\Longrightarrow$ $Y$ is $pg^{**}T_1$ space.
5. $f$ is pg******-open and $X$ is a $T_1$ space$\Longrightarrow$ $Y$ is $pg^{**}T_1$ space.
6. $f$ is strongly pg******-continuous and $Y$ is $pg^{**}T_1$ space$\Longrightarrow$ $X$ is a $T_1$ space.

**Proof:** (1) Let$x$ and $y$ be two distinct points of $X$, then $f(x)$ and $f(y)$ are distinct points of $Y$. Then there exists pg******-open sets $U_x$ and$U_y$ in $Y$ such that $f(x) \in U_x$, $f(y) \notin U_x$and$f(y) \in U_y$, $f(x) \notin U_y$. Then $f^{-1}(U_x)$ and $f^{-1}(U_y)$ are pg******- open sets in $X$ such that $x \in f^{-1}(U_x)$ , $y \notin f^{-1}(U_x)$ or $y \in f^{-1}(U_y)$ , $x \notin f^{-1}(U_y)$ . Therefore $X$ is a $pg^{**}T_1$space.

Proofs for (2) to (6) are similar to the above.

**Remark 5.24:** The property of being $pg^{**}T_1$ space, is a pg******- topological property. This follows from (3) and (4) of the above theorem.

## 6. $pg^{**}T_1$ modulo $I$ space

**Definition 6.1:** An ideal topological space $(X, \tau, I)$ is said to be $pg^{**}T_1$ *modulo I* if for every pair of points $x, y \in X$ and $x \neq y$ there exists pg******-open set $U_x, U_y$ containing $x, y$ respectively, such that$U_x \cap \{y\} \in I, U_y \cap \{x\} \in I$.

**Example 6.2:** An ideal topological space$(X, \tau, I)$ where $I = \wp(X)$ is a $pg^{**}T_1$ modulo *I* space.

**Example 6.3:** Let $X = \{a, b, c\}, \tau = \{\varphi, X, \{a\}, \{c\}, \{a, c\}\}$ and $I = \varphi$, then $(X, \tau, \varphi)$ is not$pg^{**}T_1$modulo *I*space.

**Theorem 6.4:** Every $pg^{**}T_1$ space is $pg^{**}T_1$modulo *I* space for every ideal *I*.

Proof is obvious since $\varphi \in I$.

**Remark 6.5:** If $I = \{\varphi\}$ then both $pg^{**}T_1$ space and $pg^{**}T_1$modulo *I* space happen together.

**Theorem 6.6:** Every ideal topological space which is $pg^{**}T_1$modulo *I* is $pg^{**}T_0$ modulo *I* space.

Proof follows from the definitions.

**Remark 6.7:** The converse of the above theorem is not true as seen in the following example.

**Example 6.8:** Let$X = \{a, b, c\}, \tau = \{\varphi, X, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\varphi, \{b\}\}$, then $(X, \tau, I)$ is $pg^{**}T_0$modulo $I$ but not $pg^{**}T_1$modulo *I* space.

**Theorem 6.9:** Let $I, J$ be ideals of $X$ and if $I \subseteq J$, then $(X, \tau, I)$ is $pg^{**}T_1$modulo $I$ implies $(X, \tau, J)$ is $pg^{**}T_1$modulo $J$.

**Proof:** If $x, y \in X$ and $x \neq y$, then there exists disjoint pg**-open sets $U_x, U_y$ containing $x, y$ respectively such that $U_x \cap U_y = \varphi \in I \subseteq J$. Therefore $(X, \tau, J)$ is a $pg^{**}T_1$modulo$J$ space.

**Theorem 6.10:** Let$(X, \tau, I)$ and $(Y, \sigma, J)$ be two ideal topological spaces and $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ be a bijection where $J = f(I)$ is an ideal in $Y$ then,

1. $f$ is pg**-resolute and $X$ is $pg^{**}T_1$modulo *I* space$\Rightarrow$ $Y$ is $pg^{**}T_1$modulo $J$ space.
2. $f$ is pg**-continuous and $Y$ is a $T_1$modulo $J$ space$\Rightarrow$ $X$ is a $pg^{**}T_1$modulo *I* space.
3. $f$ is continuous and $Y$ is a $T_1$modulo $J$ space$\Rightarrow$ $X$ is a $pg^{**}T_1$modulo *I* space.
4. $f$ is pg**-irresolute and $Y$ is $T_1$modulo $J$ space $\Rightarrow$ $X$ is $pg^{**}T_1$modulo *I* space.
5. $f$ is pg**-open and $X$ is a $T_1$ space$\Rightarrow$ $Y$ is $pg^{**}T_1$modulo $J$ space.
6. $f$ is open and$X$ is a $T_1$ space$\Rightarrow$ $Y$ is $pg^{**}T_1$modulo $J$ space.

**Proof:** (1) Let $y_1 \neq y_2 \in Y$. Since $f$ is a bijection there exists $x_1 \neq x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $X$ is $pg^{**}T_1$ modulo $I$ space there exists pg**- open sets $U$ and $V$ in $X$ such that $x_1 \in U, U \cap \{x_2\} \in I$ and $x_2 \in V, V \cap \{x_1\} \in I$this implies $y_1 \in f(U), f(U) \cap \{y_2\} \in J$ and$y_2 \in f(V), f(V) \cap \{y_1\} \in J$ where $f(U)$ and $f(V)$ are pg**- open in $Y$. Therefore $(Y, \sigma, J)$ is a $pg^{**}T_1$modulo$J$ space.

Proofs for (2) to (6) are similar to (1).

## 7. $pg^{**}T_2$ Space

**Definition 7.1:** A topological space $(X, \tau)$ is said to be $pg^{**}T_2$ *space* if $x, y \in X$ and $x \neq y$, there exists disjoint pg**-open sets $U_x, U_y$ containing $x, y$ respectively.

**Example 7.2:** Every discrete and indiscrete topological space is $pg^{**}T_2$ space, since every subset is pg**-open. For, if $x \neq y$ in $X$, $U = \{x\}$ and $V = \{y\}$ are disjoint pg**-open sets.

**Example 7.3:** An infinite set with cofinite topology is not $pg^{**}T_2$, since it is impossible to find two disjoint pg**-open sets.

**Theorem 7.4:** Every$T_2$ space is $pg^{**}T_2$space but not conversely.

Proof is obvious since every open set is pg**-open set.

**Example 7.5:** An indiscrete topological space $(X, \tau)$ has more than one point is $pg^{**}T_2$ but not a $T_2$ space.

**Remark 7.6:**

(i) The properties $pg^{**}T_0$, $pg^{**}T_1$ and $pg^{**}T_2$ are separation properties through pg**-open sets in increasing order of strictness. That is, we have $pg^{**}T_2 \Rightarrow pg^{**}T_1 \Rightarrow pg^{**}T_0$.

(ii) If $(X, \tau)$ is a $pg^{**}T_2$ space and $\tau^* \supseteq \tau$, then $(X, \tau^*)$ is also $pg^{**}T_2$ space.

**Theorem 7.7:** If $X$ is $pg^{**}T_2$ space then for $x \neq y \in X$ there exists a pg**-open set $U$ such that $x \in U$ and $y \notin pg^{**}cl(U)$.

**Proof:** Let $x, y$ be distinct points of $X$. Since $X$ is $pg^{**}T_2$ there exists disjoint pg**-open sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$. Therefore $V^c$ is pg**-closed set such that $pg^{**}cl(U) \subseteq V^c$. Since $y \in V$, we have $y \notin V^c$. Thus $y \notin pg^{**}cl(U)$.

**Theorem 7.8:** Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $f$ and $g$ be pg**-irresolute functions from $X$ to $Y$. If $Y$ is a $pg^{**}T_2$ space then the set $A = \{x \in X / f(x) = g(x)\}$ is pg**-closed in $X$.

**Proof:** If $y \in X - A$, then $f(y) \neq g(y)$. Since $Y$ is a $pg^{**}T_2$ space there exists pg**-open sets $U$ and $V$ such that $f(y) \in U$, $g(y) \in V$ and $U \cap V = \varphi$, this implies $y \in f^{-1}(U) \cap g^{-1}(V) = G$ ispg**-open in $X$. Consequently$G$ is a pg**-neighbourhood of $y \in X - A$and hence$X - A$ is pg**-open. Therefore $A$ is pg**-closed in $X$.

**Theorem 7.9:** Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $f$ and $g$ be pg**-continuous functions from $X$ to $Y$. If $Y$ is a $T_2$ space then the set $A = \{x \in X / f(x) = g(x)\}$ is pg**-closed in $X$.

Proof is similar to the above theorem.

**Theorem 7.10:** Let $f : (X, \tau) \to (Y, \sigma)$ be an injective map and $Y$ is $pg^{**}T_2$space. If $f$ is pg**-totally continuous then $X$ is ultra-Hausdorff.

**Proof:** Let$x$ and $y$ be any two distinct points in $X$. Since $f$ is injective,$f(x)$ and $f(y)$ are distinct points in $Y$. Since $Y$ is $pg^{**}T_2$ space there exists pg**- open sets $U_x, U_y$ such that $f(x) \in U_x$ , $f(y) \in U_y$ and $U_x \cap U_y = \varphi$ . Then $x \in f^{-1}(U_x)$ and $y \in f^{-1}(U_y)$. Since $f$ is pg**- totally continuous$f^{-1}(U_x)$and $f^{-1}(U_y)$ are clopen in $X$. Also $f^{-1}(U_x) \cap f^{-1}\left(U_y\right) = \varphi$. This implies every pair of distinct points of $X$ can be separated by disjointclopen sets. Therefore $X$ is ultra-Hausdorff.

**Theorem 7.11:** If $(X, \tau)$is a $pg^{**}T_2$ space then a sequence of points of $X$ pg**-congregates to atmost a point of $X$.

**Proof:** Let $x, y \in X$ and $x \neq y$ , suppose $\langle x_n \rangle \xrightarrow{pg^{**}} x$ and $\langle x_n \rangle \xrightarrow{pg^{**}} y$ . Since $X$ is a $pg^{**}T_2$ space there exists disjointpg**-open sets $U$ and $V$ such that $x \in U$ and $y \in V$. Since $\langle x_n \rangle \xrightarrow{pg^{**}} x$ there exists a positive integer $N$ such that $x_n \in U,\ \forall\, n \geq N$. Hence $V$ can contain only finitely many points of the sequence $\langle x_n \rangle$. Therefore $\langle x_n \rangle$ does not pg**-congregates to $y$.

**Definition 7.12:** If $f: X \to X$ is a function then define $Fix\ (f) = \{x \in X / f(x) = x\}$.

**Theorem 7.13:** If $(X, \tau)$is a $pg^{**}T_2$ space and $f$ is pg**-irresolute function of $X$ into itself then $Fix(f)$ is pg**-closed.

**Proof:** Let $Fix(f) = A$. To prove $X - A$ is pg**-open, suppose $X - A$ is empty then it is pg**-open. Presume that $X - A \neq \varphi$, then there exists $y \in X - A$. Therefore $f(y) \neq y$. Since $X$ is $pg^{**}T_2$, there exists disjoint pg**-open sets $U$ and $V$ such that $y \in U$ and $f(y) \in V$ . Therefore $U \cap f^{-1}(V)$ is a pg**-open set containing $y$ . Suppose if $x \in U \cap f^{-1}(V)$, then $f(x) \neq x$ which implies $x \notin A$. Therefore $U \cap f^{-1}(V) \subseteq X - A$. Therefore $X - A$ is pg**-open.

**Theorem 7.14:** If $(X, \tau)$is a $T_2$ space and $f$ is pg**-continuous function of $X$ into itself then $Fix(f)$ is pg**-closed.

Proof is similar to the above.

**Theorem 7.15:** Product of two $pg^{**}T_2$ space is $pg^{**}T_2$ space.

**Proof:** Let $X \times Y$ be the product of two topological spaces $X$and $Y$. Let $x$ and $y$ be any two distinct points in $X$ and $(x_1, y_1)$ and $(x_2, y_2)$ be any two distinct points of $X \times Y$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$ and since $X$ is $pg^{**}T_2$ space there exists pg**- open sets $U_x, U_y$ containing $x, y$ respectively. Consequently $U_x \times Y$ and $U_y \times Y$ are pg**- open sets containing $(x_1, y_1)$ and $(x_2, y_2)$ respectively such that $(U_x \times Y) \cap (U_y \times Y) = \left(U_x \cap U_y\right) \times Y = \varphi$. Therefore $X \times Y$ is a $pg^{**}T_2$ space.

## 8. $pg^{**}T_2$ Spaces and $pg^{**}$Compact spaces

**Theorem 8.1:** Let $(X, \tau)$be a $pg^{**}T_2$ space, then every pg**-compact subset of $X$ is pg**-closed.

**Proof:** Let $Y$ be a pg**-compact subset of $X$ and $x \in X - Y$. Then for every $y \in Y$ there exists disjointpg**-open sets $U_x$ and $V_y$ containing $x$ and $y$ respectively. Now $\{V_y / y \in Y\}$ forms a pg**-open cover for $Y$, then there exists $\{y_1, y_2, y_3, \dots, y_n\} \in Y$ such that $Y \subseteq \bigcup_{i=1}^{n} V_{y_i} = V$. Let $U = \bigcap_{i=1}^{n} U_{x_i}$, then $U$ is pg**-open.

Obviously $U \cap Y = \varphi$. Therefore $U$ is a pg**-neighbourhood of $x$ contained in $X - Y$. Therefore $X - Y$ is pg**-open and hence $Y$ is pg**-closed.

**Remark 8.2:** In theorem (8.1) $pg^{**}T_2$ property is essential. An infinite cofinite topological space is pg**multiplicative but not $pg^{**}T_2$ space, in this space every subset is pg**-compact but only finite sets are pg**-closed.

**Theorem 8.3:** If $\{X_\alpha\}$ is a collection of pg**-compact subsets of a pg**-multiplicative pg**$T_2$ space$(X, \tau)$ such that the intersection of every finite subcollection of $\{X_\alpha\}$ is nonempty, then $\cap\, X_\alpha$ is nonempty.

**Proof:** Fix a member $X_1$ of $\{X_\alpha\}$ and put $U_\alpha = X_\alpha^c$. Assume that no point of $X_1$ belongs to every $X_\alpha$. Then the sets $U_\alpha$ form an pg**- open cover of $X_1$, and since $X_1$ is pg**-compact, there are finitely many indices $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ such that $X_1 \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \dots \cup U_{\alpha_n}$. But this implies $X_1 \cap X_{\alpha_1} \cap X_{\alpha_2} \cap \dots \cap X_{\alpha_n}$ is empty, contradiction to our hypothesis. Therefore $\cap X_\alpha$ is nonempty.

**Theorem 8.4:** A pg**multiplicative space$(X,\tau)$ is $pg^{**}T_2$ if and only if two disjoint pg**-compact subsets of $X$ can be separated by disjoint pg**-open sets

**Proof:** Let $(X,\tau)$ be a $pg^{**}T_2$ space and $A, B$ be disjointpg**-compact subsets of $X$. Choose $x \in A$, then for every $y \in B$ we have $x \neq y$, since $X$ is $pg^{**}T_2$ there exists disjointpg**-open sets $U_x$ and $V_y$ containing $x$ and $y$ respectively.Now $B = \bigcup_{y\in B} \{y\} \subseteq \bigcup_{y\in B} V_y$, we get $\{V_y / y \in B\}$ forms a pg**-open cover for $B$, then there exists $\{y_1, y_2, y_3, \dots, y_n\} \in Y$ such that$B \subseteq \bigcup_{i=1}^{n} V_{y_i} = V$. Define $U_a = \bigcap_{i=1}^{n} U_{x_i}$, then $U_n$ is pg**-open. $x \in U_n$and $U_a \cap V = \varphi$. Seeing as $A = \bigcup_{x\in A} \{x\} \subseteq \bigcup_{x\in A} U_n$, we get $\{U_a / a \in A\}$ forms a pg**-open cover for $A$. Since $A$ is pg**-compact $A \subseteq \bigcup_{i=1}^{m} U_{a_i} = U$(say). Since $X$ is pg**multiplicative $U$ is pg**-open. Since $U_a \cap V = \varphi$ for every $a \in A$, we get $U \cap V = \varphi$. Therefore the pg**-open sets$U$ and $V$ are disjoint pg**-open sets containing $A, B$ respectively. Conversely assume that any two disjoint pg**-compact subsets of $X$ can be separated by disjoint pg**-open sets. Let $x \neq y \in X$ then $\{x\}$ and $\{y\}$ are disjoint pg**-compact subsets of $X$. By hypothesis there exists disjoint pg**-open sets $U$ and $V$ such that $\{x\} \subseteq U$, $\{y\} \subseteq V$. Therefore $X$ is a $pg^{**}T_2$ space.

**Theorem 8.5:** If a nonempty pg**multiplicative pg**-compact$pg^{**}T_2$ space $X$ has no pg**-isolated points then $X$ is uncountable.

**Proof:** Let $x_1 \in X$. Since $X$ has no isolated points we can choose $y \in X$ such that $x_1 \neq y$. Since $X$ is $pg^{**}T_2$ there exists disjointpg**-open sets $U_1$ and $V_1$ containing $x_1$ and $y$ respectively. Therefore $V_1$ is a pg**-open set and $x_1 \notin pg^{**}cl(V_1)$. Repeating the same process with $V_1 = X$ and $x_1 \neq x$, then we get a pg**-open set $V_2$ and $x_1 \notin pg^{**}cl(V_2)$.

In general for a nonempty pg**-open set $V_{n-1}$, we get pg**-open set $V_n$ such that $V_n \subseteq V_{n-1}$ and $x_n \notin pg^{**}cl(V_n)$. Thus we get a nested sequence of pg**-closed sets such that $pg^{**}cl(V_n) \supseteq pg^{**}cl(V_{n+1}) \supseteq \cdots$, since $X$ is pg**-compact there exists $x \in \cap\, pg^{**}cl(V_n)$. Define $f: \mathbb{N} \to X$ such that $f(n) = x_n$. We show that there exists $x \in X - f(\mathbb{N})$. $x \in \cap\, pg^{**}cl(V_n)$ but $x_n \notin pg^{**}cl(V_n)$ this implies $x \neq x_n$ for every $n$. Therefore $x \in X - f(\mathbb{N})$. $f: \mathbb{N} \to X$ is not onto and hence $X$ is uncountable.

**Theorem 8.6:** Let $(X,\tau)$ be a pg**multiplicative $pg^{**}T_2$ space. Then $X$ is pg**-locally compact if and only if each of its points is a pg**-interior point of some pg**-compact subset of $X$.

**Proof:** Let $X$be pg**-locally compact and $x \in X$. Thenthere is some pg**-compact subset $C$ of X that contains a pg**-neighbourhood $N$ of $x$. Conversely let every point $x \in X$ be a pg**-interior point of some pg**-compact subset $C$ of $X$. Then $C$ is a pg**-neighbourhood $x$. Since $C$ is pg**-compact it is pg**-closed. Therefore $X$ is pg**-locally compact.

**Theorem 8.7:** Every pg**- irresolute mapping of a pg**-compact space into a $pg^{**}T_2$ space is pg**- resolve.

**Proof:** Let $(X,\tau)$be pg**-compact space and $(Y,\sigma)$ be a $pg^{**}T_2$ space. Let $f: X \to Y$ be a pg**- irresolute map and $F$ be pg**-closed in $X$. To prove $f(F)$ is pg**-closed in $Y$. Since $F$ is a pg**-closed subset of a pg**-compact space $X$, $F$ is pg**-compact. Also $f: X \to Y$is pg**- irresolute and $F$ is pg**-compact implies $f(F)$ is pg**-compact subset of $Y$. Since $f(F)$ is pg**-compact subset of a $pg^{**}T_2$ space $f(F)$ is pg**-closed. Therefore $f$ is pg**-resolve.

**Theorem 8.8:** A one-one pg**-irresolute mapping of a pg**-compact space onto apg**multiplicative $pg^{**}T_2$ space is a pg**-homeomorphism.

**Proof:** Let $X$ be pg**-compact, $Y$pg**multiplicative $pg^{**}T_2$ space and $f$ a one-one pg**-irresolute mapping onto $Y$. In order to show that $f$ is a pg**-homeomorphism, it is only necessary to show that it carries pg**-open sets into pg**-open sets or unvaryingly pg**-closed sets into pg**-closed sets. But if $E$ is a pg**-closed subset of $X$, then $E$ is pg**-compact. Since $f$ is pg**-irresolute $f(E)$ is pg**-compact. Therefore by theorem (8.1) $f(E)$ is pg**-closed.

**Theorem 8.9:** Let $(X,\tau)$be a pg**multiplicative $pg^{**}T_2$ space. If $E$ and $F$ are subsets of $X$ and if $E$ is pg**-closed and $F$ is pg**-compact, then $E \cap F$ is pg**-compact.

**Proof:** Since $X$ is a pg**multiplicative $pg^{**}T_2$ space $E \cap F$ is pg**-closed. Also $E \cap F$is a pg**-closed subset of a pg**-compact space $F$. Therefore $E \cap F$ is pg**-compact.

## 9. $pg^{**}T_2$ modulo $I$ space

**Definition 9.1:** An ideal topological space $(X, \tau, I)$ is said to be $pg^{**}T_2$ *modulo* $I$ if for every pair of points $x, y \in X$ and $x \neq y$ there exists pg**-open set $U, V$ such that$x \in U - V, y \in V - U$ and $U \cap V \in I$.

**Example 9.2:** For any ideal $I$ an indiscrete topological space $(X, \tau, I)$ is $pg^{**}T_2$modulo $I$ space.

**Example 9.3:** Let$(X, \tau, I)$ be an infinite co finite ideal topological space with $I = \{\varphi\}$. It is not possible to find two disjoint pg**-open sets of $X$ such that $x \in U - V, y \in V - U$ and $U \cap V \in I$. Therefore $X$ is not $pg^{**}T_2$modulo $I$ space.

**Theorem 9.4:** Every $pg^{**}T_2$ space is $pg^{**}T_2$modulo $I$ space for every ideal $I$ but not conversely.

Proof is obvious since $\varphi \in I$.

**Example 9.5:** Let $X$ be an infinite ideal topological space with cofinite topology and $I = \wp(X)$, then the space is not $pg^{**}T_2$ but it is $pg^{**}T_2$modulo $I$ space.

**Remark 9.6:** If$I = \{\varphi\}$ then both $pg^{**}T_2$ space and $pg^{**}T_2$modulo $I$ space coincide.

**Theorem 9.7:** Let $(X, \tau, I)$ be $pg^{**}T_2$modulo $I$ and $J$ be an ideal of $X$ with $I \subseteq J$, then $(X, \tau, J)$ is $pg^{**}T_2$modulo $J$.

Proof is obvious.

**Theorem 9.8:** Every ideal topological space which is $pg^{**}T_2$modulo $I$ is $pg^{**}T_1$modulo $I$ space.

Proof follows from the definitions.

**Remark 9.9:** The converse of the above theorem need not be true as seen in the following example.

**Example 9.10:** Let $X = \{a, b, c\}, \tau = \{\varphi, X, \{a\}, \{c\}, \{a, c\}\}$ , $PG^{**}O(X) = \{\varphi, X, \{a\}, \{c\}, \{a, c\}\}$ and $I = \wp(X)$ then $(X, \tau, I)$ is $pg^{**}T_1$modulo $I$ but not $pg^{**}T_2$modulo $I$ space.

**Theorem 9.11:** Let $(X, \tau, I)$ and $(Y, \sigma, J)$ be two ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, J)$ be a bijection where $J = f(I)$ is an ideal in $Y$ then,

1. $f$ is pg**-resolute and $X$ is $pg^{**}T_2$modulo $I$ space$\Longrightarrow$ $Y$ is $pg^{**}T_2$modulo $J$ space.
2. $f$ is pg**-continuous and $Y$ is a $T_2$modulo $J$ space$\Longrightarrow$ $X$ is a $pg^{**}T_2$modulo $I$ space.
3. $f$ is continuous and $Y$ is a $T_2$modulo $J$ space$\Longrightarrow$ $X$ is a $pg^{**}T_2$modulo $I$ space.
4. $f$ is pg**-irresolute and $Y$ is $T_2$modulo $J$ space $\Longrightarrow$ $X$ is $pg^{**}T_2$modulo $I$ space.
5. $f$ is pg**-open and$X$ is a $T_2$ space$\Longrightarrow$ $Y$ is $pg^{**}T_2$modulo $J$ space.
6. $f$ is open and $X$ is a $T_2$ space$\Longrightarrow$ $Y$ is $pg^{**}T_2$modulo $J$ space.

**Proof:** (1) Let $y_1 \neq y_2 \in Y$. Since $f$ is a bijection there exists $x_1 \neq x_2 \in X$ such that $f(x_1) = y_1$and $f(x_2) = y_2$. Since $X$ is $pg^{**}T_2$modulo $I$ space there exists pg**- open sets $U$and $V$ in $X$ such that $x_1 \in U - V$, $x_2 \in V - U$ and $U \cap V \in I$.

This implies $y_1 \in f(U) - f(V)$, $f(V) - f(U)$ and $f(V) \cap f(V) \in J$ where $f(U)$ and $f(V)$ are pg**- open in $Y$. Therefore $(Y, \sigma, J)$ is a $pg^{**}T_2$modulo $J$ space.

Proofs for (2) to (6) are similar to (1).

## 10. pg**regular spaces

**Definition 10.1:** A $pg^{**}T_1$ space $(X, \tau)$ is said to be $pg^{**}$*regular*if $F$ is a pg**- closed set and $x \in X$ is a point such that $x \notin F$, there exists disjoint pg**- open sets $U_F, U_x$ containing $F$ and $x$respectively.

**Example 10.2:** Every indiscrete topological space is $pg^{**}$regular.

If $F$ is a pg**-closed subset of $X$ and $x \notin F$ then $\{x\}$ and $F$ are disjoint pg**- open sets containing $x$ and $F$ respectively, Since every subset of a indiscrete topological space is pg**- open.

**Example 10.3:** Any infinite co finite topological space is not $pg^{**}$regular, since it is impossible to find disjoint pg**- open sets.

 

**Theorem 10.4:** Every $pg^{**}$regular space is $pg^{**}T_2$ space.

**Proof:** Follows from $\{x\}$ is pg**- closed for all $x \in X$.

**Theorem 10.5:** Let $(X, \tau)$ be a $pg^{**}$multiplicative $pg^{**}T_1$ space, then the following are equivalent.

(i) $X$ is $pg^{**}$regular.

(ii) For every $x \in X$ and for every pg**-neighbourhood $U$ of $x$ there exists a pg**-neighbourhood $V$ of $x$ such that $pg^{**}cl(V) \subseteq U$.

(iii) For every $x \in X$ and for every pg**-closed set not containing $x$there exists pg**-neighbourhood $V$ of $x$ such that $pg^{**}cl(V) \cap F = \varphi$.

**Proof** $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$: Let $(X, \tau)$ be$pg^{**}$regular. Let $x \in X$ and $U$ be a pg**-neighbourhood of $x$, then $F = X - U$ is pg**-closed. Then there exists disjoint pg**- open sets $V$ and $W$such that $x \in V$ and $F \subseteq W$. Let $y \in F = X - U$. Therefore $y \notin pg^{**}cl(V)$. Therefore$x \in V \subseteq pg^{**}cl(V) \subseteq U$.

$(\boldsymbol{ii}) \Rightarrow (\boldsymbol{iii})$: Let $x \in X$ and $F$ be a pg**-closed set with $x \notin F$. Then $x \in X - F$ which is pg**- open. Then there exists pg**-neighbourhood $V$ of $x$ such that $pg^{**}cl(V) \subseteq X - F$. Therefore $pg^{**}cl(V) \cap F = \varphi$.

$(\boldsymbol{iii}) \Rightarrow (\boldsymbol{i})$: Let $x \in X$ and $F$ be a pg**-closed set with $x \notin F$. Then by hypothesis there exists a pg**-neighbourhood $V$ of $x$ such that $pg^{**}cl(V) \cap F = \varphi$. Therefore $F \subset X - pg^{**}cl(V) = W$.

Now $V \cap (X - pg^{**}cl(V)) \subset V \cap (X - V) = \varphi$. Therefore $V$ and $W$ are disjoint pg**- open sets containing $x$ and $F$ respectively.Therefore $X$ is $pg^{**}$regular.

**Theorem 10.6:** Every pair of points in a $pg^{**}$regular space have pg**-neighbourhoods whose pg**-closures are disjoint.

**Proof:** Let $x$ and $y$ be distinct points in $X$. Then by the definition of $pg^{**}$regularity $\{y\}$ is pg**-closed and there exists disjoint pg**- open sets $U, V$ containing $x$ and $y$ respectively. Then by theorem (10.5) there exists a pg**-neighbourhood $U_x$ of $x$ such that$x \in U_x \subseteq pg^{**}cl(U_x) \subseteq U$. Similarly there exists a pg**-neighbourhood $V_x$ of $x$ such that $x \in V_x \subseteq pg^{**}cl(V_x) \subseteq V$. Therefore $U_x$ and $V_x$ are pg**-neighbourhoods of $x$ and $y$ whose pg**-closures are disjoint.

**Theorem 10.7:** Let $A$ be a pg**-compact subset of a $pg^{**}$multiplicative $pg^{**}$regular space $(X, \tau)$ then for any pg**-open set $G$ containing $A$ there exists a pg**-closed set $F$ such that $A \subseteq F \subseteq G$.

**Proof:** If $a \in A$ then $a \in G$. Since $X$ is $pg^{**}$ regular there exists a pg**-neighbourhood $V_a$ of $a$ such that $a \in V_a \subseteq pg^{**}cl(V_a) \subseteq G$. Now $A = \underset{a \in A}{\cup}\{a\} \subseteq \underset{a \in A}{\cup} V_a$ and $\{V_a\}_{a \in A}$ forms a pg**-open cover for a pg**-compact set $A$. Hence $A \subseteq \overset{n}{\underset{i=1}{\cup}} V_{a_i}$. Now $pg^{**}cl(V_{a_i}) \subseteq G$ for al $i$, $1 \leq i \leq n$implies $F = \overset{n}{\underset{i=1}{\cup}} pg^{**}cl(V_{a_i})$. Since $X$ is $pg^{**}$multiplicative F is pg**-closed such that$A \subseteq F \subseteq G$.

**Theorem 10.8:** Let $(X, \tau)$ be a$pg^{**}$finitely multiplicative $pg^{**}$regular space. Let $A$ and $B$ be disjoint subsets of $X$ such that $A$ is pg**-closed and $B$ is pg**-compact in $X$. Then there exists disjoint pg**-open sets in $X$ containing $A$ and $B$ respectively.

**Proof:** If $b \in B$ then $b \notin A$. Since $X$ is $pg^{**}$ regularthere exists disjoint pg**-open sets $V_A, U_b$ containing $A$ and $b$respectively for each $b \in B$. Therefore$\underset{b \in B}{\cup}\{b\} \subseteq \underset{b \in B}{\cup} U_b$and $\{U_b\}_{b \in B}$forms a pg**-open cover for $B$. Since $B$ is pg**-compact $B \subseteq \overset{n}{\underset{i=1}{\cup}} U_{b_i}$. Define $U = \overset{n}{\underset{i=1}{\cup}} U_{b_i}$ which is pg**- open. Find corresponding$V_{A_i}$ for all $i$, then $A \subseteq \overset{n}{\underset{i=1}{\cap}} V_{A_i}$. Define $V = \overset{n}{\underset{i=1}{\cap}} V_{A_i}$ which is pg**-open. Therefore there exists disjoint pg**-open sets such that $A \subseteq V$ and $B \subseteq U$.

**Theorem 10.9:** $pg^{**}$closure of a pg**-compact subset of a$pg^{**}$multiplicative $pg^{**}$regular space is pg**-compact.

**Proof:** Let $(X, \tau)$ be a $pg^{**}$regular space and $A$ be a pg**-compact subset of $X$. Let $\{G_\alpha\}$ be a pg**-open cover for $pg^{**}cl(A)$. Then $\{G_\alpha\}$ is also a pg**-open cover for $A$. Since $A$ is pg**-compact $A \subseteq \overset{n}{\underset{i=1}{\cup}} G_{\alpha_i} = G$ which is pg**-open. Then by theorem (10.7) there exist a pg**-closed set $F$ such that $A \subseteq F \subseteq G$. Since $X$ is $pg^{**}$multiplicative and $F$ is pg**-closed, $pg^{**}cl(A) \subseteq pg^{**}cl(F) = F \subseteq G = \overset{n}{\underset{i=1}{\cup}} G_{\alpha_i}$. Therefore the open cover $\{G_\alpha\}$ of $pg^{**}cl(A)$ has a finite subcover. Hence $pg^{**}cl(A)$ is pg**-compact.

 

## 11. pg**normal spaces

**Definition 11.1:** A $pg^{**}T_1$ space $(X,\tau)$ is said to be $pg^{**}$*normal* if for each pair $A$ and $B$ of disjoint pg**- closed sets in $X$, there exist disjoint pg**- open sets $U_A, U_B$ containing $A$ and $B$respectively.

**Example 11.2:** Every indiscrete topological space is $pg^{**}$normal, since every subset of a indiscrete topological space ispg**-open.

**Example 11.3:** Any infinite co finite topological space is not $pg^{**}$normal, since it is impossible to find disjoint pg**-open sets.

**Theorem 11.4:** Every $pg^{**}$normal space is $pg^{**}$regular space.

**Proof:** Follows from $\{x\}$ is pg**-closed for all $x \in X$.

**Theorem 11.5:** Let $(X,\tau)$ be a $pg^{**}$multiplicative $pg^{**}T_1$ space, then $X$ is $pg^{**}$normal if and only if for every pg**-closed set $A$ and a pg**-open set$U$containing $A$ there exists a pg**-open set $V$ containing $A$ such that $pg^{**}cl(V) \subseteq U$.

**Proof:** Let $A$ be a pg**-closed set and $U$be a pg**-open set containing $A$. Then $B = X - A$ is pg**-closed and $A \cap B = \varphi$. Since $X$ is $pg^{**}$normalthere exists disjoint pg**- open sets $V, W$ containing $A$ and $B$respectively. Now $A \subseteq V \subseteq pg^{**}cl(V)$. Let $y \in X - U = B \subseteq W$ and $V \cap W = \varphi$. Therefore $y \notin pg^{**}cl(V)$. Hence $pg^{**}cl(V) \subseteq U$. Conversely let $A$ and $B$ be two pg**-closed subsets of $X$. Then $U = X - B$ is pg**-open set containing $A$. By hypothesis there exists a pg**-open set $V$ containing $A$ such that $A \subseteq V \subseteq pg^{**}cl(V) \subseteq U$. Since $X$ is $pg^{**}$multiplicative $pg^{**}cl(V)$ is pg**-closed. Therefore $X - pg^{**}cl(V) = W$ is a pg**-open set containing $B$ and $V$ is a pg**-open set containing $A$ such that $V \cap W = \varphi$. Therefore $(X,\tau)$ is $pg^{**}$normal.

**Theorem 11.6:** A $pg^{**}$multiplicative space $X$ in which every singleton set is a pg**-isolated point is $pg^{**}$normal.

**Proof:** follows from every subset is pg**-clopen.

**Theorem 11.7:** Every pg**-compact $pg^{**}$finitely multiplicative$pg^{**}T_2$ space is $pg^{**}$normal.

**Proof:** Let $X$ be a pg**-compact $pg^{**}$finitely multiplicative $pg^{**}T_2$ space. Let $A$ and $B$ be two pg**-closed subsets of $X$. Since $B$ is a pg**-closed subset of a pg**-compact space $B$ is pg**-compact, also by theorem (8.1) for every $x \in B$ there exists disjoint pg**-open sets $U_x, V_x$ such that $x \in U_x$ and $A \subseteq V_x$. Now $\{U_x / x \in B\}$ is a pg**-open cover for $B$. Then $B \subseteq \overset{n}{\underset{i=1}{\cup}} U_{x_i} = U$(say) which is pg**-open. Let $V = \overset{n}{\underset{i=1}{\cap}} V_{x_i}$ which is pg**-open. Then$V$ and $U$ are disjoint pg**-open sets containing $A$ and $B$ respectively. Also every $pg^{**}T_2$space is $pg^{**}T_1$. Hence $X$ is $pg^{**}$normal.

**Theorem 11.8:** Every metrizable space $(X,\tau)$ is $pg^{**}$normal.

**Proof:** Let $(X,\tau)$ be metrizable space with metric $d$. Let $A$ and $B$ be two pg**-closed subsets of $X$. For every $a \in A$, choose $\varepsilon_a$ such that $B(a,\varepsilon_a) \cap B = \varphi$. Correspondingly for every $b \in B$, choose $\varepsilon_b$ such that $B(b,\varepsilon_b) \cap A = \varphi$. Let $U = \underset{a \in A}{\cup} B\left(a, \frac{\varepsilon_a}{2}\right), V = \underset{b \in B}{\cup} B\left(b, \frac{\varepsilon_b}{2}\right)$. $U$ and $V$ are pg**-open, since $U$ and $V$ are open in $X$. In $z \in U \cap V$ then $z \in B\left(a, \frac{\varepsilon_a}{2}\right) \cap B\left(b, \frac{\varepsilon_b}{2}\right)$ for some $a \in A$and $b \in B$. Therefore $(a,b) \leq d(a,z) + d(z,b) \leq \frac{\varepsilon_a + \varepsilon_b}{2}$. Without loss of generality let $\varepsilon_a \leq \varepsilon_b$. Then $d(a,b) < \varepsilon_b$, this implies $a \in B(b,\varepsilon_b)$ which is a contradiction. Therefore $U \cap V = \varphi$. Since $X$ is metrizable, every singleton set is closed and hence pg**-closed. Hence $X$ is $pg^{**}$normal.

**Theorem 11.9:** In a $pg^{**}$normal space $(X,\tau)$ every pair of disjoint pg**-closed sets have pg**-neighbourhoods whose $pg^{**}$closures are disjoint.

**Proof:** Let $A$ and $B$ be disjoint pg**-closed subsets of $X$. Then by definition of $pg^{**}$normality there exist disjoint pg**- open sets $U_A, U_B$ containing $A$ and $B$respectively. Then there exists a pg**-open set $V$containing $A$ such that $A \subseteq V \subseteq pg^{**}cl(V) \subseteq U_A$. Likewise, there exists a pg**-open set $W$containing $B$ such that $B \subseteq W \subseteq pg^{**}cl(W) \subseteq U_B$. Therefore $V$ and $W$ are the required pg**-neighbourhoods.

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