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EMBEDDED RELATIONS AND VARYING DISTANCE FUNCTION IN FUZZY METRIC SPACES

T. RAMESH*

H. No: 3-10-255, Reddy Colony, Hanamkonda – Warangal, India.

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ABSTRACT

In this present paper investigation on emended relations and varying distance function in fuzzy metric spaces,

Key words: *fixed point, fixed point theorem. Fuzzy metric space, implicitly relations.*

1. INTRODUCTION

In 1994, Mishra, Sharma and Singh [9] introduced the notion of compatible maps under the name of asymptotically commuting maps in FM-spaces. Singh and Jain [17] studied the notion of weak compatibility in FM-spaces (introduced by Jungck and Rhoades [6] in metric spaces). However, the study of common fixed points of non compatible maps is also of great interest. Pant [10] initiated the study of common fixed points of on compatible maps in metric spaces. In 2002, Aamri and Moutawakil [1] studied a new property for pair of maps i.e. the so-called property (E.A), which is a generalization of the concept of non compatible maps in metric spaces. Recently, Pant and Pant [11] studied the common fixed points of a pair of non compatible maps and the property (E. A) In FM-spaces.

Recently, implicit relations are used as a tool for finding common fixed point of contraction maps (see, [2], [8], [12], [13], [15], [16]). These implicit relations guarantee coincidence point of pair of maps that ultimately leads to the existence of common fixed points of a quadruple of maps satisfying weak compatibility criterion. In 2008, Altun and Turkoglu [3] proved two common fixed point theorems on complete FM-space with an implicit relation. In [3], common fixed point theorems have been proved for continuous compatible maps of type (α) or (β).

Our objective of this chapter is to prove a common fixed point theorem by removing the assumption of continuity, relaxing compatibility to compatible maps of type (α) or (β). weak compatibility and replacing the completeness of the space with a set of alternative conditions for functions satisfying an implicit relation in FM-space.

In our paper, we deal with implicit relation used in [3]. In [3], Altun and Turkoglu used the following implicit relation: Let $I = [0,1], *$ be a continuous t-norm and F be the set of all real continuous functions $F : I^6 \to R$ satisfying the following conditions

I. F is no increasing in the fifth and sixth variables,
II. if, for some constant $k \in (0,1)$ we have

(a) $F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1$, or

(b) $F\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right), 1\right) \geq 1$

for any fixed $t > 0$ and any nondecreasing functions $u, v : (0,\infty) \to I$ with $0 \leq u(t), v(t) \leq 1$ then there exists $h \in (0,1)$ with $u(ht) \geq v(t) * u(t)$, if, for some constant $k \in (0,1)$ we have

$$F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$$

for any fixed $t > 0$ and any nondecreasing function $u : (0,\infty) \to I$ then $u(kt) \geq u(t)$.

Lemma 1.1: In a fuzzy metric space $(X, M, \star)$ limit of a sequence is unique.

Lemma 1.2: Let $(X, M, \star)$ be a fuzzy metric space. Then

I. Then for all $x, y \in X$ $M(x, y, .)$ is a non decreasing function.
II. If there exists $k \in (0,1)$ such that for all $x, y \in X$, $M(x, y, kt) \geq M(x, y, t)$ $\forall t > 0$, then $x = y$.
III. If there exists a number $k \in (0,1)$ such that

$$M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t) \ \forall \ t > 0 \text{ and } n \in N$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Corresponding Author: T. Ramesh, E-mail: greengirl_ias2001@yahoo.com*

Definition 1.3: The only $t - \text{norm}\ \star$ satisfying $r \star r = r$ for all $r \in [0,1]$ is the minimum $t - \text{norm}$ that is
$$a \star b = \min\{a, b\} \text{ for all } a, b \in [0,1].$$

2. COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPS OF Type $(\boldsymbol{\beta})$ AND Type $(\boldsymbol{\alpha})$

In this section we prove a common fixed point theorem for compatible map of type (β) in fuzzy metric space. In fact we prove the following theorem.

Theorem 2.1: Let $(X, M, \star)$ be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:

2.1(a) $P(X) \subset ST(X)$ and $Q(X) \subset AB(X)$,

2.1(b) (P, AB) is compatible of type (β) and (Q, ST) is weak compatible,

2.1(c) there exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$
$$F\begin{pmatrix} M^2(Px, Qy, kt), M^2(ABx, STy, t), M^2(Px, ABx, t), \\ M^2(Qy, STy, t), M^2(Px, STy, t), M^2(ABx, Qy, t) \end{pmatrix} \geq 1$$
Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: Let $x_0 \in X$, then from (a) we have $x_1, x_2 \in X$ such that
$$Px_0 = STx_1 \text{ and } Qx_1 = ABx_2$$

Inductively, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that for $n \in N$
$$Px_{2n-2} = STx_{2n-1} = y_{2n-1} \text{ and } Qx_{2n-1} = ABx_{2n} = y_{2n}$$

Put $x = x_{2n}$ and $y = x_{2n+1}$ in (b) then we have
$$F\begin{pmatrix} M^2(Px_{2n}, Qx_{2n+1}, kt), M^2(ABx_{2n}, STx_{2n+1}, t), M^2(Px_{2n}, ABx_{2n}, t) \\ , M^2(Qx_{2n+1}, STx_{2n+1}, t), M^2(Px_{2n}, STx_{2n+1}, t), M^2(ABx_{2n}, Q_{2n+1}, t) \end{pmatrix} > 1$$

$$F\begin{pmatrix} M^2(y_{2n+1}, y_{2n+2}, kt), M^2(y_{2n}, y_{2n+1}, t), M^2(y_{2n+1}, y_{2n}, t), \\ M^2(y_{2n+2}, y_{2n+1}, t), M^2(y_{2n+1}, y_{2n+1}, t), M^2(y_{2n}, y_{2n+2}, t) \end{pmatrix} > 1$$

$$F\begin{pmatrix} M^2(y_{2n+1}, y_{2n+2}, kt), M^2(y_{2n}, y_{2n+1}, t), M^2(y_{2n+1}, y_{2n}, t), \\ M^2(y_{2n+2}, y_{2n+1}, t), M^2(y_{2n+1}, y_{2n+1}, t), M^2\left(y_{2n}, y_{2n+1}, \frac{t}{2}\right) \star M^2\left(y_{2n+1}, y_{2n+2}, \frac{t}{2}\right) \end{pmatrix} > 1$$

From condition (a) we have
$$M^2(y_{2n+1}, y_{2n+2}, kt) \geq M^2\left(y_{2n}, y_{2n+1}, \frac{t}{2}\right) \star M^2\left(y_{2n+2}, y_{2n+1}, \frac{t}{2}\right)$$

we have
$$M^2(y_{2n+1}, y_{2n+2}, kt) \geq M^2\left(y_{2n}, y_{2n+1}, \frac{t}{2}\right)$$
That is
$$M(y_{2n+1}, y_{2n+2}, kt) \geq M\left(y_{2n}, y_{2n+1}, \frac{t}{2}\right)$$

Similarly we have
$$M(y_{2n+2}, y_{2n+3}, kt) \geq M\left(y_{2n+1}, y_{2n+2}, \frac{t}{2}\right)$$

Thus we have
$$M(y_{n+1}, y_{n+2}, kt) \geq M\left(y_n, y_{n+1}, \frac{t}{2}\right)$$

$$M(y_{n+1}, y_{n+2}, t) \geq M\left(y_n, y_{n+1}, \frac{t}{2k}\right)$$

$$M(y_n, y_{n+1}, t) \geq M\left(y_0, y_1, \frac{t}{2^{nk}}\right) \to 1 \text{ as } n \to \infty,$$

and hence $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$ for all $t > 0$.

For each $\epsilon > 0$ and $t > 0$, we can choose $n_0 \in N$ such that
$$M(y_n, y_{n+1}, t) > 1 - \epsilon \text{ for all } n > n_0.$$

For any $m, n \in N$ we suppose that $m \geq n$. Then we have

$$M(y_n, y_m, t) \geq M\left(y_n, y_{n+1}, \frac{t}{m-n}\right) \star M\left(y_{n+1}, y_{n+2}, \frac{t}{m-n}\right) \star \ldots . \star M\left(y_{m-1}, y_m, \frac{t}{m-n}\right)$$

$$M(y_n, y_m, t) \geq (1-\epsilon) \star (1-\epsilon) \star \ldots \ldots \star (1-\epsilon)(m-n)\text{times}$$

$$M(y_n, y_m, t) \geq (1-\epsilon)$$

And hence $\{y_n\}$ is a Cauchy sequence in X.

Since $(X, M, \star)$ is complete, $\{y_n\}$ converges to some point $z \in X$. Also its subsequences converges to the same point $z \in X$.

That is

$$\{Px_{2n+2}\} \to z \text{ and } \{STx_{2n+1}\} \to z \qquad 2.1\text{ (i)}$$

$$\{Qx_{2n+1}\} \to z \text{ and } \{ABx_{2n}\} \to z \qquad 2.1\text{(ii)}$$

As (P, AB) is compatible pair of type (β), we have

$$M(PPx_{2n}, (AB)(AB)x_{2n}, t) = 1, \quad \text{for all } t > 0$$

Or $\quad M(PPx_{2n}, ABz, t) = 1$

Therefore, $\quad PPx_{2n} \to ABz.$

Put $x = (AB)x_{2n}$ and $y = x_{2n+1}$ in 2.1(c) we have

$$F\begin{pmatrix} M^2(P(AB)x_{2n}, Qy, kt), M^2(AB(AB)x_{2n}, STx_{2n+1}, t), M^2(P(AB)x_{2n}, AB(AB)x_{2n}, t) \\ M^2(Qx_{2n+1}, STx_{2n+1}, t), M^2(P(AB)x_{2n}, STx_{2n+1}, t), M^2(AB(AB)x_{2n}, Qx_{2n+1}, t) \end{pmatrix} > 1$$

Taking $n \to \infty$ and 2.1(a) we get

$$M^2\big((AB)z, z, kt\big) \geq M^2\big((AB)z, z, t\big)$$

That is $\quad M\big((AB)z, z, kt\big) \geq M\big((AB)z, z, t\big)$

Therefore we have

$$ABz = z. \qquad 2.1\text{(iii)}$$

Put $x = z$ and $y = x_{2n+1}$ in 3.2.1(c) we have

$$F\begin{pmatrix} M^2(Pz, Q\,x_{2n+1}, kt), M^2(ABz, ST\,x_{2n+1}, t) \star M^2(Pz, ABz, t) \\ M^2(Q\,x_{2n+1}, ST\,x_{2n+1}, t), M^2(Pz, ST\,x_{2n+1}, t), M^2(ABz, Q\,x_{2n+1}, t) \end{pmatrix} > 1$$

Taking $n \to \infty$ (a) and using equation 2.1 (i) we have

That is $\quad M^2(Pz, z, kt) \geq M^2(Pz, z, t)$

And hence $\quad M(Pz, z, kt) \geq M(Pz, z, t)$

Therefore by using lemma 3.1.6, we get $Pz = z$

So we have $\quad ABz = Pz = z.$

Putting $x = Bz$ and $y = x_{2n+1}$ in 2.1(d), we get

$$F\begin{pmatrix} M^2(PBz, Qx_{2n+1}, kt), M^2(ABBz, STx_{2n+1}, t), M^2(PBz, ABBz, t) \\ M^2(Qx_{2n+1}, STx_{2n+1}, t), M^2(PBz, STx_{2n+1}, t), M^2(ABBz, Qx_{2n+1}, t) \end{pmatrix} > 1$$

Taking $n \to \infty$, (a) and using 2.1(i) we get

$$M^2(Bz, z, kt) \geq M^2(Bz, z, t)$$

That is $\quad M(Bz, z, kt) \geq M(Bz, z, t)$

Therefore by Lemma1.1.we have $\quad Bz = z$

And also we have $ABz = z$ implies $Az = z$

Therefore $Az = Bz = Pz = z$. 2.1 (iv)

As $P(X) \subset ST(X)$ there exists $u \in X$ such that

$$z = Pz = STu$$

Putting $x = x_{2n}$ and $y = u$ in 2.1(c) we get

$$F\begin{pmatrix} M^2(Px_{2n}, Qu, kt), M^2(ABx_{2n}, STu, t), M^2(Px_{2n}, ABx_{2n}, t) \\ M^2(Qu, STu, t), M^2(Px_{2n}, STu, t), M^2(ABx_{2n}, Qu, t) \end{pmatrix} > 1$$

Taking $n \to \infty$ and using 3.2.1(i) and 3.2.1(ii) we get

$$F\begin{pmatrix} M^2(z, Qu, kt), M^2(z, STu, t), M^2(z, z, t) \\ M^2(Qu, STu, t), M^2(z, STu, t), M^2(z, Qu, t) \end{pmatrix} > 1$$

$$M^2(z, Qu, kt) \geq M^2(z, Qu, t)$$

That is $M(z, Qu, kt) \geq M(z, Qu, t)$

we have $Qu = z$

Hence $STu = z = Qu$.

Hence (Q, ST) is weak compatible, therefore, we have

$$QSTu = STQu$$

Thus $Qz = STz$.

Putting $x = x_{2n}$ and $y = z$ in 2.1(c)we get

$$F\begin{pmatrix} M^2(Px_{2n}, Qz, kt), M^2(ABx_{2n}, STz, t), M^2(Px_{2n}, ABx_{2n}, t) \\ M^2(Qz, STz, t), M^2(Px_{2n}, STz, t), M^2(ABx_{2n}, Qz, t) \end{pmatrix} > 1$$

Taking $n \to \infty$ and using 2.1(ii) we get

$$F\begin{pmatrix} M^2(z, Qz, kt), M^2(z, STz, t), M^2(z, z, t) \\ M^2(Qz, STz, t), M^2(z, STz, t), M^2(z, Qz, t) \end{pmatrix} > 1$$

$$M^2(z, Qz, kt) \geq M^2(z, Qz, t)$$

And hence $M(z, Qz, kt) \geq M(z, Qz, t)$

we get $Qz = z$.

Putting $x = x_{2n}$ and $y = Tz$ in 2.1(c) we get

$$F\begin{pmatrix} M^2(Px_{2n}, QTz, kt), M^2(ABx_{2n}, STTz, t), M^2(Px_{2n}, ABx_{2n}, t) \\ M^2(QTz, STTz, t), M^2(Px_{2n}, STTz, t), M^2(ABx_{2n}, QTz, t) \end{pmatrix} > 1$$

As $QT = TQ$ and $ST = TS$ we have

$QTz = TQz = Tz$

And $ST(Tz) = T(STz) = TQz = Tz$.

Taking $n \to \infty$ we get

$$F\begin{pmatrix} M^2(z, Tz, kt), M^2(z, Tz, t), M^2(z, z, t) \\ M^2(Tz, Tz, t), M^2(z, Tz, t), M^2(z, Tz, t) \end{pmatrix} > 1$$

$$M^2(z, Tz, kt) \geq M^2(z, Tz, t)$$

Therefore $M(z, Tz, kt) \geq M(z, Tz, t)$

Therefore by Lemma 1.1. we have $Tz = z$

Now $STz = Tz = z$ implies $Sz = z$.

Hence $Sz = Tz = Qz = z$ 2.1(v)

Combining 2.1(iv) and 2.1(v) we have

$$Az = Bz = Pz = Sz = Tz = Qz = z$$

Hence z is the common fixed point of A, B, S, T, P and Q.

Uniqueness: Let u be another common fixed point of A, B, S, T, P and Q. Then

$$Au = Bu = Su = Tu = Pu = Qu = u$$

Putting $x = u$ and $y = z$ in 3.2.1(c) then we get

$$F\begin{pmatrix} M^2(Pu, Qz, kt), M^2(ABu, STz, t), M^2(Pu, ABu, t) \\ M^2(Qz, STz, t), M^2(Pu, STz, t), M^2(ABu, Qz, t) \end{pmatrix} > 1$$

Taking limit both side then we get

$$F\begin{pmatrix} M^2(u, z, kt), M^2(u, z, t), M^2(u, u, t) \\ M^2(z, z, t), M^2(u, z, t), M^2(u, z, t) \end{pmatrix} > 1$$

$$M^2(u, z, kt) \geq M^2(u, z, t)$$

And hence $M(u, z, kt) \geq M(u, z, t)$

we get $z = u.$

That is z is a unique common fixed point of A, B, S, T, P and Q in X.

Remark 3.2.2: If we take $B = T = I$ identity map on X in Theorem 2.1 then we get following Corollary

Corollary 2.1: Let $(X, M, \star)$ be a complete fuzzy metric space and let A, S, P and Q be mappings from X into itself such that the following conditions are satisfied:

2.1(a) $P(X) \subset S(X)$ and $Q(X) \subset A(X)$,

2.2(b) (P, A) is compatible of type (β) and (Q, S) is weak compatible,

2.3(c) there exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$

$$F\begin{pmatrix} M^2(Px, Qy, kt), M^2(Ax, Sy, t), M^2(Px, Ax, t), \\ M^2(Qy, Sy, t), M^2(Px, Sy, t), M^2(Ax, Qy, t) \end{pmatrix} \geq 1$$

Then A, S, P and Q have a unique common fixed point in X.

Remark 3.2.4: If we take weakly compatible mapping in place of compatible mapping of type (β) then we get following result.

Corollary 2.2: Let $(X, M, \star)$ be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:

2.1(a) $P(X) \subset ST(X)$ and $Q(X) \subset AB(X)$,

2.2(b) (P, AB) and (Q, ST) is are weak compatible,

2.3(c) there exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$

$$F\begin{pmatrix} M^2(Px, Qy, kt), M^2(ABx, STy, t), M^2(Px, ABx, t), \\ M^2(Qy, STy, t), M^2(Px, STy, t), M^2(ABx, Qy, t) \end{pmatrix} \geq 1$$

Then A, B, S,T, P and Q have a unique common fixed point in X.

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