International Journal of Mathematical Archive-8(6), 2017, 30-40
Available online through www.ijma.info ISSN 2229 – 5046

SIMILARITY SOLUTIONS OF SPHERICAL SHOCK WAVES IN AN IDEAL GAS WITH THERMAL RADIATION

ANOOP KUMAR*, RAJAN ARORA

Department of Applied Science and Engineering,
IIT Roorkee, Saharanpur Campus, Saharanpur – (U.P.)-247001, India.

Received On: 25-05-17; Revised & Accepted On: 16-06-17)

ABSTRACT

In this paper, a group theoretic method is used to obtain an entire class of similarity solutions to the problem of shocks propagating through in an ideal gas with thermal radiation, and to characterize analytically the state dependent form of the medium ahead for which the problem is invariant and admits similarity solutions. The arbitrary constants occurring in the expressions for the infinitesimals of the local Lie group of transformations give rise to two different cases of possible solutions i.e. with a power law and exponential shock paths. A particular case of collapse of imploding spherically symmetric shock in a medium in which the initial density obeys power law is worked out in detail. Numerical calculations have been performed to obtain the similarity exponents and the profiles of the flow variables behind the shock, and comparison is made with the Guderley's [1] results.

Keywords: *Lie group, Similarity solutions, Gas-dynamics, Shock Waves, Rankine-Hugoniot Conditions.*

1. INTRODUCTION

Nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various fields of physical and engineering interests. Many flow fields involving wave phenomena are governed by quasi linear hyperbolic system of nonlinear partial differential equations (PDEs). For nonlinear systems involving discontinuities such as shocks, we do not generally have the complete exact solutions, and we have to rely on some approximate analytical or numerical methods which may be useful to provide information to understand the physics involved therein. One of the most powerful methods to obtain the similarity solutions to such type of nonlinear PDEs is the similarity method which is based upon the study of their invariance with respect to one parameter Lie group of transformations. Indeed, with the help of symmetry generators, one can construct similarity variable which can reduce the system of partial differential equations (PDEs) to a system of ordinary differential equations (ODEs). A theoretical study of the imploding shock wave near the centre of convergence in an ideal gas was first performed by Guderley [1]. Among the extensive work that followed, we mention the contributions of Sakurai [2], Zeldovich and Raizer [3], Hayes [4], Axford and Holm [5, 6], Lazarus [7], Hafner [8], Sharma and Radha [9], Jena and Sharma [10], Conforto [11], Madhumita and Sharma [12], Sharma and Radha [13], Sharma and Arora [14], Arora et al. [15, 16] and Husain *et al.* [17] who presented high accuracy results and alternative approaches for the investigation of implosion problem.

Steeb [18] determined the similarity solutions of the Euler equations and the Navier–Stokes equations for incompressible flows using the group theoretic approach outlined in the work of Bluman and Cole [19], Ovasiannikov [20], Olver [21], Logan [22] and Bluman and Kumei [23]. In the present paper, following Bluman and Kumei [23], and in a spirit closer to Logan [22], we characterize the medium ahead of shock for which the problem is invariant and admits similarity solutions. The occurrence of arbitrary constants in the expressions for the infinitesimals of the Lie group of transformations yields different cases of solutions with a power law and exponential shock paths. We have worked out in detail one particular case of collapse of imploding spherically symmetric shock in a medium in which initial density obeys power law.

Corresponding Author: Anoop Kumar*,
Department of Applied Science and Engineering,
IIT Roorkee, Saharanpur Campus, Saharanpur – (U.P.)-247001, India.

The type of motion in which the distribution of flow variables remain similar to themselves with time and vary only as a result of changes in scale is called self-similar. For self-similar motions, the system of non-linear partial differential equations reduces to a system of ordinary differential equations in new unknown functions of the similarity variable ξ. Here, we consider the spherically symmetric motion of a polytropic gas with adiabatic index γ and use the Lie group of transformations to establish the self-similar solutions. The arbitrary constants occurring in the expressions for the infinitesimals of the local Lie group of transformations give rise to two different cases of possible solutions i.e. with a power law and exponential shock paths. The computed values of the similarity exponent (δ) are also compared with the Guderley [1] result and the computation of the flow field in the region behind the shock has been carried out to determine the effects of the ambient density exponent (θ).

2. BASIC EQUATIONS AND SHOCK CONDITIONS

The basic equations [24] describing the one-dimensional unsteady spherically symmetric motion in an ideal gas can be written as

$$\left.\begin{aligned} &\rho_t + u\,\rho_x + \rho u_x + 2\rho u / x = 0, \\ &u_t + u\,u_x + \frac{1}{\rho} p_x = 0, \\ &p_t + u\,p_x + \gamma\,p\left(u_x + \frac{2u}{x}\right) + (\gamma-1)\rho q = 0, \end{aligned}\right\} \tag{1}$$

where p is the gas pressure, ρ is the density, u is the velocity, γ is the constant specific heats ratio, the independent variable x is the radial distance from the center in spherically symmetric flows; t is the time; q is the cooling rate. Here, we assume that q is given as:

$$q(\rho,T) = q_0\,\rho^{\alpha}\,T^{\beta}, \tag{2}$$

where q_0, α and β are constants and q is defined as the total amount of heat energy liberated per unit mass per unit time over the whole frequency interval. The non-numeric subscripts denote the partial differentiation with respect to the indicated variables unless stated otherwise. The equation of state for thermally perfect gas is given in the following form:

$$p = \rho\,RT, \tag{3}$$

where T is the translational temperature and R is the specific gas constant. Now, we consider the shock speed $v = dX/dt$ propagating into an inhomogeneous medium specified by

$$u = 0, \qquad p = p_0, \qquad \rho = \rho_0(x). \tag{4}$$

The Rankine-Hugoniot jump conditions for the strong shocks are (Singh and Vishwakarma [24])

$$u = \frac{2}{\gamma+1}v, \qquad \rho = \frac{\gamma+1}{\gamma-1}\rho_0(X(t)), \qquad p = \frac{2}{\gamma+1}\rho_0(X(t))v^2, \tag{5}$$

where the suffix 0 refers to the condition of flow just ahead of the shock.

3. SIMILARITY ANALYSIS BY INVARIANCE GROUPS

Here, we suppose that there exists a solution of system (1) along a family of curves, called similarity curves for which the system (1) of partial differential equations reduces to a system of ordinary differential equations; this type of solution is called a similarity solution. In order to obtain the similarity solutions of the system (1), we derive its symmetry group such that the system is invariant under this group of transformations. The idea of the calculation is to find a one-parameter infinitesimal group of transformations (Sharma and Arora [14]):

$$\begin{aligned} &x^* = x + \varepsilon\,\chi(x,t,\rho,u,p), \qquad t^* = t + \varepsilon\psi(x,t,\rho,u,p), \\ &u^* = u + \varepsilon U(x,t,\rho,u,p), \qquad \rho^* = \rho + \varepsilon\,S(x,t,\rho,u,p), \\ &p^* = p + \varepsilon\,P(x,t,\rho,u,p), \end{aligned} \tag{6}$$

where the generators χ, ψ, U, S and P are to be determined in such a way that the system (1) of partial differential equations together with the condition (2), (3) and (5) are invariant with respect to the transformation (6); the entity ε is so small that its square and higher powers may be neglected. The existence of such a group allows the number of independent variables in the problem to be reduced by one, and thereby allowing the system (1) to be replaced by a system of ordinary differential equations.

In continuation, we shall use summation convention, and introduce the notation

$$x_1 = t,\ x_2 = x,\ u_1 = \rho,\ u_2 = u,\ u_3 = p \ \text{ and } p_j^i = \frac{\partial u_i}{\partial x_j}, \ \text{ where } i=1, 2, 3 \text{ and } j=1, 2.$$

The system of basic equations (1) which is represented as

$$F_k(x_j, u_i, p_j^i) = 0, \quad k = 1,2,3,$$

is said to be constantly conformally invariant under the infinitesimal group of transformations (6), if there exist constants $\alpha_{kr}\,(k, r = 1, 2, 3)$ such that for all smooth surfaces, $u_i = u_i(x_j)$, we have

$$\mathrm{L}\,F_k = \alpha_{kr}\ F_r\ , \tag{7}$$

$$\Rightarrow L\,F_1 = \alpha_{1r} F_r,$$

$$= \alpha_{11}F_1 + \alpha_{12}F_2 + \alpha_{13}F_3 \ \text{etc.},$$

where L is the Lie derivative in the direction of the extended vector field:

$$\mathrm{L} = \xi^j \frac{\partial}{\partial x_j} + \eta^i \frac{\partial}{\partial u_i} + \beta_j^i \frac{\partial}{\partial p_j^i},$$

with $\quad \xi^1 = \psi, \quad \xi^2 = \chi, \quad \eta^1 = S, \quad \eta^2 = U, \quad \eta^3 = P,$

and

$$\beta_j^i = \frac{\partial \eta^i}{\partial x_j} + \frac{\partial \eta^i}{\partial u_k} p_j^k - \frac{\partial \xi^l}{\partial x_j} p_l^i - \frac{\partial \xi^l}{\partial u_n} p_l^i\ p_j^n\ , \tag{8}$$

where $l = 1, 2,\ j = 1, 2,\ i = 1, 2, 3,\ \ n = 1, 2, 3$ and $k = 1, 2, 3.$ Here, repeated indices imply summation convention and β_j^i is the generalization of the derivative transformation.

Therefore, equation (7) can be written as:

$$\xi^j \frac{\partial F_k}{\partial x_j} + \eta^i \frac{\partial F_k}{\partial u_i} + \beta_j^i \frac{\partial F_k}{\partial p_j^i} = \alpha_{kr}\ F_r\ , \quad k = 1, 2, 3, \quad r = 1, 2, 3. \tag{9}$$

Substitution of β_j^i from (8) into (9) yields a polynomial equation in p_j^i. Setting the coefficients of p_j^i and $p_j^i p_j^n$ to zero, we obtain a system of first order linear partial differential equations in the generators *ψ, χ, S, U* and *P*. This system, which is called the system of the determining equations, is given by

$$\left.\begin{aligned}
&S_t + u\,S_x + \rho U_x + \frac{2}{x}\left(\rho U + u\,S - \frac{\rho u \chi}{x}\right) = \alpha_{11}\frac{2\rho u}{x} + \alpha_{13}\frac{2\gamma\, p u}{x} + \alpha_{13}(\gamma-1)\rho q,\\
&S_\rho - \psi_t - u\,\psi_x = \alpha_{11}, \qquad S_u - \rho\psi_x = \alpha_{12}, \qquad S_p = \alpha_{13},\\
&U - \chi_t + u\,S_\rho - u\ \chi_x + \rho U_\rho = \alpha_{11}\,u,\\
&S - \rho\chi_x + u\,S_u + \rho U_u = \alpha_{11}\rho + \alpha_{12}\ u + \alpha_{13}\,\gamma\, p,\\
&uS_p + \rho U_p = \alpha_{12}\frac{1}{\rho} + \alpha_{13}u,
\end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned}
&U_t + u\,U_x + \frac{1}{\rho}P_x = \alpha_{21}\frac{2\rho u}{x} + \alpha_{23}\frac{2\gamma\ p\ u}{x} + \alpha_{23}(\gamma-1)\rho q,\\
&U_\rho = \alpha_{21}, \qquad U_u - \psi_t - u\,\psi_x = \alpha_{22},\\
&u\,U_\rho + \frac{1}{\rho}P_\rho = \alpha_{21}u, \qquad U_p - \frac{1}{\rho}\psi_x = \alpha_{23},\\
&U - \chi_t + u\,U_u - u\ \chi_x + \frac{1}{\rho}P_u = \alpha_{21}\,\rho + \alpha_{22}\,u + \alpha_{23}\,\gamma\, p,\\
&\frac{-S}{\rho^2} + u\,U_p + \frac{1}{\rho}P_p - \frac{1}{\rho}\chi_x = \alpha_{22}\frac{1}{\rho} + \alpha_{23}\,u,
\end{aligned}\right\} \tag{11}$$

$$\left.\begin{array}{l}
P_t + u\,P_x + \gamma\, p\, U_x + \dfrac{2\gamma}{x}\left(u\,P + pU - \dfrac{p\,u}{x}\chi\right) + (\gamma-1)\,S\,q + (\gamma-1)\rho\left[S\,q_\rho + P\,q_p\right] = \\
\alpha_{31}\,\dfrac{2\rho u}{x} + \alpha_{33}\left(\dfrac{2\gamma\, pu}{x} + (\gamma-1)\rho\, q\right), \\
P_\rho = \alpha_{31}, \quad P_u - \gamma\, p\,\psi_x = \alpha_{32}, \quad P_p - \psi_t - u\,\psi_x = \alpha_{33}, \\
u\,P_\rho + \gamma\, p\, U_\rho = \alpha_{31}\, u, \\
\gamma\, P + \gamma\, pU_u - \gamma\, p\chi_x + u\,P_u = \alpha_{31}\,\rho + \alpha_{32}\, u + \alpha_{33}\,\gamma\, p, \\
U - \chi_t + \gamma\, pU_p + u\,P_p - u\,\chi_x = \alpha_{32}\,\dfrac{1}{\rho} + \alpha_{33}\, u.
\end{array}\right\} \tag{12}$$

Solving the above systems of determining equations, we obtain the infinitesimals of the group of transformations as follows:

$$\left.\begin{array}{l}
S = (\alpha_{11} + a)\rho, \\
U = (\alpha_{22} + a)\,u, \\
P = (2\alpha_{22} + \alpha_{11} + 3a)\,p, \\
\chi = (\alpha_{22} + 2a)\,x, \\
\psi = at + b, \\
S\,q_\rho + P\,q_p = q\,(2\,\alpha_{22} + a),
\end{array}\right\} \tag{13}$$

where a, b, α_{11} and α_{22} are the arbitrary constants. The arbitrary constants, occurring in the expressions of the infinitesimals of local Lie-group of transformations, give rise to four different cases of possible solutions.

4. SELF-SIMILAR SOLUTIONS AND CONSTRAINTS

The arbitrary constants, which become observable in the expressions for the infinitesimals of the invariant group of transformations, yield four different cases of possible solutions as discussed below:

Case-I: When $a \neq 0$ and $\alpha_{22} + 2a \neq 0$, the change of variables from (*x*, *t*) to $\left(\bar{x}, \bar{t}\right)$ defined as

$$\bar{x} = x \qquad \bar{t} = t + \frac{b}{a}, \tag{14}$$

does not change the equation (1)-(12). Thus, rewriting the set of equations (13) in the expressions of the new variables $\bar{x}$ and $\bar{t}$, and then suppressing the bar sign, we obtain

$$\left.\begin{array}{l}
S = (\alpha_{11} + a)\rho, \\
U = (\alpha_{22} + a)\,u, \\
P = (2\alpha_{22} + \alpha_{11} + 3a)\,p, \\
\chi = (\alpha_{22} + 2a)\,x, \\
\psi = at, \\
S\,q_\rho + P\,q_p = q\,(2\,\alpha_{22} + a).
\end{array}\right\} \tag{15}$$

The similarity variable and the forms of similarity solutions for ρ, u, p and q follow from the invariant surface conditions which yield

$$\left.\begin{array}{ll}
\psi\,\rho_t + \chi\,\rho_x = S, & \psi\,u_t + \chi\,u_x = U, \\
\psi\,p_t + \chi\,p_x = P, & S\,q_\rho + Pq_p = q\,(2\alpha_{22} + a).
\end{array}\right\} \tag{16}$$

The equations (16) with the help of (15) yield on integration the forms of the flow variables as

$$\left.\begin{aligned}
\rho &= t^{\left(1+\frac{\alpha_{11}}{a}\right)}\widehat{S}(\xi),\\
u &= t^{\delta-1}\widehat{U}(\xi),\\
p &= t^{(2(\delta-1)+k)}\widehat{P}(\xi),\\
q &= p^{\frac{(2\delta-3)a}{[(2\delta-1)a+\alpha_{11}]}}Q(\eta),
\end{aligned}\right\} \tag{17}$$

along with the similarity curve

$$x = \xi t^{\delta}, \tag{18}$$

where Q is an arbitrary function of η which is found as

$$\eta = \rho\, p^{-\left\{\frac{(\alpha_{11}+a)}{(2\delta-1)a+\alpha_{11}}\right\}},\ k = \frac{\alpha_{11}+a}{a} \text{ and } \delta = \frac{\alpha_{22}+2a}{a} \tag{19}$$

and $\widehat{S}, \widehat{U}$ and $\widehat{P}$ are the arbitrary functions of ξ. Since the shock is a similarity curve, and at shock ξ remains constant; without loss of generality it may be normalized to be at $\xi = 1$. The shock path $X = X(t)$, shock velocity v and the values of the density, velocity and pressure of gas at $\xi = 1$ are then given by

$$X = t^{\delta}, \tag{20}$$

$$v = \frac{\delta X}{t} = \delta t^{\delta-1}. \tag{21}$$

At the shock, we have the following conditions on the functions $\widehat{S}, \widehat{U}$ and $\widehat{P}$

$$\left.\begin{aligned}
\rho\big|_{\xi=1} &= t^{\left(1+\frac{\alpha_{11}}{a}\right)}\widehat{S}(1),\\
u\big|_{\xi=1} &= t^{\delta-1}\widehat{U}(1),\\
p\big|_{\xi=1} &= t^{(2(\delta-1)+k)}\widehat{P}(1).
\end{aligned}\right\} \tag{22}$$

The invariance of jump condition (5) for ρ suggests the following form of $\rho_0(x)$:

$$\rho_0(x) = \rho_c x^{\theta}, \tag{23}$$

where ρ_c is an arbitrary constant and $\theta = \dfrac{\alpha_{11}+a}{\delta a}$. (24)

Also, we obtain the following conditions on the functions $\widehat{S}, \widehat{U}$ and $\widehat{P}$ at the shock $(\xi = 1)$:

$$\left.\begin{aligned}
&\widehat{S}(1) = \frac{\gamma+1}{\gamma-1}\rho_c,\quad \widehat{U}(1) = \frac{2\delta}{\gamma+1},\\
&\widehat{P}(1) = \frac{2\rho_c\delta^2}{(\gamma+1)},
\end{aligned}\right\} \tag{25}$$

where ρ_c is some reference constant associated with the medium. Using (20), (21) and (23) we rewrite the equation (17) as

$$\left.\begin{aligned}
&\rho = \rho_0(X(t))\, S^*(\xi) \qquad u = v\, U^*(\xi),\\
&p = \rho_0(X(t))\, v^2 P^*(\xi),
\end{aligned}\right\} \tag{26}$$

where $S^*(\xi) = \dfrac{\widehat{S}(\xi)}{\rho_c},\ U^*(\xi) = \dfrac{\widehat{U}(\xi)}{\delta},\ P^*(\xi) = \dfrac{\widehat{P}(\xi)}{\delta^2\rho_c}$ and $\delta\theta\alpha + 2\beta\delta - 2\beta + 3 - 2\delta = 0.$

Substituting (26) in the equations of system (1) and using (18), (19), (20), (21) and (23), we obtain the following system of ordinary differential equations in S^*, U^* and P^* which on suppressing the asterisk sign becomes

$$\left.\begin{aligned}&(U-\xi)\,S' + S\theta + SU' + \frac{2SU}{\xi} = 0,\\&(U-\xi)U'S + (\delta-1)\,\delta^{-1}U\,S + P' = 0,\\&(U-\xi)\,P' + 2\,(\delta-1)\,\delta^{-1}P + P\theta + \gamma\,P\left(U' + \frac{2U}{\xi}\right) + (\gamma-1)\,S\,q_* = 0,\end{aligned}\right\} \tag{27}$$

where prime denotes the differentiation with respect to ξ and $q_* = \left(\dfrac{q_0}{R^\beta}\right)\delta^{(2\beta-3)}\rho_c^{\alpha}P^{\beta}S^{(\alpha-\beta)}.$

The jump conditions for the strong shock are:

$$U(1) = \frac{2}{\gamma+1}, \qquad S(1) = \frac{\gamma+1}{\gamma-1}, \qquad P(1) = \frac{2}{\gamma+1}. \tag{28}$$

Case-II: When $a = 0$ and $\alpha_{22} + 2a \neq 0$ or $\alpha_{22} \neq 0$, the similarity variable and the forms of similarity solutions for the flow variables readily follow from (13) and (16), and can be expressed in the following forms on suppressing the bar signs

$$\begin{aligned}&\rho = \rho_0\,(X(t))\,S^*(\xi), && u = v\,U^*(\xi),\\&p = \rho_0\,(X(t))v^2P^*(\xi), && q = p^{\left(\frac{2\delta}{2\delta+\frac{\alpha_{11}}{b}}\right)}Q(\eta),\end{aligned} \tag{29}$$

Where $\quad S^* = \dfrac{\widehat{S}(\xi)}{\rho_c},\ U^* = \dfrac{\widehat{U}(\xi)}{\delta},\ P^* = \dfrac{\widehat{P}(\xi)}{\delta^2\rho_c},\ \eta = \rho\,p^{-\left(\frac{\alpha_{11}}{2\delta b+\alpha_{11}}\right)},$

$$2\theta(\alpha-\beta) + (2\delta+\delta\theta)\,\beta - 2\delta = 0,$$

and the similarity variable ξ, the shock location $X = X(t)$, the shock velocity v and ρ_0 are given by

$$\xi = xe^{-\delta t},\ X = e^{\delta t},\ v = \delta\,e^{\delta t},\ \ \rho_0(x) = \rho_c x^\theta, \tag{30}$$

with $\quad \delta = \dfrac{\alpha_{22}}{b},\ \theta = \dfrac{\alpha_{11}}{\alpha_{22}},\ k = \dfrac{\alpha_{11}}{b} = \delta\theta.$ It may be noticed that this case leads to a class of similarity solutions with an exponential shock path given in the second equation in (30). Substituting (29) into the equations of system (1) and using (30), we obtain the following system of ordinary differential equations in S^*, U^*, and P^* which on suppressing the asterisk signs become:

$$\left.\begin{aligned}&(U-\xi)\,S' + S\theta + SU' + \frac{2SU}{\xi} = 0,\\&(U-\xi)U'S + US + P' = 0,\\&(U-\xi)P' + P(2+\theta) + \gamma\,P\left(U' + \frac{2U}{\xi}\right) + (\gamma-1)S\,q_* = 0,\end{aligned}\right\} \tag{31}$$

where prime denotes the differentiation with respect to ξ and $q_* = \left(\dfrac{q_0}{R^\beta}\right)\delta^{(2\beta-3)}\rho_c^{\alpha}P^{\beta}S^{(\alpha-\beta)}.$

The jump conditions are:

$$U(1) = \frac{2}{\gamma+1}, \quad S(1) = \frac{\gamma+1}{\gamma-1}, \quad P(1) = \frac{2}{\gamma+1}. \tag{32}$$

The system (31) is to be solved subject to the jump conditions (32) for a shock of infinite strength.

Case-III: When $a \neq 0$ and $\alpha_{22} - \alpha_{11} + a = 0$, there does not exist any similarity solution for the spherically symmetric flows.

Case-IV: When $a=0$ and $\alpha_{22}-\alpha_{11}=0$, the situation is similar to the case III in the sense that it does not permit for the existence of similarity solutions in spherically symmetric flow.

5. IMPLODING SHOCKS

Now, we consider in detail the Case I of an imploding shock for which $v >> a_0$, where a_0 is the speed of sound. For the problem of a converging shock collapsing at the center, the origin of time t is taken to be the instant at which the shock reaches the center so that $t \le 0$ in equation (27). In this regard, the definition of the similarity variable is a little modified by setting

$$X=(-t)^{\delta}, \quad \xi = x/(-t)^{\delta}, \tag{33}$$

so that the intervals of the variables are $-\infty < t \le 0,\ X \le x < \infty$ and $1 \le \xi < \infty.$ At the instant of collapse $(t=0),$ the gas velocity, pressure, density and the sound speed at any finite radius x are bounded, but with $t=0$ and finite *x*, $\xi=\infty.$ In order for the quantities u, p, ρ and *a* to be bounded when $t=0$ and x is finite, we have the following boundary conditions at $\xi=\infty.$

$$U(\infty)=0, \qquad \frac{P(\infty)}{S(\infty)}=0. \tag{34}$$

In the matrix notation system (27) can be written as

$$AW' = B, \tag{35}$$

where $W=(U,S,P)^{tr}$, and the matrix A and the column vector B can be examined by inspection of system (27). In system (27) there is an unknown parameter $\delta,$ which cannot be obtained from an energy balance or the dimensional considerations; it is computed only by solving a non-linear eigenvalue problem for a system of ordinary differential equations. The range of similarity variable is $1 \le \xi < \infty$ for the implosion problem, and system (35) can be solved for the derivatives U', S' and P' in the following form:

$$U' = \frac{\Delta_1}{\Delta}, \qquad S' = \frac{\Delta_2}{\Delta}, \qquad P' = \frac{\Delta_3}{\Delta}, \tag{36}$$

where Δ, defined as the determinant of the matrix *A*, is given by

$$\Delta = -(U-\xi)S\left[(U-\xi)^2 - \frac{\gamma P}{S}\right], \tag{37}$$

and Δ_k $(k=1,2,3)$ are the determinants obtained from Δ by replacing the *k*th column by the column vector $B,$ and are given by

$$\Delta_1 = -(U-\xi)\left[\frac{-(\delta-1)US}{\delta}(U-\xi) + P\theta + \frac{2(\delta-1)P}{\delta} + \frac{2\gamma PU}{\xi} + (\gamma-1)Sq_*\right],$$

$$\Delta_2 = S\left[\frac{-(\delta-1)US}{\delta}(U-\xi) + P\theta + \frac{2(\delta-1)P}{\delta} + \frac{2\gamma PU}{\xi} + (\gamma-1)Sq_*\right]$$
$$+ S\theta + \frac{2US}{\xi}\left[(U-\xi)^2 S - \gamma P\right],$$

$$\Delta_3 = -(U-\xi)\left[(U-\xi)S\left(-P\theta - \frac{2(\delta-1)P}{\delta} - \frac{2\gamma PU}{\xi} - (\gamma-1)Sq_*\right) + \frac{(\delta-1)US}{\delta}\gamma P\right].$$

It can be verified that Δ is negative at $\xi=1$ and positive at $\xi=\infty$ representing thereby that there exists a $\xi \in [1, \infty)$ at which Δ vanishes, and accordingly the solutions become singular. In order to get a non-singular solution in the interval $[1, \infty)$, we desire the exponent δ such that Δ vanishes only at the points where the determinant

Δ_1 is zero too. It can be checked that at points where Δ and Δ_1 vanish, the determinants Δ_2 and Δ_3 also vanish at the same time. To find the value of exponent δ in such a manner, we introduce the variable *Z* as

$$Z(\xi) = (U(\xi) - \xi)^2 - \frac{\gamma P(\xi)}{S(\xi)}, \tag{38}$$

whose derivative, in view of (36), is

$$Z' = \left\{ 2(U - \xi)(\Delta_1 - \Delta) - \frac{\gamma \Delta_3}{S} + \frac{\gamma P}{S^2}\Delta_2 \right\} / \Delta. \tag{39}$$

Equations (36), in view of (39), become

$$\frac{dU}{dZ} = \frac{\Delta_1}{\Delta_4}, \qquad \frac{dS}{dZ} = \frac{\Delta_2}{\Delta_4}, \qquad \frac{dP}{dZ} = \frac{\Delta_3}{\Delta_4}, \tag{40}$$

where $\quad \Delta_4 = 2(U - \xi)(\Delta_1 - \Delta) - \dfrac{\gamma \Delta_3}{S} + \dfrac{\gamma P}{S^2}\Delta_2,$

with $\quad \xi = U(\xi) + \left\{ Z(\xi) + \dfrac{\gamma P(\xi)}{S(\xi)} \right\}^{1/2}.$

6. GUDERLEY SOLUTION

For a converging shock wave, we use the variable ξ defined earlier as

$$x = t^{\delta}\xi, \tag{41}$$

where $-\infty < t \le 0$ and $1 \le \xi < \infty$; here we use ξ as an independent variable in place of x. In terms of this independent variable ξ, let the density, velocity and pressure be given by:

$$\rho = \rho_0\, G(\xi), \quad u = \frac{\delta x}{t} V(\xi), \quad p = \rho_0 \left(\frac{\delta x}{t}\right)^2 P(\xi). \tag{42}$$

Under the transformation (42), the given system (1) is transformed into following system:

$$\left.\begin{aligned}
&\xi \frac{G'}{G}(V-1) + \xi V' = -\theta(V-1) - 3V, \\
&\xi V'(V-1) + \frac{\xi P'}{G} = -\frac{P(2+\theta)}{G} - V\left(V - \frac{1}{\delta}\right), \\
&\xi P'(V-1) + \gamma P \xi V' = -3\gamma P V - \theta P(V-1) - 2P\left(V - \frac{1}{\delta}\right) - (\gamma - 1) G q_*
\end{aligned}\right\}. \tag{43}$$

And the shock conditions at the strong shock front are given by

$$G(1) = \frac{(\gamma+1)}{(\gamma-1)}, \quad V(1) = \frac{2}{(\gamma+1)}, \quad P(1) = \frac{2}{(\gamma+1)}. \tag{44}$$

In the matrix notation the system (43) can be written as

$$AW' = B, \tag{45}$$

where $W = (V, G, P)^{tr}$ and the matrix *A* and the column vector *B* can be identified by inspection of system (43). The unknown parameter δ appearing in system (43) is computed only by solving a non-linear eigenvalue problem for a system of ordinary differential equations. System (43) can be solved for the derivatives V', G' and P' in the following form:

$$V' = \frac{\Delta_1}{\Delta}, \quad G' = \frac{\Delta_2}{\Delta}, \quad P' = \frac{\Delta_3}{\Delta}, \tag{46}$$

where Δ, which is the determinant of the matrix *A*, is given by

$$\Delta = -\frac{(V-1)\,\xi^3}{G}\left[(V-1)^2 - \frac{\gamma P}{G}\right], \tag{47}$$

and Δ_k $(k = 1, 2, 3)$ are the determinants obtained from Δ by replacing the *k*th column by the column vector B, and are given by

$$\Delta_1 = \frac{-(V-1)\xi}{G}\left[\begin{array}{l}\left(-V(V-\frac{1}{\delta}) - \frac{P}{G}(2+\theta)\right)\xi(V-1) + \\ \frac{\xi}{G}\left(3\gamma PV + \theta P(V-\frac{1}{\delta}) + 2P(V-\frac{1}{\delta}) + (\gamma-1)G\,q_*\right)\end{array}\right],$$

$$\Delta_2 = \xi\left[\begin{array}{l}\left(-V(V-\frac{1}{\delta}) - \frac{P}{G}(2+\theta)\right)\xi(V-1) + \\ \frac{\xi}{G}\left(3\gamma PV + \theta P(V-1) + 2P(V-\frac{1}{\delta}) + (\gamma-1)G\,q_*\right)\end{array}\right]$$
$$+\left(\theta(V-1)+3V\right)\left[\begin{array}{l}\xi(V-1)\left(-3\gamma PV - \theta P(V-1) - 2P(V-\frac{1}{\delta}) - (\gamma-1)G\,q_*\right) + \\ \gamma P\xi\left(V(V-\frac{1}{\delta}) + \frac{P}{G}(2+\theta)\right)\end{array}\right],$$

$$\Delta_3 = \frac{-(V-1)\xi}{G}\left[\begin{array}{l}\xi(V-1)\left(-3\gamma PV - \theta P(V-1) - 2P(V-1) - (\gamma-1)G\,q_*\right) + \\ \gamma P\xi\left(V(V-1) + \frac{P}{G}(2+\theta)\right)\end{array}\right].$$

We introduce the variable *Z* as

$$Z(\xi) = \left[(V(\xi)-1)^2 - \frac{\gamma P(\xi)}{G(\xi)}\right], \tag{48}$$

whose derivative, in view of (46), is

$$Z' = \left[2\,(V-1)\Delta_1 - \frac{\gamma}{G}\Delta_3 + \frac{\gamma P}{G^2}\Delta_2\right]/\Delta\,. \tag{49}$$

Equations (46), in view of (49), become

$$\frac{dV}{dZ} = \frac{\Delta_1}{\Delta_4}, \qquad \frac{dG}{dZ} = \frac{\Delta_2}{\Delta_4}, \qquad \frac{dP}{dZ} = \frac{\Delta_3}{\Delta_4}, \tag{50}$$

where $\Delta_4 = \left[2\,(V-1)\Delta_1 - \frac{\gamma}{G}\Delta_3 + \frac{\gamma P}{G^2}\Delta_2\right].$

7. NUMERICAL RESULTS AND DISCUSSION

We integrate equations (40) from the shock $Z = Z(1)$ to the singular point $Z = 0,$ by choosing a trial value of $\delta,$ and compute the values of *U, S, P* and Δ_1 at $Z = 0,$ the value of δ is corrected by successive approximations in such a way that for these values, the determinant Δ_1 vanishes at $Z = 0.$ The values of $\delta,$ obtained from the numerical calculations in spherically symmetric flow and for different θ are given in Table 1. The same procedure we apply for equations in (50) to find the value of Guderley's [1] $\delta.$

θ	Computed (δ)	Guderley's [1] (δ)	% Error
0.1	0.674677	0.685196	1.00%
0.5	0.635475	0.622135	2.14%
1.0	0.590995	0.590085	0.15%
1.3	0.570308	0.570305	0.00%
1.6	0.548998	0.548998	0.00%
2.0	0.551405	0.525370	4.72%

Table-1: Similarity exponent δ for spherically symmetric flow and the ambient density exponent θ with $\gamma = 1.66,\ q_0 = 1,\ R = 8.314,\ \alpha = 0.5,\ \rho_c = 1$ and $\beta = 1.5$.

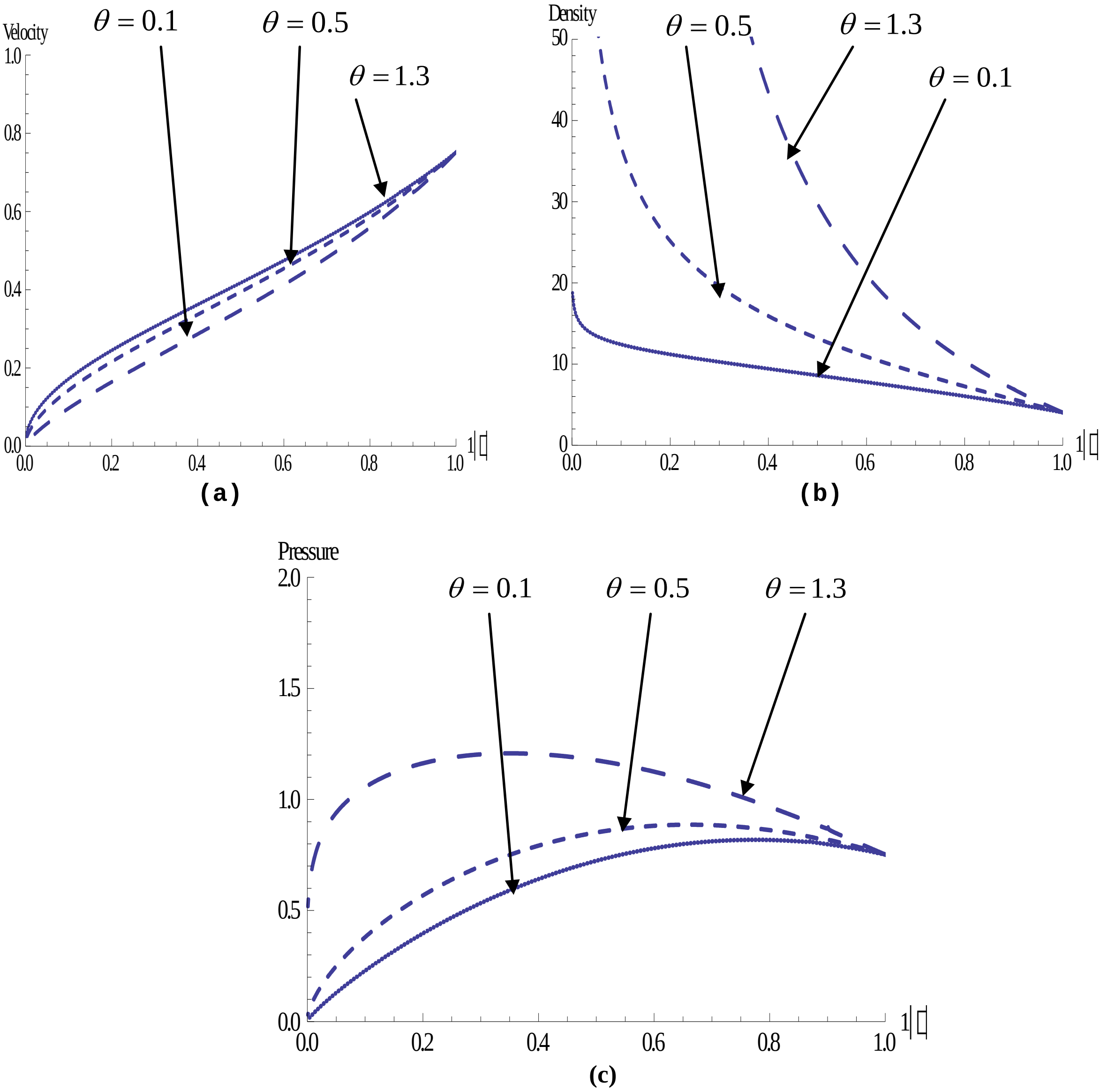

Figure-1: The profiles of velocity, density and pressure in **(a), (b)** and **(c),** respectively, behind spherical shock for $\theta = 0.1, 0.5$ and 1.3.

RESULTS AND CONCLUSION

In the present work, we have used the method of Lie group of transformations to analyze the existence of self-similar solutions for a hyperbolic system of non-linear partial differential equations governing the problem of spherical shock waves in an ideal gas with thermal radiation. The importance of using the Lie group of transformations for obtaining similarity solutions is due to the fact that the arbitrary constants occurring in the expressions for the generators of the local Lie-group of transformations give rise to different cases of possible solutions. The form of the similarity variables and similarity solutions are suggested by the method itself after suitably considering the arbitrary constants. It was observed that, depending upon arbitrary constants appearing in the infinitesimals of the transformations, we obtain different solutions with power law and exponential shock paths. In both the cases, the corresponding reduced systems of ordinary differential equations were found out. The similarity exponent δ and the flow variables can only be obtained after solving the system of ordinary differential equations numerically. We integrated the equations (40) numerically using the fourth order Runge-Kutta method for $1 \leq \xi < \infty$ and the values of velocity, density and pressure behind the shock are plotted in Figures 1 (a), (b) and (c), respectively. Figures 1 (a), (b) and (c) show that behind the shock the velocity decreases and the density increases monotonically as we move towards the center of collapse where $\xi \to \infty$, this increase in density behind the shock may be attributed to the geometrical convergence or the area contraction of the shock wave. The increase in density is further reinforced by an increase in the value of θ, and the decrease in velocity is further reinforced by an increase in the value of θ. The behavior of the pressure is more complicated: the pressure profiles behind the shock exhibit non-monotonic variations. The pressure first increases, attains a maximum value and then decreases as we move towards the center of collapse. The pressure increases with an increase in the value of θ. For imploding shocks, where the changes in flow variables may be attributed to the geometrical convergence or area contraction of the shock waves, the variations that result from the gas dynamics are evident. The computed values of the similarity exponent δ are also compared with the Guderley's [1] result for spherically symmetric case in the Table 1.

REFERENCES

1. G. Guderley, Starke kugelige und zylindrische Verdichtungsstosse in der Nahe des Kugelmittelpunktes bzw der Zylinderachse, Luftfahrtforschung, 19, 302–312 (1942).
2. A. Sakurai, On the problem of a shock wave arriving at the edge of a gas, Comm. Pure Appl. Math., 13, 353–370 (1960).
3. Y.B. Zeldovich and Y.P. Raizer, Physics of Shock Waves and High Temperature Hydrodynamic Phenomena, vol. II, Academic Press (1967).
4. W.D. Hayes, Self-similar strong shocks in an exponential medium, J. Fluid Mech., 32, 305–315 (1968).
5. R.A. Axford and D.D. Holm, Spherical shock collapse in a non-ideal medium, In Proc. Int. Symp. On Group Theoretical Methods in Mechanics, 47, Novosibirsk, USSR (1978).
6. R.A. Axford and D.D. Holm. Converging finite-strength shocks, Phys. D, 2, 194–202 (1981).
7. R.B. Lazarus, Self-similar solutions for converging shocks and collapsing cavities, SIAM J. Numer. Anal., 18, 316–371 (1981).
8. P. Hafner, Strong convergent shock waves near the centre of convergence: A power series solution, SIAM J. Appl. Math., 48, 1244–1261 (1988).
9. V.D. Sharma and Ch. Radha, Similarity solutions for converging shocks in a relaxing gas, Internat. J. Engrg. Sci., 33, 535–553 (1995).
10. J. Jena and V.D. Sharma, Self-similar shocks in a dusty gas, Internat. J. Non- Linear Mech., 34, 313–327 (1999).
11. F. Conforto, Wave features and group analysis for an axi-symmetric model of a dusty gas, Internat. J. Non-Linear Mech., 35, 925–930 (2000).
12. G. Madhumita and V.D. Sharma, Imploding cylindrical and spherical shock waves in a non-ideal medium, J. Hyperbolic Differ. Equ., 1, 521–530 (2004).
13. V.D. Sharma and R. Radha, Exact solutions of Euler equations of ideal gasdynamics via Lie group analysis, Z. Angew. Math. Phys., 59, 1029–1038 (2008).
14. V.D. Sharma and R. Arora, Similarity solutions for strong shocks in an ideal gas, Stud. Appl. Math., 114, 375–394 (2005).
15. R. Arora, A. Tomar and V.P. Singh, Similarity Solutions for Strong Shocks in a Non-Ideal Gas, Mathematical Modeling and Analysis, 17, 351-365 (2012).
16. R. Arora, M.J. Siddiqui and V.P. Singh, Similarity method for imploding strong shocks in a non-ideal relaxing gas, International Journal of Non-Linear Mechanics, 57, 1–9 (2013).
17. A. Husain, L.P. Singh and M. Singh, An approximate analytical solution of imploding strong shocks in a non-ideal gas through Lie group analysis, Chinese Physics Letter, 27, 014702 (2010).
18. S. Steeb, Similarity solutions of the Euler equation and the Navier–Stokes equation in two space dimensions, Internat. J. Theoret. Phys., 24, 255–265 (1985).
19. G.W. Bluman and J.D. Cole, Similarity Methods for Differential Equations, Springer, Berlin (1974).
20. L.V. Ovasiannikov, Group Analysis of Differential Equations, Academic, New York (1982).
21. P.J. Olver, Applications of Lie Groups to Differential Equations, Springer, New York (1986).
22. J.D. Logan, Applied Mathematics: A Contemporary Approach, Wiley-Interscience, New York (1987).
23. G.W. Bluman and S. Kumei, Symmetries and Differential Equations, Springer, New York (1989).
24. J. B. Singh and P. R. Vishwakarma, A self-similar flow behind a spherical shock wave with thermal radiation, I, Astrophysics and Space Science, 93, 261-265 (1983).

Source of support: Nil, Conflict of interest: None Declared.