A NEW PROOF FOR GANTOS’S THEOREM ON SEMILATTICE OF BISIMPLE INVERSE SEMIGROUPS

N. GHRODA*

Department of Mathematics, Faculty of Science, Al-Jabal Al-Ghrabi University, Gharian, Libya.

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ABSTRACT.

Gantos has shown that, if $S$ is a semilattice of right cancellative monoids with the (LC) condition and certain further conditions, then we can associate it with a semilattice of bisimple inverse semigroups. We show that one of Gantos’s conditions is equivalent to $S$ itself having the (LC) condition. We use this equivalence to define a simple form for the multiplication which is easier to deal with than the form which Gantos used. We provide a simple proof completely independent of Gantos’s result.

Keywords: I-orders, I-quotients, right cancellative monoid, inverse hull.

1. INTRODUCTION

An interesting concept of semigroups of left I-quotients, based on the notion of semigroups of left quotients, was developed by the author, Gould, Cegarra and Petrich, in series of papers (see [3], [7] and [8]).

Recall that a subsemigroup $S$ of a group $G$ is a left order in $G$ or $G$ is a group of left quotients of $S$ if any element in $G$ can be written as $a^{-1}b$ where $a, b \in S$. Ore and Dubreil [1] showed that a semigroup $S$ has a group of left quotients if and only if $S$ is right reversible and cancellative. By saying that a semigroup $S$ is right reversible we mean for any $a, b \in SSa \cap Sb \neq \emptyset$. A different definition proposed by Fountain and Petrich in 1985 [5] was restricted to completely 0-simple semigroups of left quotients and then shortly after to that of semigroup of left quotients by Gould [10]; this idea has been extensively developed by number of authors. A subsemigroup $S$ of a semigroup $Q$ is a left order in $Q$ if every element in $Q$ can be written as $a^{-1}b$ where $a, b \in S$ and $a^{-1}$ is an inverse of $a$ in a subgroup of $Q$. In this case we say that $Q$ is a semigroup of left quotients of $S$. Right orders and semigroup of right quotients are defined dually. If $S$ is both a left and right order in $Q$, then $S$ is an order in $Q$ and $Q$ is a semigroup of quotients of $S$.

The author and Gould in [7] have introduced the following definition of left I-orders in inverse semigroups: A subsemigroup $S$ of an inverse semigroup $Q$ is a left I-order in $Q$ and $Q$ is a semigroup of left I-quotients of $S$ if every element in $Q$ can be written as $a^{-1}b$ where $a, b \in S$ and $a^{-1}$ is the inverse of $a$ in the sense of an inverse semigroup theory. Right I-orders and semigroups of right I-quotients are defined dually. If $S$ is a left and right I-order in an inverse semigroup $Q$, we say that $S$ is an I-order in $Q$ and $Q$ is a semigroup of I-quotients of $S$. Let $S$ be a left I-order in $Q$. Then $S$ is straight in $Q$ if every $q \in Q$ can be written as $a^{-1}b$ where $a, b \in S$ and $a \mathcal{R} b \in Q$.

Clifford [1] showed that any right cancellative monoid $S$ with the (LC) condition is the $\mathcal{R}$-class of the identity of its inverse hull $\Sigma(S)$. Moreover, (in our terminology) $S$ is a left I-order in $\Sigma(S)$. By saying that a semigroup $S$ has the (LC) condition we mean for any $a, b \in S$ there is an element $c \in S$ such that $Sa \cap Sb = Sc$. Clifford established that precisely bisimple inverse monoids can be regarded as inverse hulls of right cancellative monoids $S$ satisfying the (LC) condition. The author and Gould in [7] have extended Clifford’s work to a left ample semigroup with (LC). It is worth pointing out that the inverse hull of the left ample semigroup need not be bisimple.

Gantos [11] has developed a structure for semigroups $Q$ which are semilattices $Y$ of bisimple inverse monoids $Q_{a}$, such that the set of identities elements forms a subsemigroup. His structure is determined by semigroups $S$ which are strong semilattices $Y$ of right cancellative monoids $S_{a, \alpha} \in Y$ with (LC) condition and certain morphisms satisfying two conditions.

Corresponding Author: N. Ghroda*, Department of Mathematics, Faculty of Science, Al-Jabal Al-Ghrabi University, Gharian, Libya.

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In this paper, we give another proof of this result. We show that one of Gantos’s conditions is equivalent to $S$ itself having the (LC) condition. We link this with Clifford’s result and our definition of left I-order to introduce a new aspect for such semigroups which we can read as follows: If $S$ is a semilattice of right cancellative monoids with (LC) and $S$ has (LC), then $S$ is a left I-order in a semilattice of inverse hull semigroups. Moreover, we proved that such $S$ is a left I-order in a strong semilattice of inverse hull semigroups.

In Section 2 we give some preliminaries. Section 3 contains our new proof of Gantos’s theorem.

2. PRELIMINARIES AND NOTATIONS

We begin by recalling some of the basic facts about the relations $R^*$ and $L^*$. Let $S$ be a semigroup and $a, b \in S$. We call elements $a$ and $b$ to be related by $R^*$ if and only if $a$ and $b$ are related by $R$ in some oversemigroup of $S$. Dually, we can define the relation $L^*$. An alternative description of $R^*$ is provided by the following lemma.

Lemma 2.1 [4]: Let $S$ be a semigroup and $a, b \in S$. Then the following are equivalent

(i) $a R^* b$;

(ii) for all $x, y \in S^1$ $xa = ya$ if and only if $xb = yb$.

As an easy consequence of Lemma 1.1 we have:

Lemma 2.2 [4]: Let $S$ be a semigroup, $a \in S$ and $e$ be an idempotent of $S$. Then the following conditions are equivalent:

(i) $a R^* e$

(ii) $a = ea$ and for all $x, y \in S^1$, $xa = ya$ implies that $xe = ye$.

It is well-known that Green star relations $R^*$ and $L^*$ on a semigroup $S$ are generalizations of the usual Green’s relations $R$ and $L$ on $S$, respectively.

A semigroup $S$ is left adequate if every $R^*$-class of $S$ contains an idempotent and the idempotents $E(S)$ of $S$ form a semilattice. In this case every $R^*$-class of $S$ contains a unique idempotent. We denote the idempotent in the $R^*$-class of $a$ by $a^*$. A left adequate monoid $S$ is left ample if $ae = ea$ for each $a \in S$ and $e \in E(S)$.

We can note easily that, any right cancellative monoid is left ample. By a right cancellative semigroup we mean, a semigroup $S$ such that for all $x, y \in S$

$$xz = yz \text{ implies } x = y.$$ 

Following [9], for any left ample semigroup $S$ we can construct an embedding of $S$ into the symmetric inverse semigroup $I_S$ as follows. For each $a \in S$ we let $\rho_a \in I_S$ be defined by

$$\text{dom } \rho_a = Sa^+ \text{ and } \text{im } \rho_a = Sa$$

and for any $x \in \text{dom } \rho_a$,

$$xp_a = xa.$$ 

Then the map $\theta_S: S \rightarrow I_S$ is a $(2,1)$-embedding.

The inverse hull of a left ample semigroup $S$ is the inverse subsemigroup $\Sigma(S)$ of $I_S$ generated by $\text{im } \theta_S$. If $S$ is a right cancellative monoid, then for any $a \in S$ we have $a^+ = 1$. Then $\rho_a: S \rightarrow Sa$ is defined by $xp_a = xa$ for each $x \in Sa$.

Hence $\text{dom } \rho_a = S = \text{dom } I_S$, giving that $\text{im } \theta_S \subseteq R_1$ where $R_1$ is the $R$-class of $I_S$ in $I_S$.

As in [7] we say that a $(2,1)$-morphism $\phi: S \rightarrow T$, where $S$ and $T$ are left ample semigroups with Condition (LC), is (LC)-preserving if, for any $b, c \in S$ with $Sb \cap Sc = Sw$, we have that $T(b \phi) \cap S(c \phi) = S(w \phi)$.

Let $S$ be a left I-order in an inverse semigroup $Q$. The Generalisation of Green’s relations $R^*$ and $L^*$ are on $S$. To emphasis that $R$ and $L$ are relations $Q$, we may write $R^0$ and $L^0$ or $R$ in $Q$ and $L$ in $Q$.

We will make heavy use of the following result [7, Corollary 3.10].

Lemma 2.3: [2,7] The following conditions are equivalent for a right cancellative monoid $S$:

(i) $\Sigma(S)$ is bisimple;

(ii) $S$ has Condition (LC);

(iii) $S$ is a left I-order in $\Sigma(S)$.

If the above conditions hold, then $S$ is the $R$-class of the identity of $\Sigma(S)$. Further, $\Sigma(S)$ is proper if and only if $S$ is cancellative.
Conversely, the $ℛ$-class of the identity of any bisimple inverse monoid is right cancellative with Condition (LC).

To prove our main result, we will also need the following lemma.

**Lemma 2.4:** (cf. [6]) Let $S$ be a semilattice $Y$ of right cancellative monoids $S_α$, $α ∈ Y$. Let $e_α$ denote the identity of $S_α$, $α ∈ Y$. Then

1. $e_βa_α = a_αe_β$ if $α ≥ β$;
2. $e_αa_β = e_β$ where $e_α$, $e_β$ are the identities of $S_α$ and $S_β$ respectively;
3. $E(S)$ is a semilattice;
4. The idempotents are central;
5. for any $a, b ∈ S$, $a R^* b$ in $S$ if and only if $a, b ∈ S_α$ for some $α$ in $Y$;
6. $S$ is a left ample semigroup.

**Proof:** (1) Let $e_β ∈ S_β$ and $a_α ∈ S_α$ for some $α, β ∈ Y$, where $α ≥ β$. Then $e_βa_α$ and $a_αe_β$ are in $S_α$ and $S_β$ respectively. Hence $e_βa_α = (e_βa_α)e_β = e_β(a_αe_β) = a_αe_β$.

(2) Let $e_α ∈ S_α$ and $e_β ∈ S_β$ be the identities of $S_α$ and $S_β$ respectively. From (1) it follows that $e_αe_β = e_αe_α = e_αe_β$.

Hence $(e_αe_β)e_αe_β = e_αe_β$, that is, $e_αe_β$ is an idempotent in $S_α$. But there is only one idempotent in $S_α$, so that $e_αe_β = e_α = e_αe_β$.

(3) Let $e_α ∈ S_α$ and $e_β ∈ S_β$ for some $α, β ∈ Y$. Then $e_αe_β ∈ S_α$ and from (2) we have that $e_αe_β = e_αe_αe_β = e_αe_β = e_β$.

(4) Let $e_α ∈ S_α$ and $e_β ∈ S_β$ for some $α, β ∈ Y$. Then $e_αe_β ∈ S_α$ and from (1) and (2) we get $e_αe_β = e_αe_βe_α = e_αe_β = e_αe_β = e_α = e_αe_β$.

Since $e_β$ is the identity of $S_β$, we have that $e_αe_β = a_βe_α$.

(5) Suppose that $a R^* b$ in $S$ where $a ∈ S_α$ and $b ∈ S_β$. Then $e_βa = e_βe_αa$ and so $e_βb = e_βe_αb$ which implies that $β ≤ α$. Dually, $α ≤ β$ and hence $α = β$.

Conversely, suppose that $b ∈ S_α$ and $xb = yb$ for some $x, y ∈ S$ where $x ∈ S_β$ and $y ∈ S_γ$. Then $βα = γyα$ as $xb,yb ∈ S_β = S_γ$. Thus $xb = yb = eb_α$ so that from (1) we get $xe_αb = ye_αb$, and so $xe_αb(βe_γ) = ye_αb(βe_γ)$. Now $xe_αb, ye_αb, βe_γ$ all lie in $S_β$ which is right cancellative, so that $xe_αb = ye_αb$. As in the proof of (3) we have that $e_αe_β = e_αe_β = e_α$. Hence $xe_αb = ye_αb = ye_αb = ye_αb$ and then $xe_α = ye_αb$, also, if $xb = b$, that is, $xb = e_βb$, then $xe_α = e_βe_α = e_α$. Thus $b R^* e_α$ in $S$. Hence for any $a ∈ S_α$ we have that $a R^* b$ in $S$ as required.

(6) From (3) we have that $E(S)$ is a semilattice. By (5) we deduce that each $R^*$-class contains an idempotent which must be unique as $E(S)$ is a semilattice. Notice that if $a ∈ S_α$, then $α^* = e_α$. To see that $S$ is left ample, let $a ∈ S_α$ and $e_β ∈ S_β$. We have to show that $ae_β = (ae_β)^*a$. Using (1) and the fact that $e_αe_β = e_βe_α = e_α$ as in the proof of (3) we get $(ae_β)^*a = e_βa = ae_α^*a = ae_αe_β = ae_β$ as required.

### 3. PROOF OF THE THEOREM

Gantos’s main theorem states: Let $S$ be a strong semilattice $Y$ of right cancellative monoids $S_α$, $α ∈ Y$ with (LC) condition and connecting morphisms $φ_{α,β}, α ≥ β$. Suppose in addition that (C2) holds, where (C2): if $S_αa_α ∩ S_βb_α = S_αc_α$ for all $a_α,b_α, c_α ∈ S_α$, then $S_ β(c_αφ_{ α,β}) ∩ S_β(b_αφ_{ α,β}) = S_β(c_αφ_{ α,β})$ for all $α, β ∈ Y$ with $α ≥ β$. In the terminology of Section 2 (C2) says that the connecting morphisms are (LC)-preserving. He obtained a semigroup $Q$ which is a semilattice $Y$ of bisimple inverse monoid $Q_α$, with identity $e_α, α ∈ Y$ such that $e_α, α ∈ Y$ is a subsemigroup of $Q$. In fact, $Q_α$ is the inverse hull of $S_α$ for each $α ∈ Y$. We show that (C2) is equivalent to $S$ having the (LC) condition. We then reprove Gantos’s result. In Theorems 3.13 and 3.15, we provide a simple proof completely independent of [11].

Let $Σ(S)$ be the inverse hull of left 1-quotients of a right cancellative monoid $S$ with (LC). In the rest of this section we identify $S$ with $ΣθS$, where $θS$ is the embedding of $S$ into $Σ$. We write $a^{-1}b$ short for the element $ρ_a^{-1}ρ_b$ of $Σ(S)$ where $a, b ∈ S$.
Lemma 3.2: Let $T$ be a right cancellative monoid. Then for $a, b \in T$ we have
\[ a \leq b \text{ if and only if } a = ub, \]
for some unit $u$ of $T$.

Lemma 3.3: Let $Q$ be an inverse monoid. Let $a, b, c, d \in R$. Then
\[ a^{-1}b = c^{-1}d \text{ if and only if } a = uc \text{ and } b = ud, \]
for some unit $u$.

Proof: Suppose that $a^{-1}b = c^{-1}d$ where $a, b, c, d \in R$. Since $a, b, c, d \in R$, we have that $a^{-1}R a^{-1}b = c^{-1}d R c^{-1}$ in $Q$. Then $a \leq c$ in $Q$. Since $a \leq b$, it follows that $b = aa^{-1}b = ac^{-1}d$. We claim that $ac^{-1}$ is a unit. As $a \leq c$, it follows that $ac^{-1}c c^{-1} = 1$. Since $c^{-1}R c^{-1}$ we have that $1 = ac^{-1}R ac^{-1}$ and hence $u = ac^{-1}$ is a unit, and we obtain $b = ud$. Since $u = ac^{-1}$ and $a \leq c$ we have that $uc = ac^{-1}c = a$. The converse is clear.

Theorem 3.1: Let $T = \{S_\alpha; S_\beta\}$ be a semilattice of right cancellative monoids $S_\alpha$ with identity $e_\alpha$, $\alpha \in Y$. Suppose that $S$, and each $S_\alpha$, has (LC). Then $Q = \{S_\alpha; S_\beta\}$ is a semilattice of bisimple inverse monoids (where $S_\alpha$ is the inverse hull of $S_\alpha$) and the multiplication in $Q$ is defined by: for $a^{-1}b \in S_\alpha$, $c^{-1}d \in S_\beta$,
\[ a^{-1}bc^{-1}d = (ta)^{-1}(rd) \]
where $S_\alpha b \cap S_\alpha c = S_\alpha w$ and $tb = rc = w$ for some $t, r \in S_\beta$.

Proof: By Lemma 2.3, each $S_\alpha$ is a left $I$-order in $S_\alpha$ where $S_\alpha$ is the $R$-class of the identity of $S_\alpha$. We prove the theorem by means of a sequence of lemmas. We begin by the following lemma due to Clifford.

Lemma 3.2: (cf. [2, Lemma 4.1]) Let $T$ be a right cancellative monoid. Then for $a, b \in T$ we have
\[ a \leq b \text{ if and only if } a = ub, \]
for some unit $u$ of $T$.

Lemma 3.3: Let $Q$ be an inverse monoid. Let $a, b, c, d \in R$. Then
\[ a^{-1}b = c^{-1}d \text{ if and only if } a = uc \text{ and } b = ud, \]
for some unit $u$.

Proof: Suppose that $a^{-1}b = c^{-1}d$ where $a, b, c, d \in R$. Since $a, b, c, d \in R$, we have that $a^{-1}R a^{-1}b = c^{-1}d R c^{-1}$ in $Q$. Then $a \leq c$ in $Q$. Since $a \leq b$, it follows that $b = aa^{-1}b = ac^{-1}d$. We claim that $ac^{-1}$ is a unit. As $a \leq c$, it follows that $ac^{-1}c c^{-1} = 1$. Since $c^{-1}R c^{-1}$ we have that $1 = ac^{-1}R ac^{-1}$ and hence $u = ac^{-1}$ is a unit, and we obtain $b = ud$. Since $u = ac^{-1}$ and $a \leq c$ we have that $uc = ac^{-1}c = a$. The converse is clear.

Theorem 3.2: The multiplication is well-defined.

Proof: Suppose that we have elements $a_1, b_1, a_2, b_2$ of $S_\alpha$, $c_1, d_1, c_2, d_2$ of $S_\beta$ such that
\[ a_1^{-1}b_1 = a_2^{-1}b_2 \text{ in } S_\alpha \text{ and } c_1^{-1}d_1 = c_2^{-1}d_2 \text{ in } S_\beta. \]
By Lemma 3.3,
\[ a_1 = u_1a_2, \quad b_1 = u_1b_2 \]
for some unit $u_1 \in S_\alpha$ and
\[ c_1 = v_1c_2, \quad d_1 = v_1d_2 \]
for some unit $v_1 \in S_\beta$. By definition,
\[ a_1^{-1}b_1 c_1^{-1}d_1 = (t_1a_1)^{-1}(r_1d_1) \]
where
\[ S_\alpha b_1 \cap S_\alpha c_1 = S_\alpha w_1 \text{ and } t_1 b_1 = r_1 c_1 = w_1 \]
for some $t_1, r_1, w_1 \in S_\alpha$. Also,
\[ a_2^{-1}b_2 c_2^{-1}d_2 = (t_2a_2)^{-1}(r_2d_2) \]
where
\[ S_\beta b_2 \cap S_\beta c_2 = S_\beta w_2 \text{ and } t_2 b_2 = r_2 c_2 = w_2 \]
for some $t_2, r_2, w_2 \in S_\beta$. We have to show that
\[ a_1^{-1}b_1 c_1^{-1}d_1 = a_2^{-1}b_2 c_2^{-1}d_2, \]
and to do this we need to prove that
\[ t_1a_1 = u_1t_2a_2 \quad \text{and} \quad r_1d_1 = u_2r_2d_2 \]
for some unit $u$ in $S_\alpha$, using Lemma 3.3. We aim to prove that $S_\alpha w_1 = S_\alpha w_2$. We get this if we prove that $S_\alpha b_1 = S_\alpha b_2$ and $S_\alpha c_1 = S_\alpha c_2$. Since $b_1 = u_1 b_2$, using Lemma 2.4, we have that
\[ e_\alpha b_1 = e_\alpha u_1 b_2 = (u_1 e_\alpha b_2) = (u_1 e_\alpha b_2) \]
and as $b_2 = u_1^{-1}b_1$, we have
\[ S_\alpha b_1 = S_\alpha e_\alpha b_1 = S_\alpha e_\alpha b_2 = S_\alpha b_2. \]
Similarly, $S_\alpha e_\alpha c_1 = S_\alpha e_\alpha c_2$. Hence $S_\alpha w_1 = S_\alpha w_2$ so that $w_1 \leq w_2$ in $S_\alpha$. By Lemma 3.2, $w_1 = lw_2$ for some unit $l$ in $S_\alpha$. Then
\[ w_1 = t_1 b_1 = lw_2 = l(t_2 b_2) = lt_2(u_1^{-1}b_1). \]
But, by Lemma 2.4 $a_1 R b_1$ in $S$, it follows that $t_1 a_1 = lt_2 u_1^{-1}a_1 = lt_2 a_2$. Since $w_1 = r_1 c_1 = lw_2 = lr_2 c_2 = lr_2 v_1^{-1}c_1$ and $c_1 R d_1$ in $S$, again using Lemma 2.4 we have
\[ r_1 d_1 = lr_2 v_1^{-1}d_1 = lr_2 v_1^{-1}v_1 d_2 = lr_2 d_2 \]
as required.
In order to prove the associative law we need to introduce subsidiary lemmas. The proof of the next lemma is depends only on the fact that $S_a$ is right cancellative and the proof can be found in [11].

**Lemma 3.5:** $(S_a a_a \cap S_a b_a)c_a = S_a a_a c_a \cap S_a b_a c_a$ for all $a_a, b_a, c_a \in S_a$.

In the following lemma we prove the equivalence between $S$ having the (LC) condition and (C2) mentioned in the introduction.

**Lemma 3.6:** Let $S = [Y; S_a]$ be a semilattice $Y$ of right cancellative monoids $S_a$ with the (LC) condition. Then $S$ has (LC) if and only if whenever $\beta \leq \alpha$, if $S_a a_a \cap S_a b_a = S_a c_a$ implies $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ for all $\beta \leq \alpha$. Let $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$. Then $ae_\alpha b, e_\alpha b \in S_\alpha b$ so that as $S_\alpha$ has (LC) we know that $S_\alpha e_\alpha a \cap S_\alpha e_\alpha b = S_\alpha e_\alpha c$ for some $c \in S_\alpha b$. Now, let $d \in S_\alpha a \cap S_\beta b$, say $d \in S_\gamma$ so that $\gamma \leq \alpha \beta$ and $d = ua = vb$ for some $u, v \in S$. By assumption, $S_\gamma e_\gamma a \cap S_\gamma e_\gamma b = S_\gamma e_\gamma c$. Then $S_\gamma e_\gamma a \cap S_\gamma e_\gamma b = S_\gamma e_\gamma c$. As $S_\gamma \subseteq S_\alpha$, we have that $d \in S_\gamma$, so that $\gamma \leq \alpha \beta$. Hence $\gamma \leq \alpha \beta$, and we get $d = d_{\alpha \beta}$.

From the other hand, suppose that $S$ has (LC) and let $S_a a_a \cap S_a b_a = S_a c_a$, so that $c_a = u_a b_a = v_a b_a$ for some $u_a, v_a \in S_a$. We claim that $S_a a_a \cap S_a b_a = S_a c_a$.

As $S$ has the (LC) condition there exists $d \in S_\alpha$ such that $S_a a_a \cap S_a b_a = S_d$. Then $d = k a_a = h b_a$ for some $k, h \in S$ and so $\xi \leq \alpha$. Since $c_a \in S_a a_a \cap S_a b_a$, we have that $c_a = r d$ for some $r \in S$ so that $\alpha \leq \xi$. Hence $\alpha = \xi$, that is, $d \in S_\alpha$, and so that $d = d_{\alpha \beta}$.

Thus $c_a = \alpha \beta$, we have that $c_a = (e_\alpha r) d_a \in S_a d_a$ so that $S_a c_a \subseteq S_a d_a$.

Since $d_{\alpha \beta} = k a_a = h b_a = (e_\alpha k) a_a = (e_\alpha h) b_a$, we have that $d_a \in S_a a_a \cap S_a b_a = S_a c_a$, and so $S_a d_a \subseteq S_a c_a$. Thus $S_a d_a = S_a c_a$. Hence $d_a \in S_a a_a \cap S_a a_a$, so that $d_a \in S_a a_a$. We have $S_a a_a \cap S_a b_a = S_a c_a$.

Hence our claim is established.

Now let $\beta \leq \alpha$. Since $S_\beta$ has the (LC) condition and $e_\beta a_\beta, e_\beta b_\beta \in S_\beta$ we have that $S_\beta e_\beta a_\beta \cap S_\beta e_\beta b_\beta = S_\beta w_\beta$ for some $w_\beta \in S_\beta$.

Since $w_\beta \in S_\beta e_\beta a_\beta \cap S_\beta e_\beta b_\beta \subseteq S_\alpha a_\alpha \cap S_\beta b_\beta$, we have that $w_\beta \in S_\alpha c_a$ and so $w_\beta = l c_a$ for some $l \in S$, say $l \in S_\eta$ so that $\eta \geq \beta$. Since $w_\beta = e_\beta w_\beta = e_\beta l c_a$ and $\eta \geq \beta$, it follows that $w_\beta = e_\beta w_\beta = e_\beta c_a$ by Lemma 2.4. Then $w_\beta = (e_\beta e_\beta c_a) \in S_\beta c_a$ so that $S_\beta w_\beta \subseteq S(e_\beta c_a)$.

Conversely, since $c_a = u_a c_a = v_a b_a$ and $\beta \leq \alpha$, it follows that $e_\beta c_a = e_\beta u_a e_\beta a_\beta = e_\beta v_a e_\beta b_\beta$, by Lemma 2.4. It follows that $e_\beta c_a \in S_\beta a_\beta \cap S_\beta b_\beta = S_\beta w_\beta$. Hence $S_\beta e_\beta a_\beta \subseteq S_\beta w_\beta$. Thus $S_\beta e_\beta a_\beta = S_\beta w_\beta$ as required.

**Lemma 3.7:** Let $a^{-1}_\beta b, a^{-1}_\beta e_\beta \in S_\alpha$ and $c^{-1}_\beta d, e_\beta d \in S_\beta$ where $a, b \in S_\alpha, c, d \in S_\beta$ and $e_\beta$ are the identities elements in $S_a$ and $S_\beta$ respectively. Then

(i) $a^{-1}_\beta b e_\beta d = (a e_\beta)^{-1}(b d)$,

(ii) $(a^{-1}_\beta e_\beta)(c^{-1}_\beta d) = (c a)^{-1}(d e_\beta)$.

**Proof:** (i) We have that $S_\alpha e_\beta \cap S_\beta b_\beta = S_\beta \cap S_\beta b_\beta = S_\beta b_\beta$, and $e_\beta b_\beta = b e_\beta$.
Using Lemma 2.4. We have

\[(a^{-1}b)(e_{\beta}d) = (a^{-1}b)(e_{\beta}^{-1}d) \]
\[= (e_{\alpha}a)^{-1}(e_{\alpha}bd) \]
\[= (e_{\alpha}a)^{-1}(bd). \]

(ii) We have that \(S_{\alpha\beta}c \cap S_{\alpha\beta}e_{\alpha} = S_{\alpha\beta}c \cap S_{\alpha\beta} = S_{\alpha\beta}c\) and
\[e_{\alpha\beta}c = (ce_{\alpha\beta})e_{\alpha} = (e_{\alpha\beta}c)e_{\alpha} = ce_{\alpha\beta}. \]

Using Lemma 2.4. We have
\[(a^{-1}e_{\alpha})(c^{-1}d) = (ca)^{-1}(de_{\alpha\beta})\]
as required.

**Lemma 3.8:** Let \(a^{-1}b \in \Sigma_{\alpha}, e_{\beta}d, d^{-1}e_{\beta} \in \Sigma_{\beta}\) and \(x^{-1}y \in \Sigma_{\gamma}\) where \(e_{\beta}\) is the identity element in \(S_{\beta}\) where \(a, b \in S_{\alpha}, e_{\beta}, d \in S_{\beta}\) and \(x, y \in S_{\gamma}\). Then

(i) \((a^{-1}be_{\beta}d)x^{-1}y = a^{-1}b(e_{\beta}dx^{-1}y)\);
(ii) \((a^{-1}bde_{\beta})x^{-1}y = a^{-1}b(d^{-1}e_{\beta}x^{-1}y)\).

**Proof:** (i) Let \(a^{-1}b, e_{\beta}d, x^{-1}y\) be as in the hypothesis. Then
\[(a^{-1}be_{\beta}d)x^{-1}y = (a e_{\alpha\beta})^{-1}(bd)x^{-1}y\text{ by Lemma 3.7 (i),}
\[= (t_1 a)^{-1}(r_1 y)\]
where \(t_1 b d = r_1 x = w_1\) and \(S_{\alpha\beta y}(be_{\alpha\beta y}) \cap S_{\alpha\beta y}(xe_{\alpha\beta y}) = S_{\alpha\beta y}w_1\)
for some \(t_1, r_1, w_1 \in S_{\alpha\beta y}\).

On the other hand, by definition of multiplication,
\[a^{-1}b(e_{\beta}dx^{-1}y) = a^{-1}b((t_2 e_{\beta})^{-1}r_2 y)\]
where \(t_2 d = r_2 x = w_2\) with \(S_{\alpha\beta y}(de_{\alpha\beta y}) \cap S_{\alpha\beta y}(xe_{\alpha\beta y}) = S_{\alpha\beta y}w_2\) \((1)\)
for some \(t_2, r_2, w_2 \in S_{\alpha\beta y}\) and \(t_3 b = r_3 t_2 e_{\alpha\beta y} = w_3\) with \(S_{\alpha\beta y}be_{\alpha\beta y} \cap S_{\alpha\beta y}t_2 e_{\alpha\beta y} = S_{\alpha\beta y}w_3\) \((2)\)
for some \(t_3, r_3, w_3 \in S_{\alpha\beta y}\). Using (1) and Lemma 3.6 gives
\[S_{\alpha\beta y}d \cap S_{\alpha\beta y}x = S_{\alpha\beta y}w_2\] \((3)\)
We must show that \((t_3 a)^{-1}(r_3 y) = (t_2 a)^{-1}(r_2 y)\).
By using Lemma 3.3, we have to show that \(t_3 a = ut_3 a\) and \(t_2 a = ur_3 t_2 y\) for some unit \(u\) in \(S_{\alpha\beta y}\).

Once we know \(w_1 \in L w_3 d\) in \(S_{\alpha\beta y}\), we have that \(w_1 = hw_3 d\) for some unit \(h\) in \(S_{\alpha\beta y}\) by Lemma 3.2. Hence \(t_3 b d = ht_3 b d\) so that \(t_3 e_{\alpha\beta y} b d = ht_3 e_{\alpha\beta y} b d\). Since \(t_1, h t_3\) and \(e_{\alpha\beta y} b d\) are in \(S_{\alpha\beta y}\), which is right cancellative we obtain \(t_1 = h t_3\) so that \(t_3 a = h t_3 a\).

Now,
\[w_1 = r_1 x = t_1 b d = h t_3 b d = h r_3 t_2 d = h r_3 r_2 x.\]
As \(r_1, h r_3 r_2\) and \(e_{\alpha\beta y} x\) are in \(S_{\alpha\beta y}\) again by right cancellativity in \(S_{\alpha\beta y}\) we have that \(r_1 = h r_3 r_2\) and so \(r_1 y = h r_3 r_2 y\).

Now, as \(S\) has (LC)
\[S_{\alpha\beta y}w_1 = S_{\alpha\beta y}bd \cap S_{\alpha\beta y}x\]
\[= S_{\alpha\beta y}bd \cap S_{\alpha\beta y}d \cap S_{\alpha\beta y}x\]
\[= S_{\alpha\beta y}bd \cap S_{\alpha\beta y}w_2\] by (3)
\[= S_{\alpha\beta y}bd \cap S_{\alpha\beta y}t_2 d\]
\[= S_{\alpha\beta y}bd e_{\alpha\beta y} \cap S_{\alpha\beta y}t_2 e_{\alpha\beta y}\]
\[= (S_{\alpha\beta y}b \cap S_{\alpha\beta y}t_2) de_{\alpha\beta y}\] by Lemma 3.5
\[= S_{\alpha\beta y}w_3 d\] by (2).

(ii) Let \(a^{-1}b, d^{-1}e_{\beta}, x^{-1}y\) be as in the hypothesis. Then,
\[(a^{-1}bd^{-1}e_{\beta})x^{-1}y = (t_1 a)^{-1}(r_1 e_{\beta})x^{-1}y\]
\[= (t_2 t_3 a)^{-1}(r_2 y)\]
where \(t_1 b = r_1 d = w_1\) with \(S_{\alpha\beta}(be_{\alpha\beta}) \cap S_{\alpha\beta}(de_{\alpha\beta}) = S_{\alpha\beta}w_1\) \((4)\)
for some \( t_1, r_1, w_1 \in S_{\alpha\beta} \) and \( t_2 r_1 = r_2 x = w_2 \) with
\[
S_{\alpha\beta} r_1 \cap S_{\alpha\beta} x = S_{\alpha\beta} w_2 \tag{5}
\]
for some \( t_2, r_2, w_2 \in S_{\alpha\beta} \). By (4) and Lemma 3.6 we have
\[
S_{\alpha\beta} b \cap S_{\alpha\beta} d = S_{\alpha\beta} w_1. \tag{6}
\]
On the other hand, by Lemma 3.7 (ii),
\[
a^{-1} b (d^{-1} e_\beta x^{-1} y) = a^{-1} b (x d)^{-1} (y e_\beta y) = (t_3 a)^{-1} (r_3 y e_\beta y)
\]
where
\[
t_3 b = r_3 x d = w_3, S_{\alpha\beta} (xd) \cap S_{\alpha\beta} (be_\beta y) = S_{\alpha\beta} w_3
\]
for some \( t_3, r_3, w_3 \in S_{\alpha\beta} \).

We have to show that \( (t_2 t_1 a)^{-1} (r_2 y) = (t_3 a)^{-1} (r_3 y e_\beta y) \). By using Lemma 3.3, we have to show that \( t_3 a = v t_2 t_1 a \) and \( r_3 y = v r_2 y \) for some unit \( v \) in \( S_{\alpha\beta} \).

Once we know \( w_3 \in w_2 \) in \( S_{\alpha\beta} \), we have \( w_3 = kw_2 d \) for some unit \( k \) in \( S_{\alpha\beta} \), by Lemma 3.2. Hence \( r_3 x d = k r_2 x d \) so that \( r_3 e_\beta y x d = k r_2 e_\beta y x d \). Since \( r_3, e_\beta y x d \) and \( k r_2 \) are in \( S_{\alpha\beta} \) which is right cancellative we obtain \( r_3 = k r_2 \) so that \( r_3 y = k r_2 y \). Now,
\[
w_3 = t_3 b = r_3 x d = k r_2 x d = k t_2 t_1 b.
\]
Hence \( t_3 e_\beta y b = k t_2 t_1 e_\beta y b \) where \( t_3, e_\beta y b \) and \( k t_2 t_1 \) are in \( S_{\alpha\beta} \) again by right cancellativity in \( S_{\alpha\beta} \), we have that \( t_3 = k t_2 t_1 \) and so \( t_3 a = k t_2 t_1 a \).

Now,
\[
S_{\alpha\beta} w_3 = S_{\alpha\beta} b \cap S_{\alpha\beta} x d
\]
\[
= S_{\alpha\beta} b \cap S_{\alpha\beta} x d \cap S_{\alpha\beta} d
\]
\[
= S_{\alpha\beta} x d \cap S_{\alpha\beta} w_1 \cap S_{\alpha\beta} d
\]
\[
= S_{\alpha\beta} x d \cap S_{\alpha\beta} w_2 d
\]
\[
= S_{\alpha\beta} w_2 d
\]
as required.

**Lemma 3.9:** The associative law holds in \( Q \).

**Proof:** Suppose that \( a^{-1} b \in \sum alpha, c^{-1} d \in \sum beta \) and \( s^{-1} t \in \sum y \) where \( a, b \in S_\alpha, c, d \in S_\beta \) and \( s, t \in S_y \). From Lemma 3.8, we have that
\[
a^{-1} b (c^{-1} ds^{-1} t) = a^{-1} b (c^{-1} e_\beta e_\beta d s^{-1} t)
\]
\[
= a^{-1} b (c^{-1} e_\beta e_\beta ds^{-1} t)
\]
\[
= (a^{-1} b c^{-1} e_\beta) (e_\beta d s^{-1} t)
\]
\[
= (a^{-1} b c^{-1} e_\beta) s^{-1} t
\]
\[
= (a^{-1} b c^{-1} d) s^{-1} t.
\]
From Lemmas 3.9 and 3.4 we get the proof of Theorem 3.1.

Let \( a \in S_\alpha \) and \( b \in S_\beta \) for some \( a, \beta \in Y \). By Lemmas 3.7 and 2.4,
\[
e_\alpha a e_\beta b = e_\alpha a^{-1} a e_\beta^{-1} b = (e_\alpha a e_\beta)^{-1} (ab) = e_\alpha e_\beta (ab) = ab
\]
and we get the following lemma;

**Lemma 3.10:** The multiplication on \( Q \) extends the multiplication on \( S \).

The next corollary now is clear.

**Corollary 3.11:** The semigroup \( S \) defined as above is a left I-order in \( Q = \cup a e_\beta \sum alpha \).

The following lemma shows that the 'strong' in Gantos's result is automatic.
Lemma 3.12: [7] Let \( P = \{Y; S_\alpha\} \) where each \( S_\alpha \) is a monoid with identity \( e_\alpha \), such that \( E = \{e_\alpha; \alpha \in Y\} \) is a subsemigroup of \( P \). Then \( E \) is a semilattice isomorphic to \( Y \) and \( E \) is central in \( P \).

If we define \( \phi_{\alpha,\beta}: S_\alpha \to S_\beta \) by \( a_\alpha \phi_{\alpha,\beta} = a_\alpha e_\beta \) where \( \alpha \geq \beta \), then each \( \phi_{\alpha,\beta} \) is a monoid morphism, and \( P = \{Y; S_\alpha; \phi_{\alpha,\beta}\} \).

Let \( S = \{Y; S_\alpha\} \) be a semilattice \( Y \) of of right cancellative monoids \( S_\alpha \) with identity \( e_\alpha \), \( \alpha \in Y \) such that each \( S_\alpha \), \( \alpha \in Y \) has the (LC) condition. By Lemma 2.4, \( E = \{e_\alpha; \alpha \in Y\} \) is a subsemigroup of \( S \). Hence \( S \) is a strong semilattice \( Y \) with connecting morphisms \( \phi_{\alpha,\beta}: S_\alpha \to S_\beta \) given by \( a_\alpha \phi_{\alpha,\beta} = a_\alpha e_\beta \) where \( \alpha \geq \beta \) for any \( a_\alpha \in S_\alpha \), by Lemma 3.12. In fact, every semilattice of right cancellative monoids is a strong semilattice of cancellative monoids (see [13, Exercises III.7.12]). If \( S \) has the (LC) condition, then by Corollary 3.11, \( S \) has a semigroup of left \( I \)-quotients \( Q = U_{\alpha \in Y} \sum_\alpha \) where \( \sum_\alpha \) is the inverse hull of \( S_\alpha \), \( \alpha \in Y \). It is easy to see that \( e_\alpha \) is the identity of \( \sum_\alpha \). From Lemma 3.6 and Theorem 3.11 of [7], the \( \phi_{\alpha,\beta} \)'s lift to morphisms \( \phi_{\alpha,\beta}|_\sum_\alpha: \sum_\alpha \to \sum_\beta \) and \( \phi_{\alpha,\beta}|_\sum_\gamma = \phi_{\alpha,\gamma} \) for all \( \alpha \geq \beta \geq \gamma \), and \( \phi_{\alpha,\alpha} \) is the identity on \( \sum_\alpha \). Hence \( Q \) is a strong semilattice of bisimple inverse monoids \( \sum_\alpha \)s, \( \alpha \in Y \), by Lemma 3.12. The following theorem is now clear.

Theorem 3.13: Let \( S = \{Y; S_\alpha; \phi_{\alpha,\beta}\} \) and for each \( \alpha \), let \( S_\alpha \) be a right cancellative monoid with Condition (LC) and \( \sum_\alpha \) as its inverse hull of left \( I \)-quotients. Suppose that \( S \) has the (LC) condition. Then \( S \) is a left I-order in a strong semilattice of bisimple monoids \( Q = \{Y; \sum_\alpha; \phi_{\alpha,\beta}\} \) where \( \phi_{\alpha,\beta} \)'s lift to \( \phi_{\alpha,\beta} \)'s, \( \alpha \geq \beta \).

We aim now to prove the converse of Theorem 3.1. Let \( Q \) be a semilattice \( Y \) of bisimple inverse monoids \( Q_\alpha \), (with identity \( e_\alpha \)) such that \( E = \{e_\alpha; \alpha \in Y\} \) is a subsemigroup of \( Q \). By Lemma 3.12, \( E \) is central in \( Q \). Further if we define \( \phi_{\alpha,\beta}: Q_\alpha \to Q_\beta \) by \( q_\alpha \phi_{\alpha,\beta} = q_\alpha e_\beta \) (\( \alpha \geq \beta \)), then each \( \phi_{\alpha,\beta} \) is a monoid morphism and \( Q = \{Y; Q_\alpha; \phi_{\alpha,\beta}\} \). Let \( S \) be the \( R \)-class of the identity \( e_\alpha \) in \( Q_\alpha \). Clearly, \( \phi_{\alpha,\beta}|_s: Q_\alpha \to Q_\beta \) and \( S = \{Y; Q_\alpha; \phi_{\alpha,\beta}|_s\} \) is a strong semilattice \( Y \) of right cancellative monoids \( S_\alpha \). We wish to show that \( S \) has the (LC) condition. By Lemma 3.6, to show that \( S \) has (LC) condition we have to show that \( \phi_{\alpha,\beta}|_s \) is (LC)-preserving \( (\alpha \geq \beta) \). We need the following technical lemma from [12] (see, Lemma 3.2 of [2]).

Lemma 3.14: (cf. [12, Lemma X.1.5]) Let \( Q \) be a bisimple inverse monoid and let \( R \) be the \( R \)-class of the identity. For any \( a, b, c \in R \),

\[
Ra \cap Rb = Rc \quad \text{if and only if} \quad a^{-1}ab^{-1}b = c^{-1}c.
\]

Returning to our argument before Lemma 3.14. Let \( S_\alpha a \cap S_\beta b = S_\alpha c \) where \( a, b, c \in S_\alpha \). Then, we have that \( a^{-1}ab^{-1}b = c^{-1}c \). We claim that

\[
(e_\beta a)^{-1} (e_\beta a)(e_\beta b)^{-1} (e_\beta b) = (e_\beta c)^{-1} (e_\beta c)
\]

where \( \alpha \geq \beta \).

Since \( E \) is central in \( Q \) we have

\[
(e_\beta a)^{-1} (e_\beta a)(e_\beta b)^{-1} (e_\beta b) = a^{-1}e_\beta e_\beta ab^{-1}e_\beta b = a^{-1}e_\beta ab^{-1}e_\beta b
\]

\[
= a^{-1}ae_\beta b^{-1}b
\]

\[
= e_\beta a^{-1}ab^{-1}b
\]

\[
= e_\beta c^{-1}c
\]

\[
= e_\beta c^{-1}e_\beta c
\]

\[
= (e_\beta c)^{-1} (e_\beta c)
\]

Hence our claim is established. By the above lemma \( S_\beta e_\beta a \cap S_\beta e_\beta b = S_\beta e_\beta c \) where \( \alpha \geq \beta \). Thus by Lemma 3.6, \( S \) has the (LC) condition and the following theorem is clear.

Theorem 3.15: Let \( Q \) be a semilattice \( Y \) of bisimple inverse monoids \( Q_\alpha \), (with identity \( e_\alpha \)) such that \( E = \{e_\alpha; \alpha \in Y\} \) is a subsemigroup of \( Q \). Then there is a subsemigroup \( S \) of \( Q \) with the (LC) condition which is a strong semilattice of right cancellative monoids \( S_\alpha \) where \( S_\alpha \) is the \( R^{Q_\alpha} \)-class of \( e_\alpha \). Moreover, \( S \) is a left \( I \)-order in \( Q \).

Combining Theorem 3.1 and Theorem 3.15, we get the following corollary.

Corollary 3.16: (cf. [11, Main Theorem]) Let \( S = \{Y; S_\alpha\} \) be a semilattice \( Y \) of right cancellative monoids \( S_\alpha \) with identity \( e_\alpha \), such that each \( S_\alpha \) has (LC). Suppose in addition that for any \( \alpha \geq \beta \), if \( S_\alpha a \cap S_\alpha b = S_\alpha c \), then \( S_\beta a \cap S_\beta b = S_\beta c \). For each \( \alpha \in Y \), let \( Q_\alpha \) be the inverse hull of \( S_\alpha \), so that \( Q_\alpha \) is a bisimple inverse monoid, and \( S_\alpha \) is the \( R^{Q_\alpha} \)-class of \( e_\alpha \). Then \( Q = \{Y; Q_\alpha\} \) is a semigroup of left \( I \)-quotients of \( S \), such that \( E = \{e_\alpha; \alpha \in Y\} \) is a subsemigroup.
Conversely, let $Q = [Y; Q_\alpha]$ be a semilattice $Y$ of bisimple inverse monoids $Q_\alpha$, with identity $e_\alpha$, such that $E = \{ e_\alpha : \alpha \in Y \}$ is a subsemigroup. Then $S = [Y; S_\alpha]$ is a semilattice of right cancellative monoids $R_{e_\alpha}$, such that each $R_{e_\alpha}$ has (LC) and for any $\alpha \geq \beta$, if $R_{e_\alpha} a_\alpha \cap S_{e_\alpha} R_\alpha = R_{e_\alpha} c_\alpha$, then $R_{e_\beta} a_\alpha \cap S_{e_\beta} R_\alpha = R_{e_\beta} c_\alpha$.

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