A COMMON FIXED POINT THEOREM FOR SIX MAPPINGS

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ABSTRACT

Using notion of compatibility, weak compatibility and commutatively we have generalized fixed point theorem for six mappings satisfying rational inequality.

1. INTRODUCTION AND PRELIMINARIES

The concept of common fixed point theorem for commuting mapping was given by Jungck [4]. The notion of weak commutativity was introduced by Sessa [6]. Imdad and Khan [5] has proved a common fixed point theorem for six mappings which was extension of Fisher [1] and Jeong-Rhoades [3].

Definition 1.1 [6]: A pair of self-mapping \((A, B)\) on a metric space \((X, d)\) is said to be weakly commuting if

\[ d(ABx, BAx) \leq d(Bx, Ax) \]

for all \(x \in X\). Obviously, commuting mappings are weakly commuting but the converse is not necessarily true.

Definition 1.2[4]: A pair of self mappings \((A, B)\) of a metric space \((X, d)\) is said to be compatible if

\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = t \in X \]

Obviously, weakly commuting mappings are compatible but the converse is not necessarily true.

The following theorem is given by Fisher [1]

Theorem 1.1: Let \(S\) and \(T\) be tow self mappings of a complete metric space \((X, d)\) such that for all \(x, y \in X\) either

\[ d(Sx, Ty) \leq \frac{b \left[ d(x, Ty)^2 \right] + c \left[ d(y, Sx)^2 \right]}{d(x, Ty) + d(y, Sx)} \]

If \(d(x, Ty) + d(y, Sx) \neq 0, 0 \leq b, c, b + c < 1\) or

\[ d(Sx, Ty) = 0 \text{ if } d(x, Ty) + d(y, Sx) = 0 \]

If one of \(S\) or \(T\) is continuous than \(S\) and \(T\) have a unique common fixed point.

Motivated by Fisher [2] and Imdad and Khan [5], in the present paper, an extension of theorem 1.1 is generalized for power \(n\) by improving the contraction condition and choosing suitable weak commutativity conditions.

2 MAIN RESULT

We prove the following

Theorem 2.1: Let \(A, B, S, T, I\) and \(J\) be self mappings of a complete metric space \((X, d)\) satisfying \(AB(X) \subseteq J(X), ST(X) \subseteq I(X)\) and for each \(x, y \in X\) either

\[ d(Sx, Ty) \leq \frac{b \left[ d(x, Ty)^2 \right] + c \left[ d(y, Sx)^2 \right]}{d(x, Ty) + d(y, Sx)} \]

If \(d(x, Ty) + d(y, Sx) \neq 0, 0 \leq b, c, b + c < 1\) or

\[ d(Sx, Ty) = 0 \text{ if } d(x, Ty) + d(y, Sx) = 0 \]

If one of \(S\) or \(T\) is continuous than \(S\) and \(T\) have a unique common fixed point.
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\[ d \left( ABx, STy \right) \leq \alpha_i \left[ \left( \frac{d \left( ABx, Jy \right)}{d \left( STy, Ix \right)} \right)^n + \left( \frac{d \left( STy, Ix \right)}{d \left( ABx, Jy \right)} \right)^n \right]^n \]

\[ + \alpha_2 \left[ d \left( ABx, Ix \right) + d \left( STy, Jy \right) \right] + \alpha_3 d \left( Ix, Jy \right) \]

\[ d(x_{2n}, x_{2n+2}) = \left( \frac{d \left( ABx_{2n+1}, Ix_{2n+1} \right)}{d \left( STy_{2n+1}, Jy_{2n+2} \right)} \right)^n \]

\[ \leq \alpha_i \left[ \left( \frac{d \left( ABx_{2n}, Jx_{2n} \right)}{d \left( STy_{2n}, Ix_{2n+1} \right)} \right)^n + \left( \frac{d \left( STy_{2n}, Ix_{2n+1} \right)}{d \left( ABx_{2n}, Jx_{2n} \right)} \right)^n \right]^n \]

\[ + \alpha_2 \left[ d \left( ABx_{2n} \right) + d \left( STy_{2n} \right) \right] + \alpha_3 d \left( Ix_{2n}, Jy_{2n+1} \right) \]

\[ d(x_{2n}, x_{2n+2}) \leq \frac{\alpha_i + \alpha_2 + \alpha_3}{1 - \alpha_i - \alpha_2} d(x_{2n}, x_{2n+2}) \]

Thus for every \( n \) we have \( d(x_n, x_{n+1}) \leq kd \left( z_{n-1}, z_n \right) \)

which shows that \( \{ x_n \} \) is a cauchy sequence in the complete metric space \( (X, d) \) and so has a limit point \( z \) in \( X \). hence the sequences \( ABx \) and \( STy \) are subsequences also converge to the point \( z \).

Let us now assume that \( I \) is continuous so that the sequence \( \{ I^2 x_n \} \) and \( \{ I ABx_n \} \) converges to \( I^2 \). Also in view of compatibility of \( \{ I, AB \} \), \( \{ ABIx_n \} \) converges to \( I^2 \).
\[d(ABix_{2n}, STx_{2n+1}) \leq \alpha_i \left[ \frac{d(ABix_{2n}, Jx_{2n+1})^n}{d(ABix_{2n}, Jx_{2n+1})^{n-1}} + \frac{d(STx_{2n+1}, I^2x_{2n})^n}{d(STx_{2n+1}, I^2x_{2n})^{n-1}} \right] + \alpha_3 \left[ d(ABz, Iz) + d(STx_{2n+1}, Jx_{2n+1}) \right] \]

Which on letting \( n \to \infty \) reduces to \((1 - \alpha_1 - \alpha_3) d(Iz, z) \leq 0\)

yielding thereby \( Iz = z \)

Now,
\[d(ABz, STx_{2n+1}) \leq \alpha \left[ \frac{d(ABz, Jx_{2n+1})^n}{d(ABz, Jx_{2n+1})^{n-1}} + \frac{d(STx_{2n+1}, Iz)^n}{d(STx_{2n+1}, Iz)^{n-1}} \right] + \alpha_2 \left[ d(ABz, Iz) + d(STx_{2n+1}, Jx_{2n+1}) \right] + \alpha_3 \left[ d(Iz, Jx_{2n+1}) \right] \]

On letting and using \( Iz = z \) we get
\[d(ABz, z) \leq (\alpha_1 + \alpha_2) d(ABz, z)\]

This implies \( ABz = z \)

Since \( AB(x) \subset J(x) \) then there always exists a point \( z' \) such that
\( Jz' = z \) so that \( STz = ST(Jz') \)

Now
\[d(z, STz) = d(ABz, STz) \]
\[\leq \alpha_i \left[ \frac{d(ABz, Jz')^n}{d(ABz, Jz')^{n-1}} + \frac{d(STz', Iz)^n}{d(STz', Iz)^{n-1}} \right] + \alpha_2 \left[ d(ABz, Iz) + d(STz', Jz') \right] + \alpha_3 \left[ d(Iz, Jz) \right] \]
\[\leq (\alpha_1 + \alpha_2) d(STz', z)\]

Hence, \( STz' = z = Jz' \) which shows that \( z' \) is a common point of \( AB, I, ST \) and \( J \). Now using the weak compatibility of \((ST, J)\), we have \( STz = ST(Jz') = J(STz') = Jz \) which shows that \( z \) is also a coincidence point of the pair \((ST, J)\). Now
\[d(z, STz) = d(ABz, STz) \]
\[\leq \alpha_i \left[ \frac{d(ABz, Jz)^n}{d(ABz, Jz)^{n-1}} + \frac{d(STz, Iz)^n}{d(STz, Iz)^{n-1}} \right] + \alpha_2 \left[ d(ABz, Iz) + d(STz, Jz) \right] + \alpha_3 \left[ d(Iz, Jz) \right] \]
\[\leq (\alpha_1 + \alpha_2) d(z, STz)\]

Hence \( z = STz = Jz \) which shows that \( z \) is a common fixed point of \( AB, I, ST \) and \( J \).

Now suppose that \( AB \) is continuous so that the sequences \( \{AB^2x_{2n}\} \) and \( \{ABix_{2n}\} \) converge to \( ABz \), since \((AB, I)\) are compatible it follows that \( \{IABx_{2n}\} \) also converge to \( ABz \), thus
\(d \left( AB_{2}x_{2n}, STx_{2n+1} \right) \leq \alpha_{i} \left[ \frac{d \left( AB_{2}x_{2n}, Jx_{2n+1} \right) + d \left( STx_{2n+1}, IABx_{2n} \right)}{d \left( AB_{2}x_{2n}, Jx_{2n+1} \right) + d \left( STx_{2n+1}, IABx_{2n} \right)} \right]^{n} + \frac{d \left( STz_{2n+1}, IABx_{2n} \right)}{d \left( STz_{2n+1}, IABx_{2n} \right)}^{n-1} + \alpha_{2} \left[ d \left( AB_{2}x_{2n}, IABx_{2n} \right) + d \left( STz_{2n+1}, Jx_{2n+1} \right) \right] + \alpha_{3} d \left( IABx_{2n}, Jx_{2n+1} \right)\)

which on letting \( n \to \infty \) reduces to
\(d \left( ABz, z \right) \leq \left( \alpha_{i} + \alpha_{3} \right) d \left( ABz, z \right)\)

which implies \( ABz = z \) as earlier, there exists \( z^{'} \) is \( X \) such that
\( ABz = z = Jz^{'} \) then
\(d \left( AB_{2}x_{2n}, STz^{'} \right) \leq \alpha_{i} \left[ \frac{d \left( AB_{2}x_{2n}, Jz^{'} \right) + d \left( STz^{'}- IAB_{2n} \right)}{d \left( AB_{2}x_{2n}, Jz^{'} \right) + d \left( STz^{'}- IAB_{2n} \right)} \right]^{n} + \frac{d \left( STz^{'}- IAB_{2n} \right)}{d \left( STz^{'}- IAB_{2n} \right)}^{n-1} + \alpha_{2} \left[ d \left( AB_{2}x_{2n}, IABx_{2n} \right) + d \left( STz^{'}- Jz^{'} \right) \right] + \alpha_{3} d \left( IABx_{2n}, Jz^{'} \right)\)

This on letting \( n \to \infty \) reduces to
\(d \left( z, STz^{'} \right) \leq \left( \alpha_{i} + \alpha_{3} \right) d \left( z, STz^{'} \right)\)

This gives \( STz^{'} = z = Jz^{'} \) thus \( z^{'} \) is a coincidence point of \( ST, J \). since, the pair \( ST, J \) is weakly compatible hence
\( STz = ST \left( Jz^{'} \right) = J \left( STz^{'} \right) = Jz^{'} \) which shows that \( STz = Jz^{'} \) further,
\(d \left( ABx_{2n}, STz \right) \leq \alpha_{i} \left[ \frac{d \left( ABx_{2n}, Jz \right) + d \left( STz, Ix_{2n} \right)}{d \left( ABx_{2n}, Jz \right) + d \left( STz, Ix_{2n} \right)} \right]^{n} + \frac{d \left( STz, Ix_{2n} \right)}{d \left( STz, Ix_{2n} \right)}^{n-1} + \alpha_{2} \left[ d \left( ABx_{2n}, Ix_{2n} \right) + d \left( STz, Jz \right) \right] + \alpha_{3} d \left( Ix_{2n}, Jz \right)\)

which on letting \( n \to \infty \) reduces to
\(d \left( z, STz \right) \leq \left( \alpha_{i} + \alpha_{3} \right) d \left( z, STz \right)\)
\(d \left( STz, z \right) = 0\)

it follows that \( STz = z = Jz \)

The point \( z \) therefore is in the range of \( ST \) and since \( ST \left( X \right) \subset I \left( X \right) \) there exist a point \( z^{'} \) in \( X \) such that
\( Iz^{'} = z \) thus
\(d \left( ABz^{''}, z \right) = d \left( ABz^{''}, STz \right) \leq \alpha_{i} \left[ \frac{d \left( ABz^{''}, Jz \right) + d \left( STz, Iz^{''} \right)}{d \left( ABz^{''}, Jz \right) + d \left( STz, Iz^{''} \right)} \right]^{n} + \frac{d \left( STz, Iz^{''} \right)}{d \left( STz, Iz^{''} \right)}^{n-1} + \alpha_{2} \left[ d \left( ABz^{''}, Iz^{''} \right) + d \left( STz, Jz \right) \right] + \alpha_{3} d \left( Iz^{''}, Jz \right)\)

Letting \( n \to \infty \)
\(d \left( ABz^{''}, z \right) \leq \left( \alpha_{i} + \alpha_{3} \right) d \left( ABz^{''}, z \right)\)
which shows that \( ABz^n = z \)

Also since \( (AB, I) \) are compatible and hence using weakly commuting we obtain

\[
d(ABz, Iz) = d(AB(Iz), I(ABz)) \\
\leq d(Iz^n, ABz^n) = d(z, z) = 0
\]

Therefore \( ABz = Iz = z \)

Thus we have proved that \( z \) is common fixed point of \( AB, ST, I \) and \( J \).

If the mappings \( ST \) or \( J \) is continuous instead of \( AB \) or \( I \) then proof of \( z \) is a common fixed point of \( AB, ST, I \) and \( J \) is similar.

Let \( v \) be another fixed point of \( I, J, AB \) and \( ST \)

then

\[
d(z, v) = d(ABz, STv) \\
\leq \alpha_1 \left[ \frac{d(ABz, Jv)^n + d(STv, Iz)^n}{d(ABz, Jv)^{n-1} + d(STv, Iz)^{n-1}} \right] \\
+ \alpha_2 \left[ d(ABz, Iz) + d(STv, Jv) \right] \\
+ \alpha_3 d(Iz, Jv)
\]

\[
d(z, v) \leq (\alpha_1 + \alpha_3) d(z, v) \text{ yielding thereby } z = v
\]

Finally we need to show that \( z \) is also a common fixed point of \( A, B, S, T, I \) and \( J \). For this let \( z \) be the unique common fixed point of both the pairs \( (AB, I) \) and \( (ST, J) \). Then

\[
Az = A(ABz) = A(BAz) = AB(Az), \quad Az = A(Iz) = I(Az)
\]

\[
Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz), \quad Bz = B(Iz) = I(Bz)
\]

which shows that \( Az \) and \( Bz \) is a common fixed point of \( (AB, I) \) yielding thereby \( Az = z = Bz = Iz = ABz \) in the view of uniqueness of the common fixed point of the pair \( (AB, I) \).

Similarly using the commutatively of \( (S, T), (S J) \) and \( (T, J) \) it can be shown that \( Sz = z = Tz = Jz = STz \).

Now we need to show that \( Az = Sz = Bz = Tz \) also remains a common fixed point of both the pairs \( (AB, I) \) and \( (ST, J) \). For this

\[
d(Az, Sz) = d(A(BAz), S(TSz)) \\
= d(AB(Az), ST(Sz)) \\
\leq \alpha_1 \left[ \frac{d(AB(Az), J(Sz))^n + d(ST(Sz), I(Az))^n}{d(AB(Az), J(Sz))^{n-1} + d(ST(Sz), I(Az))^{n-1}} \right]
\]
Similarly it can be shown that $B_z = T_z$. Thus $z$ is the unique common fixed of $A, B, S, T, I$ and $J$.

(ii) Suppose that $d \left( ABx, Jy \right) + d \left( STy, lx \right) = 0$ implies $d \left( ABx, STy \right) = 0$. Then we argue as follows.

Suppose that there exists an $n$ such that $z_n = z_{n+1}$. Then, also $z_{n+1} = z_{n+2}$. Suppose not. Then from (2.3) we have $0 < d \left( z_{n+1}, z_{n+2} \right) \leq k d \left( z_{n+1}, z_n \right)$ yielding thereby $z_n = z_{n+k}$ for $k = 1, 2, ...$. It then follows that there exist two point $w_1$ and $w_2$ such that $v_1 = ABw_1 = Jw_2$ and $v_2 = STw_2 = lw_1$. Since $d \left( ABw_1, Jw_2 \right) + d \left( STw_2, lw_1 \right) = 0$ from (2.2) $d \left( ABw_1, STw_2 \right) = 0$, i.e. $v_1 = ABw_1 = STw_2 = v_2$. Also note that $Jv_1 = I \left( ABw_1 \right) = AB \left( lw_2 \right) = ABv_1$. Similarly $STv_2 = Jv_2$. Define $y_1 = ABv_1, y_2 = STv_2$. Since $d \left( ABv_1, Jv_2 \right) + d \left( STv_2, Jv_2 \right) = 0$, it follow from (2.2) that $d \left( ABv_1, STv_2 \right) = 0$, i.e. $y_1 = y_2$. Thus $ABv_1 = Iv_1 = STv_2 = Jv_2$. But $v_1 = v_2$ follows $AB, I, ST$ and $J$ have a common coincidence point. Define $w = ABv_1$, it then follows that $w$ is also a common coincidence point of $AB, I, ST$ and $J$. If $ABw \neq ABv_1 = STv_1$, then $d \left( ABw, STv_1 \right) > 0$. But, since $d \left( ABw, Jv_1 \right) + d \left( STv_1, lw_1 \right) = 0$, it follows from (2.2) that $d \left( ABw, STv_1 \right) = 0$, i.e. $ABw = STv_1$, a contradiction. Therefore $ABw = ABv_1 = w$ and $w$ is a common fixed point of $AB, ST, I$ and $J$.

The rest of the proof is identical to the case (1), hence it is omitted. This completes the proof.

**Corollary 2.2:** Theorem 2.1 remains true if contraction conditions (2.1.1) and (2.1.2) are replaced by any of the following conditions:

(i) Either $d \left( ABz, STy \right) \leq \alpha_1 \left[ \left[ d \left( ABx, Jy \right) \right]^n + \left[ d \left( STy, lx \right) \right]^n \right] + \alpha_2 \left[ d \left( ABx, lx \right) + d \left( STy, Jy \right) \right]$ if $d \left( ABx, Jy \right) + d \left( STy, lx \right) \neq 0$, $\alpha_1, \alpha_2 > 0, 2\alpha_1 + 2\alpha_2 < 1$ or

\[ d \left( ABx, STy \right) = 0 \text{ if } d \left( ABx, Jy \right) + d \left( STy, lx \right) = 0 \]

(ii) Either $d \left( ABx, STy \right) \leq \alpha_1 \left[ \left[ d \left( ABx, Jy \right) \right]^n + \left[ d \left( STy, lx \right) \right]^n \right] + \alpha_3 \left( lx, Jy \right)$ if $d \left( ABx, Jy \right) + d \left( STy, lx \right) \neq 0$, $\alpha_1, \alpha_3 > 0, 2\alpha_1 + \alpha_3 < 1$ or

\[ d \left( ABx, STy \right) = 0 \text{ if } d \left( ABx, Jy \right) + d \left( STy, lx \right) = 0 \]

(iii) Either $d \left( ABx, STy \right) \leq \alpha_1 \left[ \left[ d \left( ABx, Jy \right) \right]^n + \left[ d \left( STy, lx \right) \right]^n \right] \text{ if } d \left( ABx, Jy \right) + d \left( STy, lx \right) \neq 0$, $\alpha_1 > 0, \alpha_1 < 1/2$ or

\[ d \left( ABx, STy \right) = 0 \text{ if } d \left( ABx, Jy \right) + d \left( STy, lx \right) = 0 \]

(iv) $d \left( ABx, STy \right) \leq \alpha_1 \left[ d \left( ABx, Jy \right) + d \left( STy, Jy \right) \right]$
\( d \left( ABx, STy \right) \leq \alpha_1 \left[ d \left( ABx, Jy \right) + d \left( STy, lx \right) \right] \text{ if } \alpha_1 < \frac{1}{2} \)  \hspace{1cm} (E)

\( d \left( ABx, STy \right) \leq \alpha_2 \left[ d \left( ABx, lx \right) + d \left( STy, Jy \right) \right] \text{ if } \alpha_2 < \frac{1}{2} \)  \hspace{1cm} (F)

\( d \left( ABx, STy \right) \leq \alpha_3 \left[ d \left( ABx, lx \right) + d \left( STy, Jy \right) \right] \text{ if } \alpha_3 < 1 \)  \hspace{1cm} (G)

**Proof:** Corollaries corresponding to the contraction conditions (A), (B) and (C) can be deduced directly from Theorem 2.1 by choosing \( \alpha_1 = 0, \alpha_2 = 0 \) and \( \alpha_3 = \alpha_3 = 0 \), respectively. The corollary corresponding the contraction condition (D) also follows from Theorem 2.1 by noting that

\[
\frac{\left[ d \left( ABx, Jy \right) \right]^n + \left[ d \left( STy, lx \right) \right]^n}{\left[ d \left( ABx, Jy \right) \right]^{n-1} + \left[ d \left( STy, lx \right) \right]^{n-1}} \leq \frac{\left[ d \left( ABx, Jy \right) + d \left( STy, lx \right) \right]^n}{\left[ d \left( ABx, Jy \right) + d \left( STy, lx \right) \right]^{n-1}}
\]

Finally one may note that the contraction conditions (E), (F) and (G) are special cases of the contraction condition (D).

**REFERENCES**


