Pre* regular generalized closed sets in Topological spaces

C. SURIYAKALA*1, S. SARANYA2

1M.Phil Scholar, 2Assistant Professor, 
Department of Mathematics, Aditanar College, Tiruchendur - 628215, India.

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ABSTRACT

In this paper, we introduce a new class of sets namely, p*rg-closed sets in topological spaces. Also we study some of its basic properties and investigate the relationship with the other existing closed sets in topological space. It has been proved that the class of pre* regular generalized closed set lies between the class of regular closed set and g-closed set.

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1. INTRODUCTION


In this paper, the concept of pre* regular generalized closed sets is to be introduced and studied some of their properties. It has been proved that the class of pre* regular generalized closed set lies between the class of regular closed set and g-closed set.

2. PRELIMINARIES

Throughout this paper (X, \(\tau\)) represents a topological space on which no separation axiom is assumed unless otherwise mentioned. For a subset A of a topological space X, cl(A), int(A) and X\(\setminus\)A denotes the closure, the interior and the complement of A in X respectively.

We recall the following definitions and results.

**Definition 2.1:** Let (X, \(\tau\)) be a topological space. A subset A of X is said to be generalized closed [8] (briefly g-closed) set if cl(A) \(\subseteq\) U whenever A \(\subseteq\) U and U is open in (X, \(\tau\)).

**Definition 2.2:** Let (X, \(\tau\)) be a topological space and A \(\subseteq\) X. The generalized closure of A is denoted by cl*(A) and is defined by the intersection of all g-closed sets containing A and the generalized interior of A denoted by int*(A) and is defined by union of all g-open sets contained in A.

Corresponding Author: C. Suriyakala*1, 1M.phil Scholar, Department of Mathematics, Aditanar College, Tiruchendur -628215, India.
Definition 2.3: Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be
1. a pre-open [13] set if $A \subseteq \text{int} (\text{cl}(A))$ and a pre-closed set if $\text{cl}(\text{int} (A)) \subseteq A$.
2. a semi-open [2] set if $A \subseteq \text{cl} (\text{int} (A))$ and a semi-closed set if $\text{int} (\text{cl} (A)) \subseteq A$.
3. a regular-open [17] set if $\text{int} (\text{cl} (A)) = A$ and a regular-closed set if $\text{cl} (\text{int} (A)) = A$.
4. a semi pre-open [2] set if $A \subseteq \text{cl} (\text{int} (\text{cl} (A)))$ and a semi pre-closed set if $\text{int} (\text{cl} (\text{int} (A))) \subseteq A$.
5. an $\alpha$-open [14] set if $A \subseteq \text{int} (\text{cl} (\text{int} (A)))$ and an $\alpha$-closed set if $\text{cl} (\text{int} (\text{cl} (A))) \subseteq A$.
6. a b-open [3] set if $A \subseteq \text{cl} (\text{int} (A)) \cup \text{int} (\text{cl} (A))$ and a b-closed set if $\text{int} (\text{cl} (A)) \cap \text{cl} (\text{int} (A)) \subseteq A$.
7. a pre-open [16] set if $A \subseteq \text{int}^{*} (\text{cl}(A))$ and a pre-closed set if $\text{cl}^{*} (\text{int} (A)) \subseteq A$.

Definition 2.4: Let $(X, \tau)$ be a topological space and $A \subseteq X$. The regular closure [17] of $A$ is denoted by $\text{rcl}(A)$ and is defined by intersection of all regular closed sets containing $A$.

Definition 2.5: Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be
1. a generalized semi closed [4] set if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
2. an $\alpha$-generalized closed [10] set if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
3. a regular generalized closed [15] set if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $(X, \tau)$.
4. a generalized pre closed [11] set if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $(X, \tau)$.
5. a generalized regular closed [5] set if $\text{rgcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $(X, \tau)$.
6. a generalized semi pre closed [6] set if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
7. a pre regular closed [7] set if $\text{prcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $(X, \tau)$.
8. a generalized $\alpha$-closed [10] set if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$.
9. a generalized b-closed [1] set if $\text{bcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
10. a regular generalized b-closed [12] set if $\text{rgbcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $(X, \tau)$.

The complement of the above mentioned closed sets are their respective open sets.

Remark 2.6:

Theorem 3.3: Every regular closed set is $p^{*}\text{rg}-closed$.  

Proof: Straight forward.
Remark 3.3.1: The converse of the theorem need not be true as seen from the following example.

Example 3.3.2: Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$. Here the set $\{b, c, d\}$ is $p*rg$-closed but not regular closed.

Theorem 3.4: For a topological space $(X, \tau)$, the following conditions hold.

(i) Every $p*rg$-closed set is $gs$-closed set.
(ii) Every $p*rg$-closed set is $ag$-closed set.
(iii) Every $p*rg$-closed set is $gp$-closed set.
(iv) Every $p*rg$-closed set is $gsp$-closed set.
(v) Every $p*rg$-closed set is $rg$-closed set.
(vi) Every $p*rg$-closed set is rgb-closed set.

Proof: proof is straight forward.

Remark 3.4.1: The converse of the theorem need not be true as seen from the following example.

Example 3.4.2: Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Here the set $\{c\}$ is $gs$-closed, $ga$-closed, $gp$-closed, $gr$-closed, $gsp$-closed and rgb-closed but not $p*rg$-closed.

Theorem 3.5: For a topological space $(X, \tau)$, the following conditions are hold.

(i) Every $p*rg$-closed set is $gr$-closed set.
(ii) Every $p*rg$-closed set is $gb$-closed set.
(iii) Every $p*rg$-closed set is $gab$-closed set.
(iv) Every $p*rg$-closed set is $ga$-closed set.

Proof: proof is straight forward.

Remark 3.5.1: The converse of the theorem need not true as seen from the following example.

Example 3.5.2: Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Here the set $\{b\}$ is $ga$-closed, $gab$-closed, $gr$-closed, $gb$-closed but not $p*rg$-closed.

Theorem 3.6: Every $p*rg$-closed set is $gsp$-closed.

Proof: Since every open set is $pre*$ open and from the definition 2.5 (6), we have the proof.

Remark 3.6.1: The converse of the theorem need not true as seen from the following example.

Example 3.6.2: Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Here the set $\{b\}$ is $gsp$- closed but not $p*rg$-closed.

Remark 3.8: The above discussions are summarized in the following implications.
4. Characterisation of $p^*$rg closed sets

**Theorem 4.1:** A subset $A$ of $X$ is $p^*$rg-closed if and only if $rcl(A)\setminus A$ does not contain any non-empty pre*-closed sets.

**Proof:**

**Necessity:** Suppose $F$ is a non-empty pre*-closed subset of $X$ such that $F \subseteq rcl(A)\setminus A$. Then $A \subseteq X\setminus F$ and $X\setminus F$ is pre*-open in $(X, \tau)$. Since $A$ is $p^*$rg-closed in $X$, $rcl(A) \subseteq X\setminus F$ which implies $F \subseteq X\setminus rcl(A)$. Thus, $F \subseteq \left( rcl(A) \cap (X\setminus rcl(A)) \right) = \emptyset$.

**Sufficiency:** Let $A \subseteq U$ and $U$ is pre*-open. Suppose $rcl(A)$ does not contained in $U$, then $rcl(A) \cap (X\setminus U)$ is a non-empty pre*-closed set of $rcl(A)\setminus A$, which is contradiction. Therefore, $rcl(A) \subseteq U$. Hence $A$ is $p^*$rg-closed.

**Corollary 4.1.1:** For any subset $A$ of $X$, if $A$ is $p^*$rg-closed then $rcl(A)\setminus A$ does not contain any non-empty regular closed set.

**Proof:** Let $A$ be a subset of $X$. Then by theorem 4.1, we have $rcl(A)\setminus A$ does not contain any non-empty pre*-closed set. Since every regular closed set is pre*-closed, we have $rcl(A)\setminus A$ does not contain any non-empty regular closed set.

**Theorem 4.2:** If $A$ is both pre*-open and $p^*$rg-closed set in $X$, then $A$ is pre closed set.

**Proof:** Since $A$ is pre*-open and $p^*$rg-closed in $X$, $rcl(A) \subseteq A$. Since every regular closed set is pre closed, $pcl(A) \subseteq rcl(A) \subseteq A$. But always $A \subseteq pcl(A)$. Therefore, $A = pcl(A)$. Hence $A$ is pre closed set.

**Theorem 4.3:** If $A$ is both pre*-open and $p^*$rg-closed set in $X$, then $A$ is a closed set.

**Proof:** Since $A$ is pre*-open and $p^*$rg-closed in $X$, $rcl(A) \subseteq A$. Since every regular closed set is closed, we have $cl(A) \subseteq rcl(A) = A$. Hence $cl(A) \subseteq A$. But always $A \subseteq cl(A)$ and hence $A = cl(A)$. Therefore, $A$ is closed.

**Theorem 4.4:** Let $X$ be a topological space, if a subset $A$ of $X$ is open and gr-closed then $A$ is $p^*$rg-closed set in $X$.

**Proof:** Assume that $A$ is open and gr-closed. Let $U$ be pre*-open in $X$ containing $A$. Then we have $rcl(A) \subseteq U$. Hence $A$ is $p^*$rg-closed set in $X$.

**Theorem 4.5:** Let $X$ be a topological space. If a subset $A$ of $X$ is $\pi$-open and $\pi$gr-closed then $A$ is $p^*$rg-closed set in $X$.

**Proof:** Assume that $A$ is $\pi$-open and $\pi$gr-closed. Let $U$ be pre*-open in $X$ containing $A$. Then we have $rcl(A) \subseteq U$. Thus $rcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is pre*-open. Hence $A$ is $p^*$rg-closed set in $X$.

**Theorem 4.6:** If $A$ is regular closed and $p^*$rg-closed, then $A$ is pre* closed.

**Proof:** Suppose that $A$ is regular closed and $p^*$rg-closed. Since every regular closed set is pre*closed, $p^*cl(A) \subseteq rcl(A)$. Since $A$ is regular closed, we have $rcl(A) = A$. This implies $p^*cl(A) \subseteq A$. We know that $A \subseteq p^*cl(A)$ and hence $p^*cl(A) = A$. Therefore, $A$ is pre* closed.

**Theorem 4.7:** Let $A$ be any $p^*$rg closed set. If $A$ is regular closed then $rcl(A)\setminus A$ is pre* closed.

**Proof:** Necessity: Since $A$ is regular closed set in $(X, \tau)$, $rcl(A) = A$. Then $rcl(A)\setminus A = \emptyset$, which is a pre* closed set in $(X, \tau)$.

**Remark 4.7.1:** The converse of the above theorem need not be true.

**Example 4.7.2:** Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $A = \{c, d\}$. Here $rcl(A)\setminus A = \emptyset$, which is pre* closed but $A$ is not regular closed.

**Theorem 4.8:** The union of any two $p^*$rg-closed set is $p^*$rg-closed.

**Proof:** Let $A$ and $B$ be $p^*$rg-closed sets. Let $U$ be pre*-open set in $X$ such that $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Then $rcl(A) \subseteq U$ and $rcl(B) \subseteq U$. Then $rcl(A \cup B) = rcl(A) \cup rcl(B) \subseteq U$. Therefore, $A \cup B$ is $p^*$rg-closed set.
Theorem 4.9: Arbitrary intersection of $p^{*}\text{rg}$-closed set in a topological space $X$ is $p^{*}\text{rg}$-closed.

Proof: Let $\{A_{i}\}_{i \in I}$ be any collection of $p^{*}\text{rg}$-closed set. Let $U$ be a $pre^{*}$-open set containing each $A_{i}, i \in I$. Since each $A_{i}$ is $p^{*}\text{rg}$-closed, we have $rcl(A_{i}) \subseteq U$ for each $i \in I$. Thus $\bigcap_{i \in I} rcl(A_{i}) \subseteq U$ But $rcl(\bigcap_{i \in I} A_{i}) \subseteq \bigcap_{i \in I} rcl(A_{i}) \subseteq U$, where $U$ is $pre^{*}$-open. Hence $\bigcap_{i \in I} A_{i}$ is $p^{*}\text{rg}$-closed.

Theorem 4.10: Let $A$ and $B$ be subsets such that $A \subseteq B \subseteq rcl(A)$. If $A$ is $p^{*}\text{rg}$-closed set then $B$ is $p^{*}\text{rg}$-closed set.

Proof: Let $A$ and $B$ be subsets such that $A \subseteq B \subseteq rcl(A)$. Suppose that $A$ is $p^{*}\text{rg}$-closed. Let $B \subseteq U$ and $U$ be a $pre^{*}$-open set in $X$. Since $A \subseteq B$ and $B \subseteq U$, we have $A \subseteq U$. Hence $A \subseteq U$ and $U$ is $pre^{*}$-open in $X$. Since $A$ is $p^{*}\text{rg}$-closed, we have $rcl(A) \subseteq U$. Since $B \subseteq rcl(A)$, we have $rcl(B) \subseteq rcl(rcl(A)) = rcl(A) \subseteq U$. Hence $rcl(B) \subseteq U$. Hence $B$ is $p^{*}\text{rg}$-closed set.

Theorem 4.11: For every element $x$ in the space $X$, $\{x\}$ is $pre^{*}$-closed or $X \setminus \{x\}$ is $p^{*}\text{rg}$-closed.

Proof: Suppose $\{x\}$ is not $pre^{*}$-closed. Then $X \setminus \{x\}$ is not $pre^{*}$-open implies the only $pre^{*}$-open set containing $X \setminus \{x\}$ is $X$. This implies $rcl(\{x\}) \subseteq X \setminus \{x\}$. Hence $X \setminus \{x\}$ is $p^{*}\text{rg}$-closed set relative to $X$.

Theorem 4.12: Suppose $B \subseteq A \subseteq X$. $B$ is $p^{*}\text{rg}$-closed set relative to $A$ and $A$ is $p^{*}\text{rg}$-closed set in $Y$, then $B$ is $p^{*}\text{rg}$-closed set relative to $X$.

Proof: Let $B \subseteq U$ and $U$ be a $pre^{*}$-open set in $X$. Since $B \subseteq A$ and $B \subseteq U$ then $B \subseteq A \cap U$. Since $B$ is $p^{*}\text{rg}$ closed set relative to $A$, $rcl(B) \subseteq A \cap U \subseteq U$. Therefore $A \cup rcl(B) \subseteq U$. Since $A$ is $pre^{*}$ closed and $B \subseteq A$ gives $rcl(B) \subseteq U \cap rcl(B)^{\circ}$. This implies that $rcl(B)$ contained in $U$ but not contained in $(rcl(B))^{\circ}$. Therefore, $B$ is $p^{*}\text{rg}$-closed set relative to $X$.

REFERENCES