International Journal of Mathematical Archive-8(3), 2017, 150-155
Available online through www.ijma.info ISSN 2229 – 5046

HOMOTOPY CONTINUATION METHOD FOR THREE-PARAMETER EIGENVALUE PROBLEMS

SONGITA BORUAH*
Research scholar, Dept. of Mathematics D.U, Dibrugarh Assam, India.

ARUN KUMAR BARUAH
Professor, Dept. of Mathematics D.U, Dibrugarh Assam, India.

(Received On: 14-02-17; Revised & Accepted On: 22-03-17)

ABSTRACT

This paper discussed Homotopy continuation method for three-parameter eigenvalue problems. Finally some numerical results are presented.

***Key words**: multiparameter, eigenvalue, eigenvector, Newton's method, Trace theorem.*

1.1 INTRODUCTION

Multiparameter eigenvalue problems are generalization of one-parameter eigenvalue problems and can be found when the method of separation of variables is applied to certain boundary value problems associated with partial differential equations. Much more works have been done in the field of one-parameter eigenvalue problems, both theoretically and numerically compared to two-parameter or more than two-parameter eigenvalue problems. Some works have been done theoretically in the field of multiparameter eigenvalue problems. Few authors have dealt with the multiparameter eigenvalue problems numerically mainly in two-parametic cases. Numerical methods applied to a three-parameter problems are very limited and hence some contribution in this area are always in needed.

1.2 **Three-parameter eigenvalue problem and its reduction to a system of one-parameter problems**

A three-parameter eigenvalue problems in matrix form is as follows

$$\begin{aligned} A_{10}x-\lambda_1 A_{11}x-\lambda_2 A_{12}x-\lambda_3 A_{13}x &= 0\\ A_{20}y-\lambda_1 A_{21}y-\lambda_2 A_{22}y-\lambda_3 A_{23}y &= 0\\ A_{30}z-\lambda_1 A_{31}z-\lambda_2 A_{32}z-\lambda_3 A_{33}z &= 0 \end{aligned} \tag{1.2.1}$$

Where $\lambda_i \in \mathbb{C}, i=1,\ 2,\ 3$ and

$$x\in\mathbb{C}^n\setminus\{0\}, \mathrm{A}_{10}, \mathrm{A}_{11}, \mathrm{A}_{12}, \mathrm{A}_{13}\in\mathbb{C}^{n\times n}$$

$$y\in\mathbb{C}^m\setminus\{0\}, \mathrm{A}_{20}, \mathrm{A}_{21}, \mathrm{A}_{22}, \mathrm{A}_{23}\in\mathbb{C}^{m\times m}$$

$$z\in\mathbb{C}^p\setminus\{0\}, \mathrm{A}_{30}, \mathrm{A}_{31}, \mathrm{A}_{32}, \mathrm{A}_{33}\in\mathbb{C}^{p\times p}$$

Where $\lambda_i \in \mathbb{C}, i=1,\ 2,\ 3$ are called the eigenvalues and x, y, z are called eigenvectors of the problem.

Problem (1.2.1) can be reduced to a system of three one-parameter problems:

$$\begin{aligned} \Delta_1 u &= \lambda_1\Delta_0 u\\ \Delta_2 u &= \lambda_2\Delta_0 u\\ \Delta_3 u &= \lambda_3\Delta_0 u \end{aligned} \tag{1.2.2}$$

Corresponding Author: Songita Boruah*
Research scholar, Dept. of Mathematics D.U, Dibrugarh Assam, India.

Where $\Delta_0, \Delta_1, \Delta_2, \Delta_3$ are $(\mathrm{mnp}) \times (\mathrm{mnp})$ dimensional matrices defined as

$$\Delta_0 = A_{11} \otimes A_{22} \otimes A_{33} - A_{11} \otimes A_{23} \otimes A_{32} + A_{12} \otimes A_{23} \otimes A_{31} - A_{12} \otimes A_{21} \otimes A_{33} + A_{13} \otimes A_{21} \otimes A_{32} - A_{13} \otimes A_{22} \otimes A_{31} \tag{1.2.3}$$

$$\Delta_1 = A_{10} \otimes A_{22} \otimes A_{33} - A_{10} \otimes A_{23} \otimes A_{32} + A_{12} \otimes A_{23} \otimes A_{30} - A_{12} \otimes A_{20} \otimes A_{33} + A_{13} \otimes A_{20} \otimes A_{32} - A_{13} \otimes A_{22} \otimes A_{30} \tag{1.2.4}$$

$$\Delta_2 = A_{11} \otimes A_{20} \otimes A_{33} - A_{11} \otimes A_{23} \otimes A_{30} + A_{10} \otimes A_{23} \otimes A_{31} - A_{10} \otimes A_{21} \otimes A_{33} + A_{13} \otimes A_{21} \otimes A_{30} - A_{13} \otimes A_{20} \otimes A_{31} \tag{1.2.5}$$

$$\Delta_3 = A_{11} \otimes A_{22} \otimes A_{30} - A_{11} \otimes A_{20} \otimes A_{32} + A_{12} \otimes A_{20} \otimes A_{31} - A_{12} \otimes A_{21} \otimes A_{30} + A_{10} \otimes A_{21} \otimes A_{32} - A_{10} \otimes A_{22} \otimes A_{31} \tag{1.2.6}$$

and

$$u = x \otimes y \otimes z$$

With $\otimes$ denoting the Kronecker product (or Tensor product) of two matrices discussed in (1.3).

Theorem: Let $(\lambda_1, \lambda_2, \lambda_3)$ be an eigenvalue and $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ a corresponding eigenvector of the system (1.2.1) then $(\lambda_1, \lambda_2, \lambda_3)$ is an eigenvalue of the system (1.2.2) and $u = x \otimes y \otimes z$ is the corresponding eigenvector.

1.3 The Kronecker Product

Definition 1.3.1: The Kronecker product $(.\otimes.) : \mathbb{C}^{m\times n} \times \mathbb{C}^{p\times q} \to \mathbb{C}^{mp\times nq}$ is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}$$

Where we use the standard notation $(A)_{ij} = a_{ij}$

The Kronecker product is a special case of the tensor product, and as such it inherits the properties of bilinearity and associativity, i.e.

$$(\mathrm{kA}) \otimes \mathrm{B} = \mathrm{A} \otimes (\mathrm{kB}) = \mathrm{k}(\mathrm{A} \otimes \mathrm{B})$$
$$\mathrm{A} \otimes (\mathrm{B} + \mathrm{C}) = \mathrm{A} \otimes \mathrm{B} + \mathrm{A} \otimes C$$
$$(\mathrm{A} + \mathrm{B}) \otimes \mathrm{C} = \mathrm{A} \otimes C + B \otimes C$$

1.2 Homotopy Continuation Method:

Homotopy Continuation methods are used to find a solution to a problem by constructing a new problem, simpler than the original one. Homotopy method can be used to generate a good starting value.

All Homotopy methods are based on the construction of a function H(x, t).

To solve a problem F(x)=0 with the help of homotopy method first one has to construct a parameter depending function H(x, t) where $t \in [0,1]$ such that H(x,0)=0 is the problem with known solution and H(x,1)=0 is the original problem F(x)=0.

1.3 Homotopy continuation method for three-parameter eigenvalue problem

In this paper we construct the following Homotopy

$$H : \mathrm{R}^{n_1} \times R^{n_2} \times R^{n_3} \times R \times R \times R \times [0,1] -> \mathrm{R}^{n_1} \times R^{n_2} \times R^{n_3} \times R \times R \times R$$

$$H(x, y, z, \lambda_1, \lambda_2, \lambda_3, t) = \begin{bmatrix} (1-t)W_1x + tA_{10}x - \lambda_1 A_{11}x - t\lambda_2 A_{12}x - t\lambda_3 A_{13}x \\ (1-t)W_2y + tA_{20}y - t\lambda_1 A_{21}y - \lambda_2 A_{22}y - t\lambda_3 A_{23}y \\ (1-t)W_3z + tA_{30}z - t\lambda_1 A_{31}z - t\lambda_2 A_{32}z - \lambda_3 A_{33}z \\ \frac{1}{2}(1-x^T x) \\ \frac{1}{2}(1-y^T y) \\ \frac{1}{2}(1-z^T z) \end{bmatrix}$$

Where W_i are symmetric $n_i \times n_i$ matrices, such that the eigenproblem

$$\left.\begin{aligned} W_1x &= \lambda_1 A_{11}x \\ W_2y &= \lambda_2 A_{22}y \\ W_3z &= \lambda_3 A_{33}z \end{aligned}\right\} \tag{2.2.1}$$

have n_1, n_2 and n_3 distinct eigenvalues respectively.

The solution of H(x, y, z, λ_1, λ_2, λ_3, t) = 0 is a solution of the three-parameter eigenvalue problem

$$\left.\begin{aligned} (1-t)W_1x + tA_{10}x &= \lambda_1 A_{11}x + t\lambda_2 A_{12}x + t\lambda_3 A_{13}x \\ (1-t)W_2y + tA_{20}y &= \lambda_1 tA_{21}y + \lambda_2 A_{22}y + t\lambda_3 A_{23}y \\ (1-t)W_3z + tA_{30}z &= \lambda_1 tA_{31}z + t\lambda_2 A_{32}z + \lambda_3 A_{33}z \end{aligned}\right\} \tag{2.2.2}$$

Which is equal to (1.2.1) for t=1 and equal to (2.2.1) for t=0. Here we discretize t as $0 = t_0 < t_1 < ...... < t_n = 1$ where $t(0) = t_0, t_1 = t_0 + h, t_2 = t_1 + h...$ and so on. First one has to find the solution of (2.2.2) for t=0, then taking these as the initial approximation for t=t_1. Continuing this process finally we will obtain a solution for t=1 and it will give us the solution of (1.2.1).

3.1 Numerical Example:

Consider the three-parameter eigenvalue problem

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x - \lambda_1 \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -.5 \\ 1 & -.5 & 2 \end{bmatrix} x - \lambda_2 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x - \lambda_3 \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix} x = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} y - \lambda_1 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} y - \lambda_2 \begin{bmatrix} 3 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 4 \end{bmatrix} y - \lambda_3 \begin{bmatrix} 6 & 1 & -.5 \\ 1 & 6 & 1 \\ -.5 & 1 & 7 \end{bmatrix} y = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} z - \lambda_1 \begin{bmatrix} 2 & -.5 & 0 \\ -.5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} z - \lambda_2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} z - \lambda_3 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} z = 0$$

Let us consider the homotopy

$$\left.\begin{aligned} (1-t)W_1x + tA_{10}x &= \lambda_1 A_{11}x + t\lambda_2 A_{12}x + t\lambda_3 A_{13}x \\ (1-t)W_2y + tA_{20}y &= \lambda_1 tA_{21}y + \lambda_2 A_{22}y + t\lambda_3 A_{23}y \\ (1-t)W_3z + tA_{30}z &= \lambda_1 tA_{31}z + t\lambda_2 A_{32}z + \lambda_3 A_{33}z \end{aligned}\right\} \tag{3.1.1}$$

Let $W_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $W_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

For t = 0 equation (3.1.1) becomes

$$\begin{aligned} W_1 x &= \lambda_1 A_{11} x \\ W_2 y &= \lambda_2 A_{22} y \\ W_3 z &= \lambda_3 A_{33} z \end{aligned} \qquad (3.1.2)$$

Taking the solutions of (3.1.2) as initial approximation we will solve the given problems using Newton's method

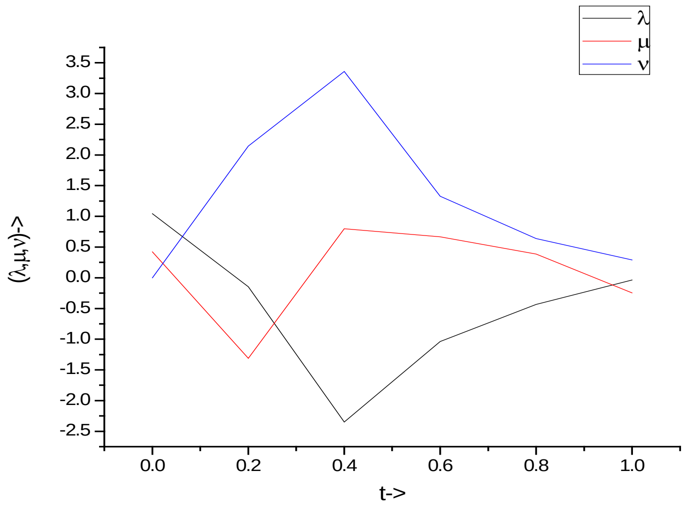
The solutions of (3.1.2) are

$\lambda_1 = (-1.0445, 1.0445, 0)$ and the corresponding eigenvectors are (.8067,-.3435,-.4808), (.0297,-.5810,-.8134), (0,-.7071,.7071)

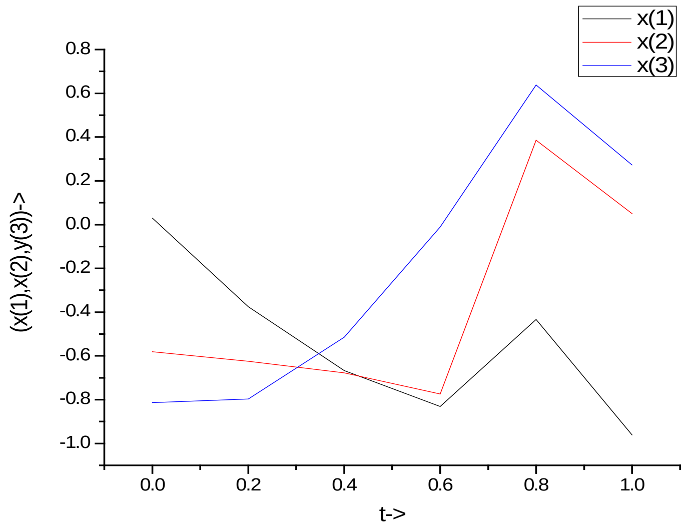
$\lambda_2 = (.42155, -.2965, 0)$ and the eigenvectors are(.9094,.4160,0), (-.5658,.8246,0), (-.7071,0,.7071)

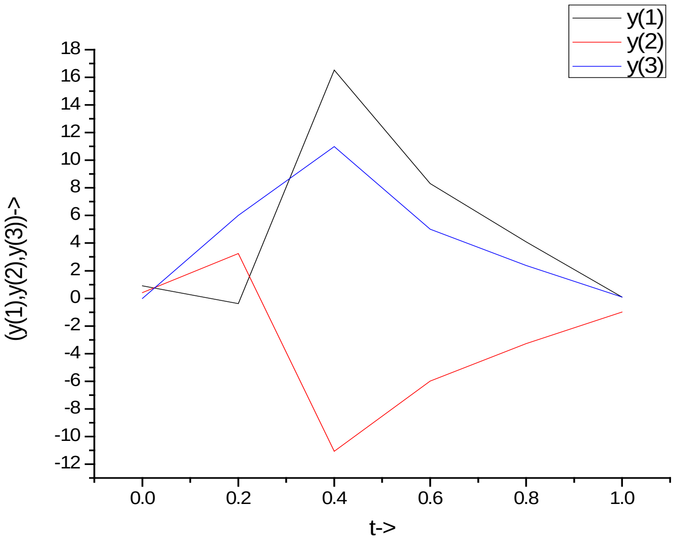
$\lambda_3 = (1.6180, -.6180, 0)$ and the eigenvectors are(.8798,-.3361,.3361), (.2607,-.6826,.6826), (-.7071,.7071,0)

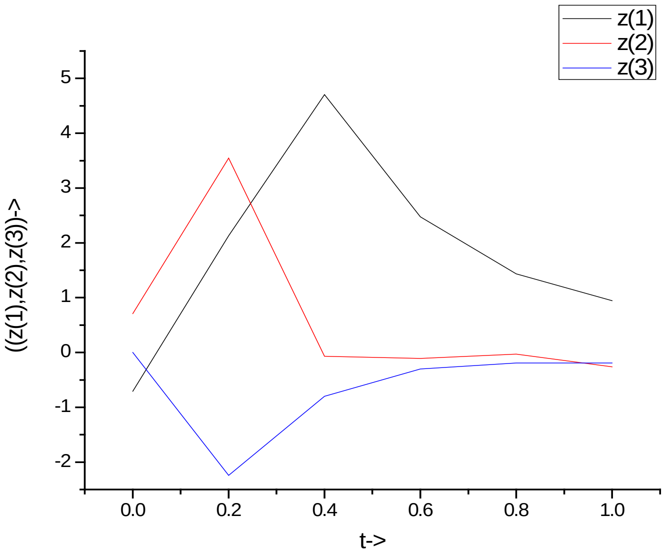
Taking $\lambda_1(0) = 1.0445, \lambda_2(0) = .4215, \lambda_3(0) = 0$ and X(0) = (.0297,-.5810,-.8134), Y(0) = (.9094,.4160,0) and Z(0) = (-.7071,.7071,0) as initial approximation we have the following eigenvalue curve



Eigenvalue (λ,μ,ν) versus t







Statistics for all 27 eigenvalues:

i	$(\lambda_1(0),\lambda_2(0),\lambda_3(0))$	$(\lambda_1(1),\lambda_2(1),\lambda_3(1))$
1	(1.0445,.4215,0)	(-.0378,-.2463,.29)
2	(1.0445,-.2965,0)	(-.01,.2434,.0603)
3	(1.0445,0,-.6180)	(.3821,.0103,.2199)
4	(1.0445,.4215,1.6180)	(-.0360,-.2543,.2875)
5	(-1.0445,.4215,0)	(-.0360,-.2543,.2875)
6	(0,.4215,-.6180)	(-.01,.2434,.0603)
7	(-1.0445,0,1.6180)	(.7953,-.1382,.0103)
8	(-1.0445,-.2965,-.6180)	(.4354,-.5188,.0334)
9	(0,.4215,1.6180)	(0,0,0)
10	(0,-.2965,1.6180)	(-.2823,.1624,.0858)
11	(1.0445,-.2965,1.6180)	(-.2823,.1624,.0858)
12	(-1.0445,.4215,1.6180)	(.1159,.3824,.0082)
13	(-1.0445,-.2965,0)	(0,0,0)
14	(-1.0445,0,-.6180)	(-.2215,.0298,.1525)
15	(1.0445,-.2965,-.6180)	(.1159,.3824,.0082)
16	(1.0445,0,1.6180)	(.0732,1.2018,-.3883)
17	(-1.0445,.4215,-.6180)	(0,0,0)
18	(-1.0445,-.2965,1.6180)	(.0732,1.2018,-.3883)
19	(1.0445,.4215,-.6180)	(0,0,0)
20	(0,-.2965,-.6180)	(0,0,0)
21	(-1.0445,0,0)	(.7953,-.1382,.0103)

22	(1.0445,0,0)	(-.2823,.1624,.0808)
23	(0,.4215,0)	(-.2215,.0298,.1525)
24	(0,-.2965,0)	(0,0,0)
25	(0,0,1.6180)	(.1159,.3824,.0082)
26	(0,0,-.6180)	(0,0,0)
27	(0,0,0)	(0,0,0)

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Source of support: Nil, Conflict of interest: None Declared.