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THE OSSERMAN SURFACES

Prof. Dr. Sci., VESELIN TOTEV VIDEV*

**Dept. Mathematics and Informatics,
Trakia University, 6000 Stara Zagora, Bulagaria, Europe Union.**

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ABSTRACT.

A hyper surface $M \subset R^{n+1}$ is pointwise Osserman surface if the eigenvalues of the Jacobi operator J(X)= R(u,X,X), where R is the curvature tensor of M, are pointwise constants, for any tangent vector X in the tangent space M_p, at any point $p \in M$. In this short note we prove that M is Osserman surface if and only if M is locally Euclidean hyper surface or hyper surface of constant sectional curvature.

Mathematical subject classification: *53 B20.*

Keywords: *Jacobi operator, hyper surface, pointwise Osserman, Weingarten operator, locally Euclidean hyper surface, hyper sphere.*

Let (*N*, *g*) be a Riemannian manifold with metric tensor *g*, and curvature tensor R', defined by the formula:

$$R'(X,Y) = \nabla'_X Y - \nabla'_Y X - \nabla_{[X,Y]},$$

where ∇' is the Levi-Civita connection on *N***,** and *X,Y* are arbitrary tangent vector fields in the tangent bundle $\aleph(N)$, on the manifold *N*.

Let (*M, g*) be an *n*-dimensional Riemannian manifold, which is isometric embedding in (*N*, *g*), and let *dim N=n+m*. If we consider locally in a neighborhood U_p, at a point $p \in M$, then we always can choose a smooth intersections $\xi_1, \xi_2, ..., \xi_m$ in the normal bundle $\aleph(M)^{\perp}$, which are independent vector fields, and which form an orthonormal basis at any point $p \in U_p$. If *X*, *Y* are smooth vector fields in the tangent bundle $\aleph(M)$, then

$$\left(\nabla'_X Y\right)_p = \left(\nabla_X Y\right)_p + \alpha\left(X,Y\right)_p \quad ,$$

where $\nabla_X Y$ is the covariant derivation, defined for the Riemannian connection ∇ on the submanifold (*M, g*), respectively α (*X*, *Y*) is the second fundamental form of (*M, g*). Also

$$\alpha\left(X,Y\right) = \sum_{i=1}^{m} h^i(X,Y)\xi_i$$

is the decomposition of α (*X*, *Y*), with respect to the orthonormal basis $\xi_1, \xi_2, ..., \xi_m \in M_p$, at a point $p \in M$.

According to the definition, for any smooth vector field $\xi \in \aleph(M)^{\perp}$, holds

$$(\nabla'_X \xi)_p = -(A_\xi(X))_p + D_\xi(X)_p,$$

where A_ξ is a linear symmetric Weingarten operator in M_p, at a point $p \in M$, such that

$$g(A_\xi(X), Y) = g(\alpha(X,Y), \xi).$$

Corresponding Author: Prof. Dr. Sci., Veselin Totev Videv*, Dept. Mathematics and Informatics, Trakia University, 6000 Stara Zagora, Bulagaria, Europe Union.

The following formulae and equations are well known) [3]:

I. The Gauss formula:

$$\nabla'_X Y = \nabla_X Y + \alpha(X,Y);$$

II. The Weingarten formula:

$$(\nabla'_X \xi) = -A_\xi(X) + D_\xi(X);$$

III. The Gauss equation:

$$g\big(R'(X,Y,Z)U\big) = g\big(R(X,Y,Z)U\big) + g\big(\alpha(X,Z)\alpha(Y,U)\big) - g\big(\alpha(Y,Z)\alpha(X,U)\big),$$

where R is the curvature tensor of the Riemannian manifold (M,g), and where X,Y,Z,U are the tangent vector fields in the tangent bundle $\aleph(M)$.

Further we consider the case, when the Riemannian manifold N coincide with Euclidean vector space R^{n+1}, and the Riemannian submanifold M is a hyper surface in R^{n+1}. Then the Gauss equation has the form [3]:

$$R(X,Y,Z) = g(AY,Z)AX - g(AX,Z)AY \quad .$$

In the next we will use the Jacobi operator

$$J(X) = R(u,X,X) \quad ,$$

which is a linear symmetric operator, defined for any unit tangent vector $X \in M_p$, at a point $p \in M$ [1], [2], [4]. Following the terminology in [2] we introduce

Definition 1: *A hyper surface $M \subset R^{n+1}$ is pointwise Osserman surface if the eigenvalues of the Jacobi operator $J(X)$ are pointwise constants, for any tangent vector $X \in M_p$, at any point $p \in M$.*

Let M be a pointwise Osserman hyper surface in the Euclidean vector space R^{n+1}.

Let A be the Weingarten operator in M_p, and let $X_1, X_2, \ldots, X_n$ be the eigenvector basis of A.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues, corresponding to the eigenvectors $X_1, X_2, \ldots, X_n$. Then for any indexes $I < j$ $(i, j = 1, 2, \ldots, n)$ holds:

$$R\left(X_i, X_j, X_k\right) = g\left(AX_j, X_k\right)AX_i - g\left(AX_i, X_k\right)AX_j =$$

$$= \begin{cases} 0 & , \quad k \neq i, j; \\ -\lambda_i\lambda_j & , \quad k = i; \\ \lambda_i\lambda_j & , \quad k = j. \end{cases} \qquad (1)$$

It is easy to see that the matrix of the Jacobi operator $J(X_1)$, with respect to the orthonormal basis $X_1, X_2, \ldots, X_n$, has the form:

$$\left(J\left(X_1\right)\right) = \begin{pmatrix} \lambda_1\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_4 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \lambda_1\lambda_n \end{pmatrix},$$

where $J\left(X_1\right)$ we consider as a linear symmetric operator in the tangent subspace

$$X_1^{\perp} = span\left\{\left(X_2, X_3, \ldots, X_n\right) \subset M_p\right\}, \ p \in M.$$

Similarly we can check that the matrix of the Jacobi operator $J\left(X_2\right)$, with respect to the orthonormal basis $X_1, X_2, \ldots, X_n$, has the form:

$$\left(J\left(X_2\right)\right)=\begin{pmatrix} \lambda_2\lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2\lambda_4 & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & \lambda_2\lambda_n \end{pmatrix},$$

where $J\left(X_2\right)$ we consider as a linear symmetric operator in the tangent subspace

$$X_2^{\perp} = span\left\{\left(X_1, X_3, ..., X_n\right) \subset M_p\right\},\ p\in M.$$

From our condition M to be pointwise Osserman hyper surface in R^{n+1}, it follows that the eigenvalues $\lambda_1\lambda_2, \lambda_1\lambda_3, ..., \lambda_{1n}$ and $\lambda_2\lambda_1, \lambda_2\lambda_3, ..., \lambda_{2n}$ of the Jacobi operators $J\left(X_1\right)$ and $J\left(X_2\right)$ coincide, hence

$$\lambda_1\lambda_s = \lambda_2\lambda_s, \quad s=3, 4, \ldots, n. \tag{2}$$

From this equality it follows that at least one $\lambda_s = 0$, for any indexes s=3, 4,…, n, and then all eigenvalues K_{sj} ($s\neq j$=1, 2,…, n), of the Jacobi operator $J\left(X_s\right)$, are equal to 0. Since we assume M to be pointwise Osserman hyper surface in R^{n+1}, then all eigenvalues of any Jacobi operator $J\left(X_s\right)$ (s=1, 2,…, n) are equal to 0. That means that the matrix of the curvature operator $\Re$ on the second exterior product $\wedge^2 M_p$, with respect to the orthonormal 2-vector basis $X_s \Lambda X_t \left(s < t,\ s, t = 1, 2, ..., n\right) \in \wedge^2 M_p$, is zero matrix, which means that $\Re \equiv 0$. From the last equality it follows that the curvature tensor R of hyper surface M is vanishing, which means that M is locally Euclidean hyper surface in R^{n+1}[3]. If all $\lambda_s \neq 0$, for any indexes s = 3, 4,…, n in the equalities (2), then $\lambda_1=\lambda_2$ and if these values are equal to 0, then all eigenvalues of the Jacobi operators $J\left(X_1\right)$ and $J\left(X_2\right)$ are equal to 0. Now from the assumption M to be pointwise Osserman hyper surface in R^{n+1}, it follows that all eigenvalues of any Jacobi operator $J\left(X_s\right)$ (s=1,2,…,n), are equal to 0, and then M is locally Euclidean hyper surface in R^{n+1}, again. If $\lambda_1 = \lambda_2 \neq 0$, then using all Jacobi operators $J\left(X_s\right)$ (s=3,4,…,n), we get $\lambda_1 = \lambda_2 = ... = \lambda_n \neq 0$, which means that M is hyper surface of constant sectional curvature[3]. Thus we prove

Theorem 1: *M is pointwise Osserman hyper surface in* R^{n+1} *if and only if one of the following cases is true*:
1) *M is a locally Euclidean hyper surface*;
2) *M is a hyper surface of constant sectional curvature* .

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