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pg**- Connected space

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ABSTRACT

*In this paper we introduce pg**- connected space, pg**-component, pg**- connected modulo I space andestablish results about the relation between them.*

Key words: *pg**- connected space, pg**-component, pg**- connected modulo I space.*

1. INTRODUCTION

Levine [3] introduced the class of g-closed sets in 1970. Veerakumar[7] introduced g*-closed sets. P M Helen [5] introduced g**-closed sets. A.S.Mashhour, M.E Abd El. Monsef and S.N.EI.Deeb [5] introduced a new class of pre-open sets in 1982. Ideal topological spaces have been first introduced by K.Kuratowski [2] in 1930. The purpose of this paper is to introduce pg**- connected space, pg**-component and pg**- connected modulo I space and investigate their properties.

2. PRELIMINARIES

Definition 2.1: A subset A of a topological space (X,τ) is called a pre-open set [4] if $A \subseteq int(cl(A)$ and a pre-closed set if $cl(int(A)) \subseteq A$.

Definition 2.2: A subset A of topological space (X,τ) is called

1. generalized closed set (g-closed) [3] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).
2. g*-closed set [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X,τ).
3. g**-closed set [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g*-open in(X,τ).
4. pg**- closed set[6] if $pcl(A) \subseteq U$whenever $A \subseteq U$ and U is g*-open in(X,τ).

Definition 2.3: A function $f: (X,\tau) \rightarrow (Y,\sigma)$ is called

1. pg**-irresolute[6] if $f^{-1}(V)$ is a pg**-closed set of (X,τ) for every pg**-closed set V of (Y,σ).
2. pg**-continuous[6] if $f^{-1}(V)$ is a pg**-closed set of $(X,\tau)$for every closed set V of (Y,σ).
3. pg**-resolute[6] if $f(U)$ is pg**- open in Y whenever U is pg**- open in X.

Definition 2.4: An ideal [2] I on a nonempty set X is a collection of subsets of X which satisfies the following properties.(i) $A \in I$, B $\in I \Longrightarrow A \cup B \in I$ $(ii) A \in I, B \subset A \Longrightarrow B \in I$. A topological space (X,τ) with an ideal Ion X is called an ideal topological space and is denoted by (X,τ,I).

3. pg**- Connected space

Definition 3.1: Let X be a topological space. A pg**-separation of X is a pair A and B of disjoint nonempty pg**- open subsets of X whose union is X. The space X is said to be pg**- Connected if there does not exist a pg**-separation of X. If there exist a pg**-separation then X is said to be pg**-disconnected.

Note: If $X = A \cup B$ is a pg**-separation then $A^c = B$ and $B^c = A$ and hence A and B are pg**- closed.

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Remark 3.2: A space X is pg**- connected if and only if the only subsets of X that are both pg**- open and pg**- closed in X are the empty set and X itself.

Proof is obvious.

Example 3.3: An infinite set with finite complement topology is pg**- connected since it is impossible to find two disjoint pg**- open sets.

Example 3.4: Any indiscrete topological space $(X, \tau)$with more than one point is pg**- disconnected since every subset is pg**- open.

Theorem 3.5: Every pg**- connected space is connected but not conversely.

Proof: Obvious, since every open set is pg**- open.

Theorem 3.6: Every pg**- connected space is g**- connected but not conversely.

Proof: Obvious, since every g**- open set is pg**- open.

Example 3.7: The space in example (3.4) is connected but not pg**- connected.

Example 3.8: The space $X = \{a, b, c\}$ with topology $\tau = \{\varphi, X, \{a, c\}\}$ is g**-connected but not pg**- connected.

Example 3.9: $\mathbb{R}$ withusualtopology is connected and g**- connected but not pg**- connected.

Since $\mathbb{Q}$ and $\mathbb{Q}^c$ are pg**- open but not open and g**- open.

Theorem 3.10: Let(X, τ) be a topological space. The following conditions are equivalent:

(i) X is pg**- connected.

(ii) If A and B are disjoint pg**- open subsets of X with $X = A \cup B$, then either $A = \varphi$ (hence $B = X$) or $B = \varphi$ (hence $A = X$).

(iii) If C and D are disjoint pg**- closed subsets of X with $X = C \cup D$, then either $C = \varphi$ (hence $D = X$) or $D = \varphi$ (hence $C = X$).

Proof:
$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$: Let X be pg**- connected and let A and B be pg**- open subsets of X with $X = A \cup B$ and $A \cap B = \varphi$. Since $A = X \setminus B$, A is also pg**- closed, so either $A = \varphi$ or $A = X$, (ii) follows.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$: Assume (ii) and let G be a subset of X which is both pg**- open and pg**- closed and hence $X \setminus$ G is also both pg**- open and pg**- closed. Since $X = G \cup X \setminus$ G, (ii) gives that either $G = \varphi$ or $G = X$.

$(\boldsymbol{ii}) \Leftrightarrow (\boldsymbol{iii})$: This follows from the fact that if A and B are disjoint pg**- open sets with $X = A \cup B$, then A and B are also pg**- closed. Similarly if A and B are disjoint pg**- closed sets with $X = A \cup B$, then A and B are also pg**- open.

Definition 3.11: Let Y be a subset of a topological space X. A pg**- separation of Y is a pair of disjoint nonempty pg**- open subsets$A$ and B of X whose union is Y. The space Y is said to be pg**- connected if there does not exist a pg**- separation of.Y is said to be pg**- disconnected if there exist a pg**- separation of Y.

Theorem 3.12: If the sets A and B form a pg**-separation of X, and if Y is a pg**- open and pg**- connected subset of X, then Y lies entirely within either A or B.

Proof: $X = A \cup B$ is a pg**- separation of X. Suppose Y intersects both A and B then
$Y = (A \cap Y) \cup (B \cap Y)$ is a pg**- separation of Y which is a contradiction.

Theorem 3.13: Let C be a pg**-connected subset of a topological space X and let D be a subset such that $C \subset D \subset pg^{**}cl(C)$, then D is pg**-connected.

Proof: Suppose D is pg**-disconnected, then $D = A \cup B$ is a pg**-separation of D. Since C is pg**-connected and $C \subset D = A \cup B$, then either $C \subset A$ or $C \subset B$. To be specific, that C is disjoint from B. This implies $pg^{**}cl(C) \cap B = \varphi$, and $D \subset pg^{**}cl(C)$. Therefore $D \cap B = \varphi$, this is not true. Hence D is pg**-connected.

Theorem 3.14: Let C be a pg**-connected subset of a topological space X. Then $pg ** cl(c)$ is also pg**-connected.

Proof follows from taking $D = pg ** cl(C)$ in theorem (3.13).

Theorem 3.15: If C is a pg**-dense subset of a topological space (X, τ) and if C is also pg**-connected, then X is pg**-connected.

Proof: Follows from $pg^{**}cl(C) = \mathrm{X}$.

Theorem 3.16: Let (X, τ) and (Y, σ) be two topological spaces and $f : (\mathrm{X}, \tau) \to (\mathrm{Y}, \sigma)$ be a function. Then,

1. f is onto, pg**- continuous and X is pg**- connected ⟹ Y is connected.
2. f is onto, continuous and X is pg**- connected ⟹ Y is connected.
3. f is strongly pg**- continuous and X is connected ⟹ Y is pg**- connected.
4. f is onto and pg**- resolute then Y is pg**- connected ⟹ X is connected.
5. f is a bijection and open then Y is pg**- connected ⟹X is connected.
6. f is onto, pg**- irresolute and X is pg**- connected ⟹ Y is pg**- connected.
7. f is a bijection and pg**- resolute then Y is pg**- connected ⟹ X is pg**- connected.

Proof: (1) Suppose $Y = A \cup B$ is a separation of Y then $X = f^{-1}(Y) = f^{-1}(A) \cup f^{-1}(B)$ is a pg**- separation of X which is a contradiction. Therefore Y is connected.

Proofs for (2) to (7) are similar to the above proof.

Remark 3.17: The property of being "pg**- connected" is a pg**- topological property. This follows from (6) and (7) of theorem (3.16).

Theorem 3.18: A topological space (X, τ) is pg**- disconnected if and only if there exists a pg**- continuous map of X onto discrete two point space $Y = \{0, 1\}$.

Proof: (X, τ) is pg**- disconnected and $Y = \{0, 1\}$ is a space with discrete topology. Let $X = A \cup B$ be a pg**- separation of X. Define $f : \mathrm{X} \to \mathrm{Y}$ such that $f(A) = 0$ and $f(B) = 1$. Obviously f is onto, pg**- continuous map.

Conversely, let $f : \mathrm{X} \to \mathrm{Y}$ be pg**- continuous, onto map. Then $X = f^{-1}(0) \cup f^{-1}(1)$ is a pg**- separation of X.

Theorem 3.19: The union of a collection $\{A_\alpha\}$ of pg**- connected subsets of X that have a point p in common is pg**- connected.

Proof: Let $\cup A_\alpha = B \cup C$ be pg**- separation of $\cup A_\alpha$. Then B and C are disjoint non empty pg**- open sets in X. $p \in \cap A_\alpha \Longrightarrow p \in B$ or $p \in C$. Assume that $p \in B$. Then by theorem (3.12), A_α lies entirely within B for all α (since $p \in B$). Therefore C is empty which is a contradiction.

Corollary 3.20: Let $\{A_n\}$ be a sequence of pg**- open pg**- connected subsets of X such that $A_n \cap A_{n+1} \neq \varphi$, for all n. Then $\cup A_n$ is pg**- connected.

Proof: This can be proved by induction on n. By theorem (3.14), the result is true for $n = 2$. Assume that the result to be true when n = k. Now to prove the result when $n = k + 1$. By the hypothesis $\bigcup_{i=1}^{k} A_i$ is pg**- connected. Now $(\bigcup_{i=1}^{k} A_i) \cap A_{k+1} \neq \varphi$. Therefore $\bigcup_{i=1}^{k+1} A_i$ is pg**- connected. By induction hypothesis the result is true for all n.

Corollary 3.21: Let $\{A_\alpha\}_{\alpha \in A}$ be an arbitrary collection of pg**-open pg**-connected subsets of X. Let A be a pg**- open pg**- connected subset of X. If $A \cap A_\alpha \neq \varphi$, forall α then $A \cup (\cup A_\alpha)$ is pg**- connected.

Proof: Suppose that $A \cup (\cup A_\alpha) = B \cup C$ be a pg**- separation of the subset $A \cup (\cup A_\alpha)$. Since $A \subseteq B \cup C$, by theorem (3.10) $A \subseteq B$ or $A \subseteq C$. Without loss of generality assume that $A \subseteq B$. Let $\alpha \in A$ be arbitrary. $A_\alpha \subseteq B \cup C \Longrightarrow A_\alpha \subseteq B$ or $A_\alpha \subseteq C$. But $A \cap A_\alpha \neq \varphi \Longrightarrow A_\alpha \subseteq B$. Since α is arbitrary, $A_\alpha \subseteq B, \forall$. Hence $A \cup (\cup A_\alpha) \subseteq B$, contradicting the fact that C is nonempty. Therefore $A \cup (\cup A_\alpha)$ is pg**- connected.

Definition 3.22: A space (X, τ) is said to be *totally pg**- disconnected* if its only pg**- connected subsets are one point sets.

Example 3.23: Let (X, τ) be an indiscrete topological space with more than one point. Here all subsets are pg**- open. If $A = \{x_1, x_2\}$ then $A = \{x_1\} \cup \{x_2\}$ is a pg**-separation of A. Therefore any subset with more than one point is pg**- disconnected. Hence (X, τ) is totally pg**- disconnected.

Example 3.24: An infinite set with finite complement topology is not totally pg**- disconnected.

Remark 3.25: Totally pg**- disconnectedness implies pg**- disconnectedness.

Definition 3.26: A point $x \in X$ is said to be in pg**-boundary of A ($pg^{**}Bd(A)$) if every pg**- open set containing x intersects both A and $X - A$.

Example 3.27: Any infinite subset A of $\mathbb{R}$ whose complement is also infinite has every real number as its pg**- boundary point.

Theorem 3.28: Let (X, τ) be a topological space and let A be a subset of X. If C is pg**- open pg**- connected subset of X that intersects both A and $X - A$ then C intersects $pg^{**}Bd(A)$.

Proof: Given that $C \cap A \neq \varphi$ and $C \cap A^c \neq \varphi$. Now $C = (C \cap A) \cup (C \cap A^c)$ is a nonempty disjoint union. Suppose both are pg**- open then it is a contradiction to the fact that C is pg**- connected. Hence either $C \cap A$ or $C \cap A^c$ is not pg**- open. Suppose that $C \cap A$ is not pg**- open. Then there exist $x \in C \cap A$ which is not pg**-interior point of $C \cap A$. Let U be a pg**- open set containing x. Then $U \cap C$ is a pg**- open set containing x and hence $(U \cap C) \cap (C \cap A)^c \neq \varphi$. This implies U intersects both A and A^c and therefore $x \in pg^{**}Bd(A)$. Hence $C \cap pg^{**}Bd(A) \neq \varphi$.

Next we extend the intermediate value theorem for pg**- connected space.

Theorem 3.29: (Generalisation of Intermediate value theorem) Let $f : X \to \mathbb{R}$ be a pg**- continuous map, where X is pg**- connected space and $\mathbb{R}$ with usual topology. If x, y are two points of X and $a = f(x)$ and $b = f(y)$ then for every real number r between a and b, there exists a point c of X such that $f(c) =$ r.

Proof: Assume the hypothesis of the theorem. Suppose there is no point c of X such that $f(c) =$ r, then $A = (-\infty, r)$ and $B = (r, \infty)$ are disjoint open sets in $\mathbb{R}$ and $X = f^{-1}(A) \cup f^{-1}(B)$ which is a pg**-separation of X, contradicting the fact that X is pg**- connected. Therefore there exists $c \in X$ such that $f(c) =$ r.

Remark 3.30: The above theorem holds even if,

- f is continuous and X is pg**- connected.
- f is pg**- irresolute and X is pg**- connected.
- f is strongly pg**- continuous and X is pg**- connected.
- f is strongly pg**- continuous and X is connected.

4. pg**-components

Definition 4.1: Let(X, τ) be a topological space. Define an equivalence relation on X by setting $x \sim y$ if and only if there exists a pg**- connected subset of X containing both x and y. The equivalence classes are called *pg**- components* of X. pg**- component containing x is denoted by $C_x = \{y \in X / \ y \sim x\}$.

(i) $x \sim x$, since $\{x\}$ is pg**- connected. Hence $\sim$ is reflexive.

(ii) If $x \sim y$, then there exists a pg**- connected subset of X containing both x and y and hence $y \sim x$. Therefore $\sim$ is symmetric.

(iii) Let $x \sim y$ and $y \sim z$. Then there exists a pg**- connected subset $A$of X containing both x and y and a pg**- connected subset B of X containing both $y$and z. Since A and B are pg**- connected have a point y in common $A \cup B$ is a pg**- connected subset of X containing x, y and z. Therefore $\sim$ is transitive.

Example 4.2: Let (X, τ) be an indiscrete topological space with more than one point. Then each pg**- component of X consists of a single point.

Theorem 4.3: Any two pg**- components are either identical or disjoint.

Proof: Follows from the definition of pg**- component.

Theorem 4.4: The pg**-components of X are pg**-connected subsets of X whose union is X, such that each nonempty pg**-connected subset of X intersects only one of the pg**-components.

Proof: Each pg**- connected subset A of X intersects only one of the pg**- components. For, if A intersects the pg**- components C_1 and C_2 of X, say in points x_1 and x_2 then $x_1 \sim x_2$, this implies $C_1 = C_2$. To prove the pg**- component C is pg**- connected, choose a point $x_0 \in C$.

Now for every $x \in C$, $x_0 \sim x$. Therefore there exists a pg**- connected subset A_x containing x and x_0, implies $A_x \subset C$. Therefore $\underset{x \in C}{\overset{U}{}} A_x = C$. Since A_x are pg**- connected subsets having the point x_0 in common, C is pg**- connected.

Corollary 4.5: C_x is the union of all pg**- connected sets containing x.

Theorem: C_x is the largest pg**- connected set containing x. If there is another pg**- connected subset A of X such that $x \in A$, then $A \subset C_x$.

Proof: Let $t \in A \Longrightarrow x, t \in A$, where A is pg**- connected, this implies $t \sim x$. Therefore $t \in C_x$ and hence $\subset C_x$. Hence C_x is the largest pg**- connected set containing x.

Theorem 4.6: Let(X, τ) be a topological space, then the following are true.

(i) Each point in X is contained in exactly one pg**-component of X.
(ii) Each pg**- connected subset of X is contained in a pg**-component of X.
(iii) A pg**- connected subset of X which is pg**-clopen is a pg**-component of X.
(iv) If C is pg**- component of X then $C = pg^{**}cl(C)$. If (X, τ) is a pg**-multiplicative space then every pg**-component is pg**-closed.

Proof:

(i) Let $x \in X$ and consider the collection $\{C_i\}$ of all pg**- connected subsets of X containing x, this collection is non empty since $\{x\}$ itself is pg**- connected. $C = \cup\, C_i$ is a maximal pg**- connected subset of X which contains x and therefore a pg**-component of X. Suppose C^* is another pg**-component of X containing x, it clearly among the C_i's and is therefore contained in C, since C^* is also pg**-component we must have $C = C^*$.
(ii) A pg**- connected subset of X is contained in the pg**-component which contains any one of its points.
(iii) Let A be a pg**- connected subset of X which is pg**-clopen, then (by (ii)) A is contained in some pg**-component C. If A is a proper subset of C, then $C \cap A$ and $C \cap A^c$ forms a pg**-separation of C which is a contradiction to the fact that C, being a pg**-component, is pg**- connected. Therefore $A = C$.
(iv) If the pg**-component $C \neq pg^{**}cl(C)$ then its pg**-closure $(pg^{**}cl(C))$ is a pg**- connected (3.14) subset of X which properly contains C, this is the contradiction to the maximality of C as pg**- connected subset of X. Hence $C = pg^{**}cl(C)$. If X is a pg**-multiplicative space, then C is pg**-closed.

Theorem 4.7: Let X be a totally pg**- disconnected space. Then $C_x = \{x\}$, where C_x is a pg**-component of x.

Proof: Let X be a totally pg**- disconnected space, and then its only pg**- connected subsets are one point sets. Suppose $y \epsilon C_x$ such that $x \neq y$ then C_x is not pg**- connected which is contradiction to the fact that the pg**- components of X are pg**- connected subsets of X (4.4). Therefore in a totally pg**- disconnected space the pg**- component of $x$is$\{x\}$.

5. pg**- connected modulo *I*

Definition 5.1: Let (X, τ, I) be an ideal topological space then $X = A \cup B$ is said to be pg**-separation modulo I if A and B arenon empty pg**- open sub sets of X such that $A \cap B \in I$.(X, τ, I) is said to be pg**- connected modulo I if there is no pg**-separation modulo I for X.

Definition 5.2: Let Y be a subset of X. $Y = A \cup B$ is said to be pg**-separation modulo I of Y if A and B are non empty pg**- open sub sets of X and $A \cap B \in I$.

If there is no pg**-separation modulo I for Y then we say Y is pg**- connected modulo I subset.

Theorem 5.3: $X = A \cup B$ is pg**-separation of X implies $X = A \cup B$ is a pg**-separation modulo I of X for any ideal I.

Proof: It follows since $\varphi \in I$.

Theorem 5.4: (X, τ, I) is pg**- connected modulo I for some ideal I implies (X, τ) is pg**- connected. Equivalently If (X, τ) is pg**- disconnected then (X, τ, I) is pg**- disconnected modulo I for some ideal I.

Proof follows from theorem (5.3).

Remark 5.5: The converse is false as seen in the following example.

Example 5.6: Let(X, τ) be infinite cofinite topological space and $I = \wp(X)$. Then X is pg**- connected. On the other hand $X - \{x\}, X - \{y\}$ are pg**-open and non empty, and $(X - \{x\}) \cup (X - \{y\})$ is a pg**-separation modulo I of X. Therefore (X, τ, I) is not pg**- connected modulo I.

Theorem 5.7: Let (X, τ, I) be an ideal topological space, $X = A \cup B$ is a pg**-separation modulo I of X and Y is pg**-open pg**- connected subset of X modulo I then Y lies entirely within either A or B.

Proof: $X = A \cup B$ is a pg**-separation of X modulo I .Therefore A and B are non emptypg**- open sets and $A \cap B \in I$.

Now $Y = (Y \cap A) \cup (Y \cap B)$, $(Y \cap A)$ and $(Y \cap B)$ are pg**-open sets and $(Y \cap A) \cap (Y \cap B) = Y \cap (A \cap B) \in I$. If $(Y \cap A)$ and $(Y \cap B)$ are both non empty then $Y = (Y \cap A) \cup (Y \cap B)$ is a pg**-separation of Y modulo I which is a contradiction. Therefore $(Y \cap A) = \varphi$ or $(Y \cap B) = \varphi$ and hence Y lies entirely within either A or B.

Theorem 5.8: Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, J)$ be a bijection where $J = f(I)$, then

1. f is pg**- continuous and X is pg**- connected modulo $I \Longrightarrow Y$ is connected modulo J .
2. f is continuous and X is pg**- connected modulo $I \Longrightarrow Y$ is connected modulo J.
3. f is strongly pg**- continuous and X is connected $\Longrightarrow Y$ is pg**- connected modulo J .
4. f is pg**-resolute then Y is pg**- connected modulo $J \Longrightarrow X$ is connected modulo I.
5. f is a bijection and open then Y is pg**- connected modulo $J \Longrightarrow X$ is connected modulo I .
6. f is pg**- irresolute and X is pg**- connected modulo $I \Longrightarrow Y$ is pg**- connected modulo J.
7. f is pg**-resolute then Y is pg**- connected modulo $J \Longrightarrow X$ is pg**- connected modulo I .

Proof: (1) Assume that Y is not connected modulo J. Let $Y = A \cup B$ be a pg**- separation modulo J. Therefore A and B are nonempty pg**- open subsets of Y such that $A \cap B \in J$. Then $X = f^{-1}(A) \cup f^{-1}(B)$ is a pg**- separation modulo I since $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) \in I$ which is a contradiction. Therefore Y is connected modulo J.

Proofs for (2) to (7) are similar to the above proof.

REFERENCES

1. James R. Munkres, Topology, Ed.2, PHI Learning Pvt. Ltd. New Delhi, 2011.
2. K.Kuratowski, TopologyI.Warrzawa 1933.
3. N.Levine, Generalized closed sets in topology, Rend. Circ. Math. Palermo, 19(1970), 89-96.
4. A.S.Mashhour, M.E.Abd EI-Monsef and S.N.EI-Deeb, On pre-continuous and weak pre-continuous mappings, Proc. Math. And Phys. Soc. Egypt, 53(1982), 47-53.
5. Pauline Mary Helen M, g**-closed sets in Topological spaces, IJMA, 3(5), (2012), 1-15.
6. PunithaTharani. A, Priscilla Pacifica. G, pg**-closed sets in topological spaces, IJMA, 6(7), (2015), 128-137.
7. M.K.R.S. Veerakumar, Mem. Fac. Sci. Koch. Univ. Math., 21(2000), 1-19.

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