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FRACTIONAL INTEGRAL TRANSFORMATIONS OF MITTAG- LEFFLER TYPE E-FUNCTION WITH MULTIVARIABLE POLYNOMIAL AND ALEPH FUNCTION

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ABSTRACT

In Present paper, we study of various fractional integral transformations of Mittag-Leffer type E-function with Multivariable Polynomial $S_V^{U_1,\dots,U_l}(x_1,\dots,x_l)$ *and Aleph function in series. Also find results for earlier defined Mittag- Leffler type functions.*

***Key words**: Aleph functions in series, Multivariable Polynomial, Fractional Integral transformations, Mittag- Leffler type function.*

1. INTRODUCTION

It is found by recent works of several authors that the Mittag-Leffler (M-L) function is the solution of fractional differential and integral equations. Unified M-L function named by E-function [1]. Now we will study Erdelyi-Kober, Riemann Liouville and other fractional integral transformation of newly defined M-L type E-function.

In this Paper we use the following definitions
Riemann-Liouville Fractional integral Operator $(I_{c+}^{\theta}\Psi)(x)$ [6]

$$(I_{c+}^{\theta}\Psi)(x)=\frac{1}{\Gamma\theta}\int_c^x (x-t)^{\theta-1}\Psi(t)\,dt \tag{1}$$

Where $\theta\in C;\Re(\theta)>0.$

Erdelyi-Kober fractional integral operator $(\Xi_{0+}^{\eta,\theta})(x)$ [6]

$$\left(\Xi_{0+}^{\eta,\theta}f\right)(x)=\frac{x^{-\eta-\theta}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{\theta}f(t)\,dt \tag{2}$$

Where $\eta,\theta\in C;\Re(\eta)>0$ and $\Re(\theta)>0.$

The Multivariable polynomial $S_V^{U_1,\dots,U_l}(x_1,\dots,x_l)$ introduced by Srivastava and Garg (1987) [8, p.686, eq. (1.4)] is defined in the following manner:

$$S_V^{U_1,\dots,U_l}(x_1,\dots,x_l)=\sum_{\substack{R_1,\dots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}} (-V)_{\sum_{i=1}^{l}U_iR_i}\, A\left(V,R_1,\dots,R_l\right)\frac{x_i^{R_i}}{R_i\,!} \tag{3}$$

Where $V=0,1,2\dots$ and $U_1,\dots,U_l$ an arbitrary positive integers and the coefficients $A(V,R_1,\dots,R_l)$ are arbitrary constants (real or complex).

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In 1903, Gosta Mittag-Leffler [5], introduced the function $E_\alpha(z)$, defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+1)} z^n \tag{4}$$

Where $z, \alpha \in C; \Re(\alpha) \geq 0$ and $|z| > 0$.

In 1905, Wiman [9] extended (4) in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+\beta)} z^n \tag{5}$$

Where $z, \alpha, \beta \in C; \Re(\alpha) \geq 0, \Re(\beta) \geq 0$.

In 2000, Kiryakova [4], has studied about "Multi index M-L function" defined by

$$\left[E_{(1/\rho_i),(\mu_i)}(z)\right] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\mu_1 + n/\rho_1)\ldots\Gamma(\mu_m + n/\rho_m)} z^n \tag{6}$$

Where m>1, is an integer, $\rho_1,\ldots,\rho_m > 0$ and $\mu_1,\ldots,\mu_m$ are arbitrary real numbers.

In 2010, Saxena and Nishimoto [7], studied an extension of M-L type function as

$$E_{\gamma,k}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);z\right] = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}}{\prod_{j=1}^{m}(\alpha_j n+\beta_j)} \frac{z^n}{n!} \tag{7}$$

Where $z, \alpha_j, \beta_j, \gamma \in C; \sum_{j=1}^{m} \Re(\alpha) > \Re(k) - 1, j = 1,\ldots,m$ and $\Re(k) > 0$.

In 2012, Kalla , Haidey and Virchenko [3] , showed Multi parameter M-L type function in the following form

$$\left[HE_{\mu_1,\ldots,\mu_\nu}^{\lambda_1,\ldots,\lambda_\nu}(z)\right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{\left\{\prod_{j=1}^{\nu}\Gamma\left(1+\mu_j+\lambda_j n\right)\right\}} \left(\frac{z}{\Lambda}\right)^{\Lambda n+M} \tag{8}$$

Where $\mu_j \in C, \lambda_j > 0, j = 1,2,\ldots,\nu; \sum_{j=1}^{\nu}\mu_j = M$ and $\sum_{j=1}^{\nu}\lambda_j = \Lambda$.

Chaurasia [2] gave Series representation of the Aleph function.

$$\aleph_{P_i,Q_i,c_i,r}^{M,N}[z] = \sum_{G=1}^{M} \sum_{g=0}^{\infty} \frac{(-1)^g \Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\, B_G} z^{-s} \tag{9}$$

With $s = \eta_{G,g} = \frac{b_G + g}{B_G}$, $P_i < Q_i$, $|z| < 1$

and
$$\Omega_{P_i,Q_i,c_i,r}^{M,N}(s) = \frac{\prod_{j=1}^{M}\Gamma\left(b_j + B_j s\right)\prod_{j=1}^{N}\Gamma\left(1-a_j-A_j s\right)}{\sum_{i=1}^{r} c_i \prod_{j=M+1}^{Q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)\prod_{j=N+1}^{P_i}\Gamma\left(a_{ji}+A_{ji}s\right)} \tag{10}$$

2. MITTAG- LEFFLER TYPE E-FUNCTION

In 2014, Bhatter and Faisal [1], defined a unified M-L type E-function as follows

$$ {}_{\tau}E_k^h(z) = {}_{\tau}E_k^h\left[z \,\Big|\, \begin{matrix} (\rho,a);(\gamma_i,q_i,s_i)_{1,h} \\ (\alpha,\beta);(\delta_j,p_j,r_j)_{1,k} \end{matrix} \right] = {}_{\tau}E_k^h\left[z \,\Big|\, \begin{matrix} (\rho,a);(\gamma_i,q_i,s_i),\ldots,(\gamma_h,q_h,s_h) \\ (\alpha,\beta);(\delta_1,p_1,r_1),\ldots,(\delta_k,p_k,r_k) \end{matrix} \right] $$

$$ = \frac{\left\{(\gamma_1)_{q_1 n}\right\}^{s_1}\left\{(\gamma_2)_{q_2 n}\right\}^{s_2}\ldots\left\{(\gamma_h)_{q_h n}\right\}^{s_h}(-1)^{\rho n} z^{an+\tau}}{\left\{(\delta_1)_{p_1 n}\right\}^{r_1}\left\{(\delta_2)_{p_2 n}\right\}^{r_2}\ldots\left\{(\delta_k)_{p_k n}\right\}^{r_k}\Gamma(\alpha n+\beta)} \tag{11} $$

Where $z,\alpha,\beta,\gamma_i,\delta_j \in C, \Re(\alpha)\ge 0, \Re(\beta)>0, \Re(\gamma_i)>0, \Re(\delta_j)>0$ and $\Re(q_i)\ge 0$.

$$ \Re\left(p_j\right)\ge 0; s_i, r_j, a, \tau \in R; \rho\in\{0,1\}, \left(\sum_{i=0}^{h} q_i s_i < \sum_{j=1}^{k} p_j r_j + \Re(\alpha)\right) \quad \text{or} $$

$$ \left(\sum_{i=0}^{h} q_i s_i = \sum_{j=1}^{k} p_j r_j + \Re(\alpha) \text{ when } \prod_{i=1}^{h}(q_i)^{q_i s_i}\left[\alpha^{\alpha}\prod_{j=1}^{k}\left(p_j\right)^{p_j r_j}\right]^{-1}\left|z^{\alpha}\right|<1\right) \tag{12} $$

Where $i=1,2,\ldots,h;\ j=1,2,\ldots,k$.

3. THE IMAGE OF M-L TYPE E-FUNCTION UNDER THE RIEMANN – LIOUVILLE (R-L) OPERATOR I_{c+}^{θ}

Theorem 3.1:

If convergence condition (3), (9) and (12) are satisfied also $\theta\in C$ and $\Re(\theta)>0$ then the R-L transform I_{c+}^{θ} of the E-functions is

$$ \left[I_{C+}^{\theta}\left\{{}_{\tau}E_k^h(t-c)\, S_V^{U_1,\ldots,U_l}\left((t-c)^{R_1},\ldots,(t-c)^{R_l}\right)\aleph_{P_i,Q_i,c_i,r}^{M,N}(t-c)\right\}\right](x) $$

$$ = \frac{1}{\left(\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_{\theta}} \sum_{\substack{R_1,\ldots,R_l=0 \\ \sum_{i=1}^{l}U_iR_i\le V}} (-V)_{\sum_{i=1}^{l}U_iR_i}\, A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!} $$

$$ \times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}(x-c)^{\sum_{i=1}^{l}R_i-\eta_{G,g}} $$

$$ \times {}_{\tau+\theta}E_{k+1}^{h+1}\left[(x-c)\,\Big|\, \begin{matrix} (\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,a,1) \\ (\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\tau+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,a,1) \end{matrix}\right] \tag{13} $$

Proof:

$$ \left[I_{C+}^{\theta}\left\{{}_{\tau}E_k^h(t-c)\, S_V^{U_1,\ldots,U_l}\left((t-c)^{R_1},\ldots,(t-c)^{R_l}\right)\aleph_{P_i,Q_i,c_i,r}^{M,N}(t-c)\right\}\right](x) $$

$$ = \left[\frac{1}{\Gamma\theta}\int_c^x (x-t)^{\theta-1}\sum_{n=0}^{\infty}\Phi(n)(t-c)^{an+\tau}\sum_{\substack{R_1,\ldots,R_l=0 \\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}\, A(V,R_1,\ldots,R_l)\right. $$

$$ \left.\times\prod_{i=1}^{l}\frac{(t-c)^{R_i}}{R_i!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}(t-c)^{-\eta_{G,g}}\right]dt $$

Where

$$\Phi(n)=\frac{\left\{(\gamma_1)_{q_1 n}\right\}^{s_1}\left\{(\gamma_2)_{q_2 n}\right\}^{s_2}\cdots\left\{(\gamma_1)_{q_h n}\right\}^{s_h}(-1)^{\rho n}}{\left\{(\delta_1)_{p_1 n}\right\}^{r_1}\left\{(\delta_1)_{p_2 n}\right\}^{r_2}\cdots\left\{(\delta_1)_{p_k n}\right\}^{r_k}\Gamma(\alpha n+\beta)} \tag{14}$$

$$=\left[\frac{1}{\Gamma\theta}\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}\,A(V,R_1,\ldots,R_l)\prod_{i=1}^{l}\frac{1}{R_i!}\sum_{n=0}^{\infty}\Phi(n)\right.$$

$$\left.\times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}\int_c^x(x-t)^{\theta-1}(t-c)^{(\alpha n+\sum_{i=1}^{l}R_i-\eta_{G,g}+1)-1}\,dt\right] \tag{15}$$

By using this formula $\left[\int_a^b(z-a)^{\mu-1}(b-z)^{\nu-1}dz=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}(b-a)^{\mu+\nu-1}\right]$ after simplification by the Beta –Gamma function formula than we get the required result (13).

Special Cases:

I. R-L transform I^{θ}_{c+} of the M-L function (6)

$$\left[I^{\theta}_{C+}\left\{E_{(1/\rho_i),(\mu_i)}(t-c)\,S_V^{U_1,\ldots,U_l}\left((t-c)^{R_1},\ldots,(t-c)^{R_l}\right)\aleph^{M,N}_{P_i,Q_i,c_i,r}(t-c)\right\}\right](x)$$

$$=\frac{1}{\left(\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_{\theta}}\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}\,A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\frac{1}{\prod_{j=1}^{m-1}\Gamma(\mu_j)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}(x-c)^{\sum_{i=1}^{l}R_i-\eta_{G,g}}$$

$$\times{}_{\theta}E^{1}_{m}\left[(x-c)\,\middle|\,\begin{matrix}(0,1);\,(\sum_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\\(1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),\ldots,(\mu_{m-1},1/\rho_{m-1},1),(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\end{matrix}\right] \tag{16}$$

II. R-L transform I^{θ}_{c+} of the M-L function (7)

$$\left[I^{\theta}_{C+}\left\{E_{\gamma,k}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);t\right]S_V^{U_1,\ldots,U_l}(t^{R_1},\ldots,t^{R_l})\,\aleph^{M,N}_{P_i,Q_i,c_i,r}(t)\right\}\right](x)$$

$$=\frac{1}{\left(\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_{\theta}}\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}\,A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\frac{1}{\prod_{j=1}^{m-1}\Gamma(\beta_j)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}(x)^{\sum_{i=1}^{l}R_i-\eta_{G,g}}$$

$$\times {}_{\theta}E^{2}_{m+1}\left[x \middle| \begin{matrix} (0,1);(\gamma,k,1),\ (\sum_{i=1}^{l} R_i - \eta_{G,g}+1,1,1) \\ (1,1);(\beta_1,\alpha_1,1),\ldots,(\beta_m,\alpha_m,1),(\theta+\sum_{i=1}^{l} R_i - \eta_{G,g}+1,1,1) \end{matrix} \right] \tag{17}$$

III. R-L transform I^{θ}_{c+} of the M-L function (8)

$$\left[I^{\theta}_{C+}\left\{ \left(HE^{\lambda_1,\ldots,\lambda_\nu}_{\mu_1,\ldots,\mu_\nu};t \right) S_V^{U_1,\ldots,U_l}(t^{R_1},\ldots,t^{R_l}) \aleph^{M,N}_{P_i,Q_i,c_i,r}(t) \right\} \right](x)$$

$$= \frac{1}{\left(M+\sum_{i=1}^{l} R_i - \eta_{G,g}+1 \right)_{\theta}} \sum_{\substack{R_1,\ldots,R_l=0 \\ \sum_{i=1}^{l} U_i R_i \le V}} (-V)_{\sum_{i=1}^{l} U_i R_i} A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times \frac{1}{\prod_{j=1}^{\nu-1}\Gamma\left(1+\mu_j\right)} \sum_{G=1}^{M}\sum_{g=0}^{\infty} \frac{(-1)^g \Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G} x^{\theta+R_1+\ldots+R_l-\eta_{G,g}}$$

$$\times {}_{M}E^{1}_{\nu}\left[\frac{x}{\Lambda} \middle| \begin{matrix} (1,\Lambda);(M+\sum_{i=1}^{l} R_i - \eta_{G,g}+1,\Lambda,1) \\ (\lambda_\nu,1+\mu_\nu),\ldots,(1+\mu_1,\lambda_1,1),\ldots,(1+\mu_{\nu-1},\lambda_{\nu-1},1),(M+\sum_{i=1}^{l} R_i - \eta_{G,g}+\theta+1,\Lambda,1) \end{matrix} \right] \tag{18}$$

IV. If we Substitute Multivariable Polynomial $S_V^{U_1,\ldots,U_l}(x_1,\ldots,x_l)$ and Aleph function is unity, then we get the results reduce in [1].

4. THE IMAGE OF M-L TYPE E-FUNCTION UNDER ERDELYI- KOBER (E-K) OPERATOR $\Xi^{\eta,\theta}_{0+}$

Theorem 4.1: If convergence condition (3), (9) and (12) are satisfied also $\eta,\theta \in C, \Re(\eta) > 0$ *and* $\Re(\theta) > 0$ then the E-K transform $\Xi^{\eta,\theta}_{0+}$ of the E-function is

$$\left[\Xi^{\eta,\theta}_{0+}\left\{ {}_{\tau}E^{h}_{k}(t)\, S_V^{U_1,\ldots,U_l}\left(t^{R_1},\ldots,t^{R_l}\right) \aleph^{M,N}_{P_i,Q_i,c_i,r}(t) \right\} \right](x)$$

$$= \frac{1}{\left(\tau+\theta+\sum_{i=1}^{l} R_i - \eta_{G,g}+1 \right)_{\eta}} \sum_{\substack{R_1,\ldots,R_l=0 \\ \sum_{i=1}^{l} U_i R_i \le V}} (-V)_{\sum_{i=1}^{l} U_i R_i} A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times \sum_{G=1}^{M}\sum_{g=0}^{\infty} \frac{(-1)^g \Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G} x^{\sum_{i=1}^{l} R_i - \eta_{G,g}}$$

$$\times {}_{\tau}E^{h+1}_{k+1}\left[x \middle| \begin{matrix} (\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\tau+\theta+\sum_{i=1}^{l} R_i - \eta_{G,g}+1,a,1) \\ (\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\tau+\theta+\eta+\sum_{i=1}^{l} R_i - \eta_{G,g}+1,a,1) \end{matrix} \right] \tag{19}$$

Proof:

We get the E-K transform $\Xi^{\eta,\theta}_{0+}$ of the E-function as follows

$$\left[\Xi^{\eta,\theta}_{0+}\left\{ {}_{\tau}E^{h}_{k}(t)\, S_V^{U_1,\ldots,U_l}\left(t^{R_1},\ldots,t^{R_l}\right) \aleph^{M,N}_{P_i,Q_i,c_i,r}(t) \right\} \right](x)$$

$$=\left[\frac{x^{-\eta-\theta}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{\theta}\sum_{n=0}^{\infty}\Phi(n)t^{an+\tau}\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{t^{R_1}}{R_1!}\cdots\frac{t^{R_l}}{R_l!}\right.$$

$$\left.\times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!B_G}t^{-\eta_{G,g}}\right]dt \tag{20}$$

$$=\left[\frac{x^{-\eta-\theta}}{\Gamma(\eta)}\sum_{n=0}^{\infty}\Phi(n)\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{t^{R_1}}{R_1!}\cdots\frac{t^{R_l}}{R_l!}\right.$$

$$\left.\times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!B_G}\left[\int_0^x (x-t)^{\eta-1}t^{(\theta+an+\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1)-1}dt\right]\right]$$

$$=\left[\frac{x^{-\eta-\theta}}{\Gamma(\eta)}\sum_{n=0}^{\infty}\Phi(n)\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{t^{R_1}}{R_1!}\cdots\frac{t^{R_l}}{R_l!}\right. \tag{21}$$

$$\left.\times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!B_G}\beta\left(\theta+an+\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,\eta\right)x^{\eta+\theta+an+\tau+\sum_{i=1}^{l}R_i-\eta_{G,g}}\right]dt$$

By using the Beta –Gamma function, we get the result (19).

Special Cases:

I. E-K transform $\Xi_{0+}^{\eta,\theta}$ of the M-L type function (6)

$$\left[\Xi_{0+}^{\eta,\theta}\left\{E_{(1/\rho_i),(\mu_i)}(t)\;S_V^{U_1,\ldots,U_l}(t^{R_1},\ldots,t^{R_l})\;\aleph^{M,N}_{P_i,Q_i,c_i,r}(t)\right\}\right](x)$$

$$=\frac{1}{\left(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_{\eta}}\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\frac{1}{\prod_{j=1}^{m-1}\Gamma(\mu_j)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!B_G}x^{\sum_{i=1}^{l}R_i-\eta_{G,g}}$$

$$\times{}_0E_m^1\left[x\left|\begin{matrix}(0,1);(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\\ (1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),\ldots,(\mu_{m-1},1/\rho_{m-1},1),(\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\end{matrix}\right.\right] \tag{22}$$

II. E-K transform $\Xi_{0+}^{\eta,\theta}$ of the M-L type function (7)

$$\left[\Xi_{0+}^{\eta,\theta}\left\{E_{\gamma,k}\left[(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);t\right]S_V^{U_1,\ldots,U_l}(t^{R_1},\ldots,t^{R_l})\;\aleph^{M,N}_{P_i,Q_i,c_i,r}(t)\right\}\right](x)$$

$$=\frac{1}{\left(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_\eta}\sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}\,A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\frac{1}{\prod_{j=1}^{m-1}\Gamma(\beta_j)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}\,x^{\sum_{i=1}^{l}R_i-\eta_{G,g}}$$

$$\times{}_0E^2_{m+1}\left[x\,\middle|\,\begin{matrix}(0,1);(\gamma,k,1),\ (\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\\(1,1);(\beta_1,\alpha_1,1),\ldots,(\beta_m,\alpha_m,1),(\theta+\eta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,1,1)\end{matrix}\right]\tag{23}$$

III. E-K transform $\Xi^{\eta,\theta}_{0+}$ of the M-L type function (8)

$$\left[\Xi^{\eta,\theta}_{0+}\left\{\left(HE^{\lambda_1,\ldots,\lambda_\nu}_{\mu_1,\ldots,\mu_\nu};t\right)S^{U_1,\ldots,U_l}_V(t^{R_1},\ldots,t^{R_l})\aleph^{M,N}_{P_i,Q_i,c_i,r}(t)\right\}\right](x)$$

$$=\frac{1}{\left(M+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1\right)_\eta}\sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}\,A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}$$

$$\times\frac{1}{\prod_{j=1}^{\nu-1}\Gamma\left(1+\mu_j\right)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}\,x^{R_1+\ldots+R_l-\eta_{G,g}}$$

$$\times{}_ME^1_\nu\left[\frac{x}{\Lambda}\,\middle|\,\begin{matrix}(1,\Lambda);(\gamma,k,1),\ (M+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+1,\Lambda,1)\\(\lambda_\nu,1+\mu_\nu),\ldots,(1+\mu_1,\lambda_1,1),\ldots,(1+\mu_{\nu-1},\lambda_{\nu-1},1),(M+\eta+\sum_{i=1}^{l}R_i-\eta_{G,g}+\theta+1,\Lambda,1)\end{matrix}\right]\tag{24}$$

IV. If we Substitute Multivariable Polynomial $S^{U_1,\ldots,U_l}_V(x_1,\ldots,x_l)$ and Aleph function is unity, then we get the results reduce in [1].

5. THE IMAGE OF THE M-L TYPE E- FUNCTION UNDER A GENERALIZED INTEGRAL OPERATOR.

Theorem 5.1: If convergence condition (3), (9) and (12) are fulfilled also $\eta,\theta,\sigma\in C,\Re(\eta).>0,\Re(\sigma)>0,\Re(\theta)>0$ and $t,x,\upsilon\in R$ then

$$\int_t^x(x-s)^{\eta-1}(s-t)^{\theta-1}{}_\tau E^h_k\{\nu(s-t)^\sigma\}S^{U_1,\ldots,U_l}_V\left((s-t)^{R_1},\ldots,(s-t)^{R_i}\right)\aleph^{M,N}_{P_i,Q_i,c_i,r}(s-t)ds$$

$$=\sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}\,A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}$$

$$\times\beta(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+\sigma\tau,\eta)\ (x-t)^{\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}-1}$$

$$\times {}_{\tau}E_{k+1}^{h+1}\left[\nu(x-t)^{\sigma}\,\middle|\,\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau+\sum_{i=1}^{l}R_i-\eta_{G,g},a\sigma,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\eta+\theta+\sigma\tau+\sum_{i=1}^{l}R_i-\eta_{G,g},a\sigma,1)\end{matrix}\right] \tag{25}$$

Proof: The theorem is proved as follows

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}{}_{\tau}E_k^h\{\nu(s-t)^{\sigma}\}S_V^{U_1,\ldots,U_l}\left((s-t)^{R_1},\ldots,(s-t)^{R_l}\right)\aleph_{P_i,Q_i,c_i,r}^{M,N}(s-t)ds$$

$$=\left[\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}\sum_{n=0}^{\infty}\Phi(n)\left\{\nu(s-t)^{\sigma}\right\}^{an+\tau}\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\right.$$

$$\left.\times\prod_{i=1}^{l}\frac{(s-t)^{R_i}}{R_i!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}(s-t)^{-\eta_{G,g}}ds\right]$$

$$=\left[\sum_{n=0}^{\infty}\Phi(n)\,\nu^{an+\tau}\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\right.$$

$$\left.\times\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}\int_t^x (x-s)^{\eta-1}(s-t)^{\theta+\sum_{i=1}^{l}R_i+\sigma an+\sigma\tau--\eta_{G,g}-1}ds\right] \tag{26}$$

By using this formula $\left[\int_a^b (z-a)^{\mu-1}(b-z)^{\nu-1}dz=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}(b-a)^{\mu+\nu-1}\right]$ after simplification by the Beta –Gamma function formula than we get the required result (25).

Corollary 5.2: If convergence conditions (9) and (3) are satisfied also $\eta,\theta,\sigma\in C,\Re(\eta).>0,\ \Re(\sigma)>0,\Re(\theta)>0$ and $x,\upsilon\in R,t=0$ in theorem (5.1) then

$$\int_0^x (x-s)^{\eta-1}s^{\theta-1}{}_{\tau}E_k^h\{\nu s^{\sigma}\}S_V^{U_1,\ldots,U_l}(s^{R_1},\ldots,s^{R_l})\aleph_{P_i,Q_i,c_i,r}^{M,N}(s)ds$$

$$=\sum_{\substack{R_1,\ldots,R_l=0\\ \sum_{i=1}^{l}U_iR_i\le V}}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{g!\,B_G}$$

$$\times\beta(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}+\sigma\tau,\eta)\ x^{\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}-1}$$

$$\times {}_{\tau}E_{k+1}^{h+1}\left[\nu x^{\sigma}\,\middle|\,\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau+\sum_{i=1}^{l}R_i-\eta_{G,g},a\sigma,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\eta+\theta+\sigma\tau+\sum_{i=1}^{l}R_i-\eta_{G,g},a\sigma,1)\end{matrix}\right] \tag{27}$$

Corollary 5.3: If convergence conditions (9) and (3) are satisfied also $\theta,\sigma\in C,\ \Re(\sigma)>0, \Re(\theta)>0$ and $x,\upsilon\in R, t=0, \eta=1,$ in theorem (5.1) then

$$\int_0^x s^{\theta-1}\,{}_{\tau}E_k^h\{\nu s^{\sigma}\}S_V^{U_1,\ldots,U_l}(s^{R_1},\ldots,s^{R_l})\aleph^{M,N}_{P_i,Q_i,c_i,r}(s)ds$$

$$=\sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}$$

$$\times\left(\frac{x^{\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}}}{\sigma\tau+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}}\right){}_{\tau}E_{k+1}^{h+1}\left[\nu t^{\sigma}\left|\begin{matrix}(\rho,a);(\gamma_i,q_i,s_i)_{1,h},(\theta+\sigma\tau+\sum_{i=1}^{l}R_i-\eta_{G,g},a\sigma,1)\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k},(\eta+\theta+\sigma\tau+\sum_{i=1}^{l}R_i-\eta_{G,g},a\sigma,1)\end{matrix}\right.\right]\quad(28)$$

Special Cases:

I. General integral transform of the M-L type function (6)

$$\int_t^x(x-s)^{\eta-1}(s-t)^{\theta-1}E_{(1/\rho_i),(\mu_i)}\{\nu(s-t)^{\sigma}\}S_V^{U_1,\ldots,U_l}\left((s-t)^{R_1},\ldots,(s-t)^{R_l}\right)\aleph^{M,N}_{P_i,Q_i,c_i,r}(s-t)ds$$

$$=\sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}$$

$$\times\frac{\beta(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},\eta)}{\prod_{j=1}^{m-1}\Gamma(\mu_j)}(x-t)^{\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}-1}$$

$$\times{}_0E_m^1\left[\nu(x-t)^{\sigma}\left|\begin{matrix}(0,1),(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},\sigma,1)\\(1/\rho_m,\mu_m);(\mu_1,1/\rho_1,1),\ldots,(\mu_{m-1},1/\rho_{m-1},1),(\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},\sigma,1)\end{matrix}\right.\right]\quad(29)$$

II. General integral transform of the M-L type function (7)

$$\int_t^x(x-s)^{\eta-1}(s-t)^{\theta-1}E_{\gamma,k}\{(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m);s-t\}S_V^{U_1,\ldots,U_l}\left((s-t)^{R_1},\ldots,(s-t)^{R_l}\right)\aleph^{M,N}_{P_i,Q_i,c_i,r}(s-t)ds$$

$$=\sum_{R_1,\ldots,R_l=0}^{\sum_{i=1}^{l}U_iR_i\le V}(-V)_{\sum_{i=1}^{l}U_iR_i}A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}$$

$$\times\frac{\beta(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},\eta)}{\prod_{j=1}^{m}\Gamma(\beta_j)}(x-t)^{\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}-1}$$

$$\times{}_0E_{m+1}^2\left[(x-t)\left|\begin{matrix}(0,1),(\gamma,k,1),(\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},1,1)\\(1,1);(\beta_1,\alpha_1,1),\ldots,(\beta_m,\alpha_m,1),(\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},1,1)\end{matrix}\right.\right]\quad(30)$$

III. General integral transform of the M-L type function (8)

$$\int_t^x (x-s)^{\eta-1}(s-t)^{\theta-1}\,HE^{\lambda_1,\ldots,\lambda_\nu}_{\mu_1,\ldots,\mu_\nu}\left\{\nu(s-t)^\sigma\right\} S_V^{U_1,\ldots,U_l}\left((s-t)^{R_1},\ldots,(s-t)^{R_l}\right)\aleph^{M,N}_{P_i,Q_i,c_i,r}(s-t)\,ds$$

$$= \sum_{\substack{R_1,\ldots,R_l=0 \\ \sum_{i=1}^{l} U_i R_i \le V}} (-V)_{\sum_{i=1}^{l} U_i R_i} \; A(V,R_1,\ldots,R_l)\frac{1}{R_1!}\cdots\frac{1}{R_l!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-1)^g\,\aleph^{M,N}_{P_i,Q_i,c_i,r}(s)}{g!\,B_G}$$

$$\times\frac{\beta(\theta+\sigma M+\sum_{i=1}^{l}R_i-\eta_{G,g},\eta)}{\prod_{j=1}^{\nu-1}\Gamma(1+\mu_j)}(x-t)^{\eta+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g}-1}$$

$$\times\, {}_M E^1_\nu\left[\frac{\nu(x-t)^\sigma}{\Lambda}\middle|\begin{matrix}(1,\Lambda);(\gamma,k,1),\ (\sigma M+\theta+\sum_{i=1}^{l}R_i-\eta_{G,g},\sigma\Lambda,1)\\ (\lambda_\nu,1+\mu_\nu),\ldots,(1+\mu_1,\lambda_1,1),\ldots,(1+\mu_{\nu-1},\lambda_{\nu-1},1),(\sigma M+\eta+\sum_{i=1}^{l}R_i-\eta_{G,g}+\theta,\sigma\Lambda,1)\end{matrix}\right] \quad (31)$$

IV. If we Substitute Multivariable Polynomial $S_V^{U_1,\ldots,U_l}(x_1,\ldots,x_l)$ and Aleph function is unity, then we get the results reduce in [1].

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