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SOME COMMON FIXED POINT THEOREM FOR CONE METRIC SPACE

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ABSTRACT

In this paper, we proof some fixed point and common fixed point theorem for Cone metric space.

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Let X be a Real Banach Space and P a subset of X, P is called a cone if P satisfy followings conditions;

(i) P is closed, nonempty and $P \neq 0$
(ii) $ax + by \in P$ for all $x, y \in P$ and non negative real numbers a, b
(iii) $P \cap (-P) = \{0\}$

Given a cone $P \subset X$, we define a partial ordering $\leq$ on X with respect to P by $y - x \in P$.

We shall write $x << y$ if $(y - x) \in \operatorname{int} P$, denoted by $\|.\|$ the norm on X. the cone P is called normal if there is a number $k > 0$ such that for all $x, y \in X$

$$0 \leq x \leq y \text{ implies that } \|x\| \leq k\|y\| \tag{A}$$

The least positive number k satisfying the above condition (A) is called the normal constant of P.

The authors showed that there is no normal cones with normal constant $M < 1$ and for each $k > 1$

there are cone with normal constant $M > k$.

The cone P is called regular if every increasing sequence which is bounded from the above is convergent, that is if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \ldots \ldots \ldots . \leq y$ for some $y \in X$,

then there is $x \in X \lim_{n\to\infty} \|x_n - x\| = 0$.

The cone P is regular iff every decreasing sequence which is bounded from below is convergent.

Definition: 1 let X be a nonempty set and X is a real Banach Space, d is a mapping from X into itself such that, d satisfying following conditions,

$d_1 : d(x, y) \geq 0 \;\; \forall x, y \in X$
d_2: $d(x, y) = 0$ iff $x = y$
d_3: $d(x, y) = d(y, x)$
d_4: $d(x, y) \leq d(x, z) + d(z, y)$

Then d is called a cone metric on X and (X, d) is called cone metric space.

Definition: 2 Let A and S be two mapping of a cone metric space (X, d) then it is said to be compatible if, $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

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$\lim_{n\to\infty} Ax_n = t$ and $\lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Let A and S be two self mapping of a cone metric space (X, d) then it is said to be weakly compatible, if they commute at coincidence point, that is $Ax = Sx$ implies that,

$ASx = SAx$ for $x \in X$.

It is easy to see that compatible mapping commute at there coincidence points. It is note that a compatible maps are weakly compatible but converges need not be true.

Theorem: 1.1 Let (X, d) be a complete cone metric space and P a normal cone with normal Constant k. Suppose that the mapping T, from X into itself satisfy the condition,

$$d(Tx, Ty) \le \alpha\ d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma\,[d(x, Ty) + d(y, Tx)] \tag{1}$$

For all $x, y \in X$ and $\alpha, \beta, \gamma \ge 0$ such that $0 \le \alpha + \beta + \gamma < 1$. Then T has unique fixed point in X.

Proof: For any arbitrary x_0, in X, we choose $x_1, x_2 \in X$ such that,

$$Tx_0 = x_1 \text{ and } Tx_1 = x_2$$

In general we can define a sequence of elements of X such that,

$$x_{2n+1} = Tx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}$$

Now,

$$d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n}, Tx_{2n+1})$$

From (1)

$$(Tx_{2n}, Tx_{2n+1}) \le \alpha\ d(x_{2n}, x_{2n+1}) + \beta\ [d(x_{2n}, Tx_{2n}) + d(x_{2n+1}Tx_{2n+1})] + \gamma\,[d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n})]$$

$$d(x_{2n+1}, x_{2n+2}) \le \alpha\ d(x_{2n}, x_{2n+1}) + \beta\ [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma\,[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})]$$

$$d(x_{2n+1}, x_{2n+2}) \le \alpha\ d(x_{2n}, x_{2n+1}) + \beta\ [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma\,.\,d(x_{2n}, x_{2n+2})$$

By using triangle inequality, we get,

$$d(x_{2n+1}, x_{2n+2}) \le \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right] d(x_{2n}, x_{2n+1})$$

Similarly we can show that,

$$d(x_{2n}, x_{2n+1}) \le \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right] d(x_{2n-1}, x_{2n})$$

In general we can write,

$$d(x_{2n+1}, x_{2n+2}) \le \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right]^{2n+1} d(x_0, x_1)$$

On taking $\left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right] = \theta$

$$d(x_{2n+1}, x_{2n+2}) \le \theta^{2n+1}\ d(x_0, x_1)$$

For $n \le m$, we have

$$d(x_{2n}, x_{2m}) \le d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \cdots \ldots \ldots \ldots \ldots + d(x_{2m-1}, x_{2m})$$

$$d(x_{2n}, x_{2m}) \le \{\theta^n + \theta^{n+1} + \theta^{n+2} + \cdots \ldots \ldots + \theta^m\}\ d(x_0, x_1)$$

$$d(x_{2n}, x_{2m}) \le \frac{\theta^n}{1-\theta}\ d(x_0, x_1)$$

$$\|d(x_{2n}, x_{2m})\| \le \frac{\theta^n}{1-\theta}\ k\,\|\,d(x_0, x_1)\,\| \quad \text{as } n \to \infty$$

$\lim_{n\to\infty}\|d(x_{2n}, x_{2m})\| \to 0$

In this way

$\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) \to 0$, as $n \to \infty$

Hence $\{x_n\}$ is a Cauchy sequence which converges to $u \in X$

Hence (X, d) is complete cone metric space.

Thus $x_n \to u$ as $n \to \infty$, $Tx_{2n} \to u$ and $T_{2n+1} \to u$ as $n \to \infty$,

u is fixed point of T in X.

Uniqueness: Let us assume that, v is another fixed point of T in X different from v. then,

$$Tu = u \text{ and } Tv = v$$

$$d(u, v) = d(Tu, Tv)$$

From (1)

$$d(Tu, Tv) \le \alpha\ d(u, v) + \beta\,[d(u, Tu) + d(v, Tv)] + \gamma\,[d(u, Tv) + d(v, Tu)]$$

$$d(Tu, Tv) \le (\alpha + 2\gamma).\, d(u, v)$$

Which contradiction u is unique fixed point of T in X.

Theorem: 2 Let (X, d) be a complete cone metric space and P a normal cone with normal constant k. Suppose that S and T, be the mapping from X into itself satisfies the condition,

$$d(Sx, Ty) \le \alpha\ d(x, y) + \beta[d(x, Sx) + d(y, Ty)] + \gamma\,[d(x, Ty) + d(y, Sx)] \tag{2}$$

For all $x, y \in X$ and non negative α, β, γ, such that $0 \le \alpha + \beta + \gamma < 1$. Then S and T have unique fixed point in X. further more if, $ST = TS$ then it have unique common fixed point in X.

Proof: For any arbitrary x_0, in X, we choose $x_1, x_2 \in X$ such that,

$$Sx_0 = x_1 \text{ and } Tx_1 = x_2$$

In general we can define a sequence of elements of X such that,

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}$$

Now,

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

From (1)

$$d(Sx_{2n}, Tx_{2n+1}) \le \alpha\ d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, Sx_{2n}) + d(x_{2n+1}Tx_{2n+1})] + \gamma\,[d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})]$$

$$d(x_{2n+1}, x_{2n+2}) \le \alpha\ d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + [d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})]$$

$$d(x_{2n+1}, x_{2n+2}) \le \alpha\ d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma\,.\, d(x_{2n}, x_{2n+2})$$

By using triangle inequality, we get,

$$d(x_{2n+1}, x_{2n+2}) \le \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right] d(x_{2n}, x_{2n+1})$$

Similarly we can show that,

$$d(x_{2n}, x_{2n+1}) \le \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right] d(x_{2n-1}, x_{2n})$$

In general we can write,

$$d(x_{2n+1}, x_{2n+2}) \le \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right]^{2n+1} d(x_0, x_1)$$

On taking $\left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right] = \theta$

$$d(x_{2n+1}, x_{2n+2}) \le \theta^{2n+1}\ d(x_0, x_1)$$

For $n \leq m$, we have

$$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \cdots \ldots \ldots \ldots \ldots .. + d(x_{2m-1}, x_{2m})$$

$$d(x_{2n}, x_{2m}) \leq \{\theta^{n} + \theta^{n+1} + \theta^{n+2} + \cdots \ldots \ldots .. + \theta^{m}\} \; d(x_0, x_1)$$

$$d(x_{2n}, x_{2m}) \leq \frac{\theta^n}{1-\theta} \; d(x_0, x_1)$$

$$\|d(x_{2n}, x_{2m})\| \leq \frac{\theta^n}{1-\theta} \; k \, \| d(x_0, x_1) \|$$

as $n \to \infty$, $\lim_{n\to\infty} \|d(x_{2n}, x_{2m})\| \to 0$

In this way

$$\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) \to 0 \quad \text{as } n \to \infty$$

Hence $\{x_n\}$ is a Cauchy sequence which converges to $u \in X$

Hence (X, d) is complete cone metric space.

Thus $x_n \to u$ as $n \to \infty$

$$Sx_{2n} \to u \quad \text{and} \quad T_{2n+1} \to u \quad \text{as} \quad n \to \infty$$

u is fixed point of S and T in X.

Since $ST = TS$ this give,

$$u = Tu = TSu = STu = Su = u$$

u is common fixed point of S and T.

Uniqueness: Let us assume that, v is another fixed point of S and T in X different from v. then,

$$Tu = u \text{ and } Tv = v \text{ also } Su = u \text{ and } Sv = v$$

$$vd(u, v) = d(Su, Tv)$$

From (2)

$$d(Su, Tv) \leq \alpha \; d(u, v) + \beta[d(u, Su) + d(v, Tv)] + \gamma\,[d(u, Tv) + d(v, Su)]$$

$$d(Su, Tv) \leq (\alpha + 2\gamma).\, d(u, v)$$

Which contradiction

u is unique fixed point of S and T in X.

Theorem: 3 Let (X, d) be a complete cone metric space and P a normal cone with normal constant k. Suppose that S, R and T, be the mapping from X into itself satisfies the condition,

$$d(SRx, TRy) \leq \alpha \; d(x, y) + \beta[d(x, SRx) + d(y, TRy)] + \gamma\,[d(x, TRy) + d(y, SRx)] \tag{3}$$

For all $x, y \in X$ and non negative α, β, γ, such that $0 \leq \alpha + \beta + \gamma < 1$. Then S, R and T has unique fixed point in X. furthermore either $SR = RS$ or $TR = RT$ then it have unique common fixed point in X.

Proof: For any arbitrary x_0, in X, we choose x_1, $x_2 \in X$ such that,

$$SRx_0 = x_1 \text{ and } TRx_1 = x_2$$

In general we can define a sequence of elements of X such that,

$$x_{2n+1} = SRx_{2n} \quad \text{and} \quad x_{2n+2} = TRx_{2n+1}$$

Now,

$$d(x_{2n+1}, x_{2n+2}) = d(SRx_{2n}, TRx_{2n+1})$$

From (3)

$$d(SRx_{2n}, TRx_{2n+1}) \leq d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, SRx_{2n}) + d(x_{2n+1}TRx_{2n+1})] + \gamma\,[d(x_{2n}, TRx_{2n+1}) + d(x_{2n+1}, SRx_{2n})\,]$$

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha\; d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma\,[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})\,]$$

$$d(x_{2n+1}, x_{2n+2}) \leq \alpha\; d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + \gamma\,.\,d(x_{2n}, x_{2n+2})$$

By using triangle inequality, we get,

$$d(x_{2n+1}, x_{2n+2}) \leq \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right] d(x_{2n}, x_{2n+1})$$

Similarly we can show that,

$$d(x_{2n}\,, x_{2n+1}) \leq \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right] d(x_{2n-1}, x_{2n}\,)$$

In general we can write,

$$d(x_{2n+1}, x_{2n+2}) \leq \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right]^{2n+1} d(x_0, x_{\,1})$$

On taking $\left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right] = \theta$

$$d(x_{2n+1}, x_{2n+2}) \leq \theta^{2n+1}\; d(x_0, x_{\,1})$$

For $n \leq m$, we have

$$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + \cdots \ldots \ldots \ldots \ldots .. + d(x_{2m-1}, x_{2m})$$

$$d(x_{2n}, x_{2m}) \leq \{\theta^{n} + \theta^{n+1} + \theta^{n+2} + \cdots \ldots \ldots .. + \theta^{m}\,\}\, d(x_0, x_1)$$

,

$$d(x_{2n}, x_{2m}) \leq \frac{\theta^n}{1-\theta}\, d(x_0, x_1)$$

$$\|d(x_{2n}, x_{2m})\| \leq \frac{\theta^n}{1-\theta}\; k\, \|\, d(x_0, x_1)\,\|$$

as $n \to \infty$, $\lim_{n\to\infty} \|d(x_{2n}, x_{2m})\| \to 0$

In this way

$$\lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) \to 0 \text{ as } n \to \infty$$

Hence $\{x_n\}$ is a Cauchy sequence which converges to $u \in X$

Hence (X, d) is complete cone metric space.

Thus $x_n \to u$ as $n \to \infty$

$$SRx_{2n} \to u \quad \text{and} \quad TR_{2n+1} \to u \quad \text{as} \quad n \to \infty$$

u is fixed point of S and T in X.

Since $ST = TS$ this give,

$$u = Tu = TSu = STu = Su = u$$

u is common fixed point of S and T.

Uniqueness: Let us assume that, v is another fixed point of S and T in X different from v. then,

$$Tu = u \text{ and } Tv = v \text{ also } Su = u \text{ and } Sv = v$$

$$d(u, v) = d(Su, Tv)$$

From (3)

$$d(Su, Tv) \leq \alpha\; d(u, v) + \beta[d(u, Su) + d(v, Tv)] + \gamma\,[d(u, Tv) + d(v, Su)\,]$$

$$d(Su, Tv) \leq (\alpha + 2\gamma).\; d(u, v)$$

Which contradiction

u is unique fixed point of S and T in X.

Theorem: 4 Let (X, d) be a complete cone metric space and P a normal cone with normal constant k. Suppose that A, B, S and T, be the mapping from X into itself satisfies the condition,
(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,
(ii) $\{A, S\}$ and $\{B, T\}$ are weakly compatible.
(iii) S or T is continuous.
(iv) $d(Ax, By) \leq \alpha\ d(Sx, Ty) + \beta[d(Sx, Ax) + d(Ty, By)] + \gamma\,[d(Sx, By) + d(Ty, Ax)]$

For all $x, y \in X$ and non negative α, β, γ, such that $0 \leq \alpha + \beta + \gamma < 1$. Then A, B, S and T have unique fixed point in X.

Proof: For any arbitrary x_0 in X we define the sequence $\{x_n\}$ and $\{y_n\}$ in X, such that,

$$Ax_{2n} = Tx_{2n+1} = y_{2n} \text{ and } Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$$

for all $n = 0, 1, 2, \ldots..$

Now

$$d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1})$$

From (iv)

$$D(Ax_{2n}, Bx_{2n+1}) \leq \alpha\ d(Sx_{2n}, Tx_{2n+1}) + \beta[d(Sx_{2n}, Ax_{2n}) + d(Tx_{2n+1}, Bx_{2n+1})] + \gamma\,[d(Sx_{2n}, Bx_{2n+1}) + d(Tx_{2n+1}, Ax_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq \alpha\, d(y_{2n-1}, y_{2n}) + \beta[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + \gamma\,[d(y_{2n-1}, y_{2n+1})$$

$$d(y_{2n}, y_{2n+1}) \leq \left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right]^{2n+1} d(y_0, y_1)$$

On taking $\left[\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right] = \theta$ and For $n \leq m$, we have

$$d(y_{2n}, y_{2m}) \leq \{\theta^n + \theta^{n+1} + \theta^{n+2} + \cdots \ldots \ldots .. + \theta^m\}\, d(y_0, y_1)$$

$$d(y_{2n}, y_{2m}) \leq \frac{\theta^n}{1-\theta}\ d(y_0, y_1)$$

$$\|d(y_{2n}, y_{2m})\| \leq \frac{\theta^n}{1-\theta}\ k\, \|d(y_0, y_1)\|$$

as $n \rightarrow \infty$

$$\lim_{n\rightarrow\infty} \|d(y_{2n}, y_{2m})\| \rightarrow 0$$

Hence $\{y_n\}$ is a Cauchy sequence which converges to $u \in X$, By the continuity of S and T $\{x_n\}$ is also convergent sequence which converges to $u \in X$, Hence (X, d) is complete cone metric space. u is fixed point of A, B, S and T.

Since $\{A, S\}$ and $\{B, T\}$ are weakly compatible, implies that u is common fixed point of A, B, S and T

Uniqueness: Let us assume that, v is another fixed point of A, B, S and T in X different from v. then,

$$Au = u \text{ and } Av = v \text{ also } Bu = u \text{ and } Bv = v$$

$$d(u, v) = d(Au, Bv)$$

From (iv)

$$d(Au, Bv) \leq \alpha\ d(Su, Tv) + \beta[d(Su, Au) + d(Tv, Bv)] + \gamma\,[d(Su, Bv) + d(Tv, Au)]$$

$$d(Au, Bv) \leq (\alpha + 2\gamma).\, d(u, v)$$

Which contradiction

u is unique fixed point of A, B, S and T in X

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