On b\# generalized Closed Sets in Topological Spaces  

K. ABSANA BANU*1, Dr. S. PASUNKILIPANDIAN2  

1M.phil scholar, Aditanar College of Arts and Science, Tiruchendur - (T.N), India.  

2Associate professor Department of Mathematics,  
Aditanar College of Arts and Science, Tiruchendur - (T.N), India.  

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ABSTRACT  

In this paper a new class of generalized closed sets, namely b\#g-closed sets is introduced in topological spaces. We prove that this class lies between the class of b#-closed sets and the class of bg- closed sets. Also we find some basic properties and characterizations of b\# g –closed sets. 

Keywords: g-closed, gb –closed sets, b#-closed sets, b\#g-closed set.  

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1. INTRODUCTION  


2. PRELIMINARIES  

Throughout this paper X denotes a topological space on which no separation axiom is assumed. For any subset A of X, cl(A) denotes the closure of A and int(A) denotes the interior of A in the topological space X. Further X \ A denotes the complement of A in X. 

The following definitions and results are very useful in the subsequent sections.  

**Definition 2.1** A subset A of a space X is called  
(i) $\alpha$-open [4] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ and $\alpha$-closed if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$,  
(ii) semi-open [8] if $A \subseteq \text{cl}(\text{int}(A))$ and semi-closed if $\text{int}(\text{cl}(A)) \subseteq A$,  
(iii) pre-open [4] if $A \subseteq \text{int}(\text{cl}(A))$ and pre-closed if $\text{cl}(\text{int}(A)) \subseteq A$,  
(iv) semi-pre-open [5] or $\beta$-open [1] if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$ and semi-pre-closed or $\beta$-closed if $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$,  
(v) regular open [7] if $A = \text{int}(\text{cl}(A))$ and regular closed if $A = \text{cl}(\text{int}(A))$.  

**Definition 2.2**: Let $(X,\tau)$ be a topological space and $A \subseteq X$. The $b^\#$-closure of $A$, denoted by $b^\#\text{cl}(A)$ and is defined by the intersection of all $b^\#$-closed sets containing $A$.  

**Definition 2.3**: Let $(X,\tau)$ be a topological space and $A \subseteq X$. The $b^\#$-interior of $A$, denoted by $b^\#\text{int}(A)$ and is defined by the union of all $b^\#$-open sets contained in $A$.  

**Corresponding Author: K. Absana banu*1,  
1M.phil scholar, Aditanar College of Arts and Science, Tiruchendur - (T.N), India.**
Definition 2.4: A subset A of space X is said to be
(i) $b$-open [4] if $A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$ and $b$-closed if $\text{cl}(\text{int}(A)) \cap \text{int}(\text{cl}(A)) \subseteq A$,
(ii) $b^g$-open [19] if $A = \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$ and $b^g$-closed if $A = \text{cl}(\text{int}(A)) \cap \text{int}(\text{cl}(A))$,
(iii) a p-set [17] if $\text{cl}(\text{int}(A)) \cap \text{int}(\text{cl}(A)) \subseteq A$,
(iv) a q-set [18] if $\text{int}(\text{cl}(A)) \subseteq \text{cl}(\text{int}(A))$,
(v) $\pi$-open [20] if A is a finite union of regular open sets.

Lemma 2.5 [5]: Let A be a subset of a space X. Then
(i) $\text{scl}(A) = A \cup \text{int}(\text{cl}(A))$,
(ii) $\text{pcl}(A) = A \cup \text{cl}(\text{int}(A))$,
(iii) $\text{spcl}(A) = A \cup \text{int}(\text{cl}(\text{int}(A)))$.

Definition 2.6: A subset A of a space X is called
(i) generalized closed [9] (briefly g-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X,
(ii) generalized semi-pre-closed [6] (briefly gsp-closed) if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X,
(iii) $\pi$-generalized pre-closed [15] (briefly $\pi$ gp-closed) if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\pi$-open in X,
(iv) regular weakly generalized closed [13] (briefly rwg-closed) if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X,
(v) generalized $b$-closed set [2] (briefly gb-closed) if $\text{bcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X,
(vi) regular generalized $b$-closed set [11] (briefly rgb-closed) if $\text{bcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X,
(vii) $\pi$-generalized $b$-closed set [3] (briefly $\pi$gb-closed) if $\text{bcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\pi$-open in X,
(viii) $\pi$-generalized $b^g$-closed set [10] (briefly $\pi$gb$^g$-closed) if $\text{bcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $\pi$-open in X,
(ix) regular generalized $b^g$-closed set [14] (briefly rg-$b^g$-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X,
(x) $\pi$-generalized $b^g$-closed set [3] (briefly $\pi$gb$^g$-closed) if $\text{int}(\text{bcl}(A)) \subseteq U$ whenever $A \subseteq U$ and U is $\pi$-open in X.

The complements of the above mentioned closed sets are their respective open sets.

Remark 2.7:

\[ \text{Regular-closed} \]
\[ \downarrow \]
\[ \text{Closed} \quad \text{b-closed} \quad \text{gb-closed} \quad \text{spcl}(A) \]

Lemma 2.8[4]: Let A be a sub set of a space X. Then $\text{bcl}(A) = \text{scl}(A) \cup \text{pcl}(A)$.

3. $b^g$-generalized closed set:

Definition 3.1: Let X be a space. A subset A of X is called $b^g$-generalized closed (briefly $b^g$g-closed) if $b^g \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $b^g$-open.

Theorem 3.2: Every $b^g$-closed set is $b^g$g-closed.

Proof: Let A be a $b^g$-closed set in X. Let $A \subseteq U$ where U is $b^g$-open. Since A is $b^g$-closed, $b^g \text{cl}(A) = A \subseteq U$. Thus we have $b^g \text{cl}(A) \subseteq U$. Therefore A is $b^g$g-closed set.

Remark 3.3: The converse of the above Theorem need not be true.

Example 3.4: Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}, \{a, c, d\}, X\}$. Consider $A = \{b\}$. A is not a $b^g$-closed, However A is a $b^g$g-closed.

Theorem 3.5: Every $b^g$g-closed set is gb-closed.

Proof: Let A be $b^g$g-closed set in X. Let $A \subseteq U$ where U is open. Thus U is $b^g$-open. Since A is $b^g$g-closed, $b^g \text{cl}(A) \subseteq U$. But $\text{bcl}(A) \subseteq b^g \text{cl}(A)$. Thus we have $\text{bcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is $b^g$-open. Therefore A is gb-closed set.
Remark 3.6: The converse of the above Theorem need not be true.

Example 3.7: Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}, \{a, c, d\}, X\}$. Consider $A = \{c\}$. $A$ is not a $b^g$-closed, However $A$ is a gb-closed.

Theorem 3.8: Every $b^g$-closed set is $\pi$gb-closed.

Proof: proof is straightforward

Remark 3.9: The converse of the above theorem need not be true.

Example 3.10: Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}, \{a, c, d\}, X\}$. Consider $A = \{a\}$. $A$ is not a $b^g$-closed, However $A$ is a $\pi$gb-closed.

Theorem 3.11: Every $b^g$-closed set is rgb-closed.

Proof: proof is straightforward

Remark 3.12: The converse of the above theorem need not be true.

Example 3.13: Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}, \{a, c, d\}, X\}$. Consider $A = \{a\}$. $A$ is not a $b^g$-closed, However $A$ is a $\pi$gb-closed.

Theorem 3.14: The following example shows that $b^g$-closed sets independent from $\alpha$ -closed set, $g\alpha$-closed set, g-closed set, rg-closed set, rwg-closed set.

Example 3.15: Let $X = \{a, b, c, d\}$ and $Y = \{a, b, c, d\}$ be the topological spaces.

(i) consider $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, Y\}$. Then the set $\{c\}$ is an $\alpha$-closed set but not $b^g$-closed, and also the set $\{a\}$ is an $b^g$-closed but not $\alpha$-closed.

(ii) consider $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, Y\}$. Then the set $\{d\}$ is an $g\alpha$-closed set but not $b^g$-closed set in $X$, and also the set $\{b, c\}$ is an $b^g$-closed but not $g\alpha$-closed.

(iii) consider $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}, \{a, c, d\}, X\}$. Then the set $\{c, d\}$ is an g-closed set but not $b^g$-closed set in $X$, and also the set $\{d\}$ is an $b^g$-closed but not g-closed.

(iv) consider $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}, \{a, c, d\}, X\}$. Then the set $\{a, d\}$ is an rg-closed set but not $b^g$-closed set in $X$, and also the set $\{b\}$ is an $b^g$-closed but not rg-closed.

(v) consider $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}, \{a, c, d\}, X\}$. Then the set $\{b, c\}$ is an rwg-closed set but not $b^g$-closed set in $X$, and also the set $\{d\}$ is an $b^g$-closed but not rwg-closed.

Theorem 3.16: Let $A$ be a subset of a topological space $X$. Then $\text{cl}(\text{int}(A)) \cap \text{int}(\text{cl}(A)) \subseteq \text{bcl}(A) \subseteq b^g \text{cl}(A)$.

Proof: Obvious.

Theorem 3.17:

(i) If $A$ is a $p$-set, then $\text{cl}(\text{int}(A)) \subseteq b^g \text{cl}(A)$,

(ii) If $A$ is a $q$-set, then $\text{int}(\text{cl}(A)) \subseteq b^g \text{cl}(A)$,

(iii) If $A$ is a $t$-set, then $\text{int}(A) \subseteq b^g \text{cl}(A)$.

Proof: Let $A$ be a $p$-set. Then $\text{cl}(\text{int}(A)) \subseteq \text{int}(\text{cl}(A))$. That is $\text{cl}(\text{int}(A)) = \text{cl}(\text{int}(A)) \cap \text{int}(\text{cl}(A))$. Therefore by Theorem 3.16, $\text{cl}(\text{int}(A)) \subseteq \text{bcl}(A)$. This proves (i). Similarly the proof of (ii),(iii).
Remark 3.18:

\[ \text{A} \rightarrow \text{B} \text{ means A imply B.} \quad \text{A} \not\rightarrow \text{B} \quad \text{means A does not imply B.} \quad \text{A} \leftrightarrow \text{B} \quad \text{means A and B are independent.} \]

4. CHARACTERIZATION

**Theorem 4.1.** Suppose \( A \) is a p-set and \( b^\# g \)-closed. Then

(i) \( A \) is \( \pi gp \)-closed,
(ii) \( A \) is \( \pi gb^* \)-closed,
(iii) \( A \) is \( gsp \)-closed.

**Proof:** Let \( A \) be a p-set and \( b^\# g \)-closed in \( X \). Then by using Theorem 3.16 (i) \( \text{cl(int}(A)) \subseteq b^\# \text{cl}(A) \). Let \( A \subseteq U \) and \( U \) is \( \pi \)-open. Then \( b^\# \text{cl}(A) \subseteq U \). This implies \( \text{cl(int}(A)) \subseteq U \). That is \( A \cup \text{cl(int}(A)) \subseteq U \). Hence \( pcl(A) \subseteq U \). Hence \( A \) is \( \pi gp \)-closed. This proves (i). Similarly the Proof of (ii) and (iii).

**Theorem 4.2:** Suppose \( A \) is a q-set and \( b^\# g \)-closed. Then \( A \) is \( \pi gs \)-closed.

**Proof:** Let \( A \) be a q-set and \( b^\# g \)-closed in \( X \). Then by using Theorem 3.16 (ii) \( \text{int(cl}(A)) \subseteq b^\# \text{cl}(A) \). Let \( A \subseteq U \) and \( U \) is \( \pi \)-open. Then \( b^\# \text{cl}(A) \subseteq U \). This implies \( \text{int(cl}(A)) \subseteq U \). That is \( A \cup \text{int}(\text{cl}(A)) \subseteq U \). Hence \( scl(A) \subseteq U \). Hence \( A \) is \( \pi gs \)-closed.

**Remark 4.3:** Union and intersection of any two \( b^\# g \)-closed need not be \( b^\# g \)-closed.

**Example 4.4:** Let \( X = \{a, b, c, d\} \) with \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\} \). Then the sets \( \{a\} \) and \( \{b, c\} \) is \( b^\# g \)-closed but \( \{a, b, c\} \) is not \( b^\# g \)-closed. And also \( \{b, c\} \) and \( \{a, c, d\} \) is \( b^\# g \)-closed. But \( \{c\} \) is not \( b^\# g \)-closed.

**Theorem 4.5:** If \( A \) and \( B \) are two \( b^\# g \)-closed set in \( X \) such that either \( A \subseteq B = B \) or \( B \subseteq A \) both intersection and union of two \( b^\# g \)-closed set is \( b^\# g \)-closed.

**Proof:** Let \( A \) and \( B \) are two \( b^\# g \) closed set in a topological space \( X \). Since, \( A \subseteq B = B \subseteq A \) or \( A \cup B = B \). Since \( A \) and \( B \) are \( b^\# g \) closed sets then \( A \cup B = b^\# g \) closed. Similarly \( A \cap B = A \) or \( A \cap B \) then \( A \cap B \) is \( b^\# g \) closed.

**Theorem 4.6:** A set \( A \) is \( b^\# g \)-closed set if and only if \( b^\# \text{cl}(A) \subseteq A \) contains no non-empty \( b^- \)-closed sets.

**Proof:**

**Necessity:** Suppose that \( F \) is a non-empty \( b^- \)-closed subset of \( X \) such that \( F \subseteq b^\# \text{cl}(A) \). Then \( F \subseteq b^\# \text{cl}(A) \) and \( X \setminus F \) is \( b^- \)-open in \( X \). Since \( A \) is \( b^\# g \)-closed in \( X \), \( b^\# \text{cl}(A) \subseteq X \setminus F \), \( F \subseteq X \setminus b^\# \text{cl}(A) \). Thus \( F \subseteq b^\# \text{cl}(A) \cap (X \setminus b^\# \text{cl}(A)) = \emptyset \).
Sufficiency: A ⊆ U and U is b-open. Suppose b#cl(A) is not contain U, then b#cl(A) ∩ Uc is a non - empty b- closed set of b#cl(A):A, which is a contradiction. Therefore b#cl(A) ⊆ U and hence A is b#g-closed.

Theorem 4.7: If A is b#g closed. set and A⊆B⊆b#cl(A) then B is b#g closed. subset of X.

Proof: Let A be any b#g-closed. Set and B be any subset of X such that A⊆B⊆b#cl(A)

Let U be any b-open such that B⊆U. Since A⊆B, then A⊆U.Since A is b#g closed.

Then b#cl(A)⊆U. Since B⊆b#cl(A), then b#cl(B)⊆b#cl(A)⊆U.Therefore b#cl(A)⊆U. Hence B is b#g-closed.

Theorem 4.8: Let A be b#g-closed. Then A is b#g-closed if and only if b#cl(A):A is b#g closed.

Proof: Let A be a topological space (X, τ). Suppose A is b#g-closed. Then b#cl(A)=A. This implies b#cl(A):A=Φ, which is b#g closed. Conversely suppose that b#cl(A):A is b#g closed. Since A is b#g-closed, by above theorem 4.6, b#cl(A) does not contains any non-empty b-closed set. Therefore b#cl(A):A=Φ. Hence b#cl(A)=A. Thus A is b#g-closed.

Theorem 4.9: If a subset A of X is b#g-closed set in X then b#cl(A):A contains no non-empty Closed set.

Proof: using 4.6, we get the proof

Theorem 4.10: For every element x in a space X, X-{x} is a b#g- closed or b-open.

Proof: Suppose X-{x} is not b-open. Then X is the only b-open set containing X-{x}. This implies b#cl(X-{x})⊆X. Hence X-{x} is b#g closed.

Theorem 4.11: If A is both b-open and b#g-closed set in X, then A is b#g-closed set.

Proof: Since A is b-open and b#g-closed in X, b#cl(A)⊆A. But always A⊆b#cl(A). Therefore A= b#cl(A). Hence A is b#g-closed.

Theorem 4.12: Every subset is b#g-closed in X if and only if every b-open set is b#g-closed.

Proof: Let A be a b-open in X, by hypothesis A is b#g-closed in X. By theorem 4.11, A is a b#g-closed set conversely Let A be a subset of X and U a b-open set such that A⊆U. Then by hypothesis U is b#g-closed. This implies that b#cl(A)⊆b#cl(U)=U. Hence A is b#g-closed.

CONCLUSION

The present chapter has introduced a new concept called b#g-closed set in a topological spaces. It also analyzed some of properties. The implication shows the relationship between b#g-closed sets and the other existing sets.

REFERENCES