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FLOW OF HERSCHEL-BULKLEY FLUID FLOW THROUGH AN ARTERY WITH THE EFFECT OF STENOSIS AND POST STENOTIC DILATATION

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ABSTRACT

The steady flow of Herschel-Bulkley fluid flow through an artery with both stenosis and dilatations have been studied. The expressions for the velocity (u), plug core velocity(u_p), volumetric flow rate(Q), the pressure drop (Δp), Resistance to the flow $(\bar{\lambda})$ and wall shear stress (τ_h) have been derived by considering mild stenosis. It is found that the resistance to the flow increases with the height of stenosis, length, power law index, yield stress but it decrease with wall shear stress, stenotic dilatation. The flow resistance of the Herschel-Bulkley fluid is more than the Newtonian fluid.

Keywords: *Herschel-Bulkley fluid, stress ratio parameter, yield stress, stenosis, dilatation.*

INTRODUCTION

Recently blood flow through arteries with stenosis has attracted the attention of researchers. In medical terms, Stenosis refers to an abnormal and unnatural growth in the arterial wall thickness that appears under diseased conditions in any location of the cardiovascular system, such as narrowing of any body, tube, orfice or passage. It may result in serious consequences such as cerebral strokes, myocardial infarction leading to heart failure etc. by reducing or occluding the blood supply. The deposition of the cholesterol and proliferation of convective tissue form plaques that enlarge and restrict the blood flow. When such events occur the flow characteristics in the vicinity of the resulting protuberances may be significantly altered. Hence the studies of blood flow through stenotic arteries help scientists to understand cardiovascular diseases and allow improved diagnostics of these diseases.

Several attempts (Forrester and Young [1], Young [2], Macdonald [3]) have been made to understand the flow characteristics of blood through arteries by assuming blood as Newtonian. The assumption of Newtonian behavior of blood is acceptable for high shear rate flow that is the case of flow through larger tubes. In small vessels, blood exhibits shear-dependent viscosity and requires a finite yield stress before it commence, thereby making the non-Newtonian nature of blood is an important factor in modeling. (Majhi and Nair [4], Blair and spanner [5] Shukla *et al.*, [6]).

Herschel-Bulkley fluid is also a non-Newtonian fluid with yield stress which is more general in the sense that it contains two parameters such as the yield stress and power law index. Further, in small diameter tubes blood behaves like Herschel-Bulkley fluid (Chaturani and samy [7]). A.K.Singh and D.P. Singh [8] studied the flow of casson fluid through a radially non-symmetric stenosed artery.

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In all the above models, the studies considered the effect of single and multiple stenosis. Post stenotic dilatation of the coronary occurs at high flow rates (Tandon *et al.*, [9]) possibly due to the shear stress following the constriction (Kawaguti and Hamano [10]). Although studies of Bingham flow (Pincombe and Mazumadar [11], Sanjeev Kumar and Chandra Shekhar Diwakar [12]) through vessels with post-stenotic dilatation have been conducted. Priyadharshini and Ponalagusamy [13] investigated the blood flow through a tapered artery with stenosis and dilatation by treating blood as Herschel-Bulkley fluid.

With this motivation an attempt is made in this paper to investigate effect of stenosis and post stenotic dilatation on Herschel-Bulkley fluid.

MATHEMATICAL FORMULATION

Consider the steady flow of Herschel-Bulkley fluid through a circular artery containing multiple abnormal segments as shown in Fig 1.

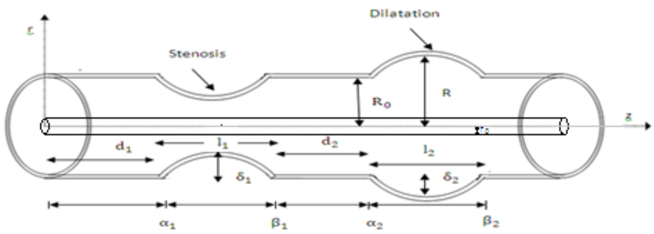

Figure-1: Geometry of arterial segment under consideration.

The equations describing the geometry of the wall, as shown in Fig. 1, are

$$h = \frac{R(z)}{R_0} = \begin{cases} 1 - \frac{\delta_i}{2R_0}[1 + Cos\frac{2\pi}{l_i}(z - \alpha_i - \frac{l_i}{2}), & \text{for } \alpha_i \le z \le \beta_i \\ 1\,, & \text{Otherwise} \end{cases} \qquad (1)$$

Where δ_i is the maximum distance of the i^{th} abnormal segment projects into the lumen and is negative for the aneurysms and positive for stenosis. R is the radius of the artery at dilatation , R_0 is the radius of the normal artery, l_i is the length of the ith abnormal segment, α_i is the distance from the origin to the start of the ith abnormal segment and is given by

$$\alpha_i = \left(\sum_{j=1}^{i}(d_j + l_j)\right) - l_i \qquad (2)$$

And β_i is the distance from the origin to the end of the i^{th} abnormal segment

$$\beta_i = \left(\sum_{j=1}^{i}(d_j + l_j)\right) \qquad (3)$$

Where d_i is the distance separating the start of the i^{th} abnormal segment from the end of the $(i-1)^{\text{th}}$, or from the start of the segment if $i = 1$.

The general equations, considering a mild stenosis in an artery of circular cross section under the conditions

$$\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) = -\frac{\partial p}{\partial z} \qquad (4)$$

Where τ_{rz} is the shear stress for H-B fluid, is given by

$$\tau_{rz} = (\frac{-\partial u}{\partial r})^n + \tau_0, \text{ if } \tau_{rz} \ge \tau_o \qquad (5)$$

$$\frac{\partial u}{\partial r} = 0, \text{ if } \tau_{rz} < \tau_0 \qquad (6)$$

Where (r, z) are cylindrical polar co-ordinates with z measured along the tube axis and r is measured along the normal to the axis of the tube. ‘p’ is pressure , ‘τ_{rz}’ is shear stress and ‘τ_o’ is yield stress and ‘u’ is the velocity of the fluid.

The boundary conditions are

(i) τ_{rz} is finite at $r = 0$ (7)

(ii) $u = 0, at\ r = h(z)$ (8)

SOLUTION

The solution of equation (4) under the boundary conditions (7) and (8) the velocity is obtained as

$$u = \frac{h^{(k+1)}P^k}{2^k(k+1)}\{\left(1 - \frac{2\tau_o}{hP}\right)^{k+1} - \left(\frac{r}{h} - \frac{2\tau_o}{hP}\right)^{k+1} \quad \text{for } r_0 \le r \le h \qquad (9)$$

Where $P = -\frac{\partial p}{\partial z}, k = \frac{1}{n}$

Using the condition (6), the upper limit of the plug flow region is obtained as

$$r_0 = \frac{2\tau_0}{P} \tag{10}$$

and using the condition $\tau_{rz} = \tau_h \; at \; r = h$

$$\frac{r_o}{h} = \frac{\tau_o}{\tau_h} = \tau, \quad 0 < \tau < 1 \tag{11}$$

Taking $r = r_0$ in Eq. (9), the plug core velocity is

$$u_p = \frac{h^{(k+1)}p^k}{2^k(k+1)}\left(1 - \frac{2r_o}{hp}\right)^{k+1} \qquad \text{for } 0 \le r \le r_0 \tag{12}$$

The volume flow rate Q is defined as

$$Q = 2\left[\int_0^{r_o} u_p r\, dr + \int_{r_o}^{h} ur\, dr\right] \tag{13}$$

On integrating,

$$Q = A\left[(k+2)(k+3)\left(1 - \frac{r}{h}\right)^{k+1} - 2(k+3)\left(1 - \frac{r_o}{h}\right)^{k+2} + 2\left(1 - \frac{r_o}{h}\right)^{k+3}\right) \tag{14}$$

Where $A = \frac{h^{(k+3)}p^k}{2^k(k+1)(k+2)(k+3)}$

From eq (14), $\frac{dp}{dz} = -P = \frac{2Q^{\frac{1}{k}}[(k+1)(k+2)(k+3)]^{\frac{1}{k}}}{h^{1+\frac{3}{k}}\{(k+2)(k+3)(1-\tau)^{k+1} - 2(1-\tau)^{k+2}(k+2+\tau)\}^{\frac{1}{k}}}$ (15)

When $k = 1, \; \tau_0 \to 0$ Eq. (15) reduces to the results of Young [2].

The pressure drop Δp across the stenosis between $z = 0$ to $z = l$ is obtained by integrating eq (15), as

$$\Delta p = \int_0^l \frac{dp}{dz}\, dz \tag{16}$$

$$\Delta p = \int_0^l \frac{2Q^{\frac{1}{k}}[(k+1)(k+2)(k+3)]^{\frac{1}{k}}}{h^{1+\frac{3}{k}}\{(k+2)(k+3)(1-\tau)^{k+1} - 2(1-\tau)^{k+2}(k+2+\tau)\}^{\frac{1}{k}}}\, dz \tag{17}$$

Introducing the following non-dimensional quantities

$$\bar{z} = \frac{z}{L}, \quad \bar{\delta} = \frac{\delta}{R_0}, \quad \bar{R}(z) = \frac{R(z)}{R_0}, \quad \bar{P} = \frac{P}{\left(\mu UL/R_0^2\right)},$$

$$\bar{\tau_0} = \frac{\tau_0}{\mu\left(\frac{U}{R_0}\right)}, \quad \bar{\tau}_{rz} = \frac{\tau_{rz}}{\mu\left(\frac{U}{R_0}\right)}, \quad \bar{Q} = \frac{Q}{\pi R_0^2 U}$$

In eq. (17), we finally get (after dropping the bars)

$$\Delta p = \int_0^1 \frac{2Q^{\frac{1}{k}}[(k+1)(k+2)(k+3)]^{\frac{1}{k}}}{h^{1+\frac{3}{k}}\{(k+2)(k+3)(1-\tau)^{k+1} - 2(1-\tau)^{k+2}(k+2+\tau)\}^{\frac{1}{k}}}\, dz \tag{18}$$

The resistance to the flow, λ, is defined by

$$\lambda = \frac{\Delta p}{Q} = \frac{1}{Q}\int_0^1 \frac{2Q^{\frac{1}{k}}[(k+1)(k+2)(k+3)]^{\frac{1}{k}}}{h^{1+\frac{3}{k}}\{(k+2)(k+3)(1-\tau)^{k+1} - 2(1-\tau)^{k+2}(k+2+\tau)\}^{\frac{1}{k}}}\, dz \tag{19}$$

The pressure drop in the absence of stenosis ($h = 1$) is denoted by Δp_N, is obtained from eq.(18) as

$$\Delta p_N = \int_0^1 \frac{2Q^{\frac{1}{k}}[(k+1)(k+2)(k+3)]^{\frac{1}{k}}}{\{(k+2)(k+3)(1-\tau)^{k+1} - 2(1-\tau)^{k+2}(k+2+\tau)\}^{\frac{1}{k}}}\, dz \tag{20}$$

The resistance to the flow in the absence of stenosis is denoted by λ_N is obtained from eq. (20) as

$$\lambda_N = \frac{\Delta p_N}{Q} \tag{21}$$

The normalized resistance to the flow denoted by $\bar{\lambda}$ is given by

$$\bar{\lambda} = \frac{\lambda}{\lambda_N} \tag{22}$$

The wall shear stress is given by

$$\tau_h = -\frac{h}{2}\frac{dp}{dz} \tag{23}$$

RESULTS AND ANALYSIS

The expressions for the velocity (u), plug core velocity(u_p), volumetric flow rate(Q), the pressure drop (Δp), impedance $(\bar{\lambda})$ and wall shear stress (τ_h) are given by eqs.(9), (12), (14), (18), (22) and (22) respectively. Using Mathematica 9.0 computer codes are developed to study the influence of various parameters on impedance $\bar{\lambda}$ and wall shear stress (τ_h). The results are displayed graphically in Figs. 2-17.

It is noticed that the resistance to the flow increases with the height, length of the stenosis and power law index $(n = \frac{1}{k})$, but it decreases in stenotic dilatation Figs.2-9.

From Figs.10-12, the resistance to the flow increases with the height of stenosis and yield stress but it decreases with the height of stenotic dilatation.

It is analyzed that the resistance to the flow increases with the height of stenosis, decreases with wall shear stress and stenotic dilatation Figs.13-15.

It can also be observed from Fig.16 &17 that the resistance of the Herschel-Bulkley fluid is more than the Newtonian fluid.

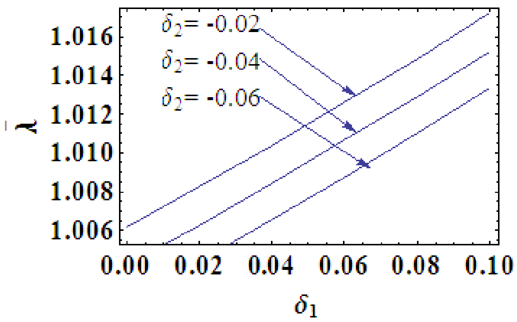

Figure-2: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different δ_2 $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, k = 2000, \tau = 0.02)$

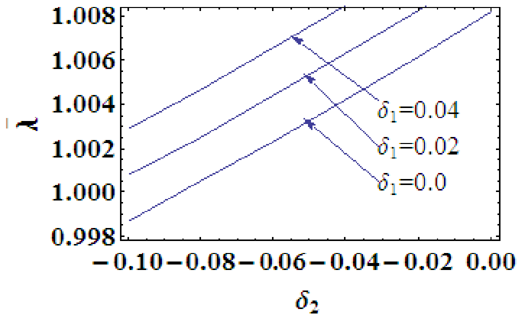

Figure-3: Variation of flow resistance $\bar{\lambda}$ with δ_2 for different δ_1 $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, k = 2000, \tau = 0.02)$

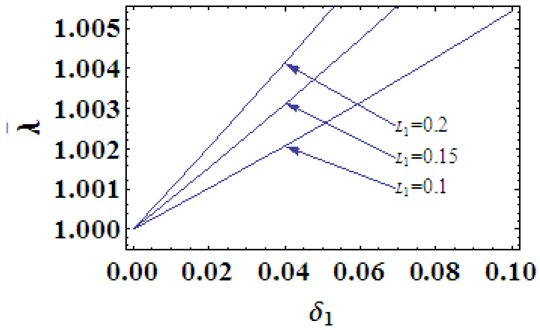

Figure-4: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different L_1 $(d_1 = 0.2, d_2 = 0.2, L_2 = 0.2, L = 1, Q = 0.1, k = 2000, \delta_2 = 0.0, \tau = 0.02)$

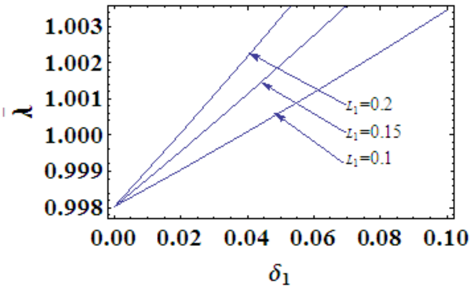

Figure-5: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different L_1 $(d_1 = 0.2, d_2 = 0.2, L_2 = 0.2, L = 1, Q = 0.1,$ $k = 2000, \delta_2 = -0.02, \tau = 0.02)$

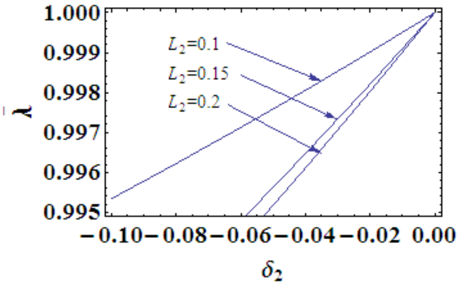

Figure-6: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different L_2 $(d_1 = 0.2, d_2 = 0.2, L_1 = 0.2, L = 1, Q = 0.1,$ $k = 2000, \delta_1 = 0.0, \tau = 0.02)$

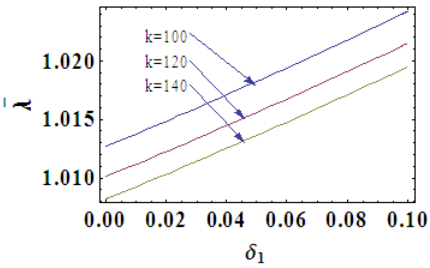

Figure-7: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different k $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1,$ $Q = 0.1, \tau = 0.02, \delta_2 = 0.0)$

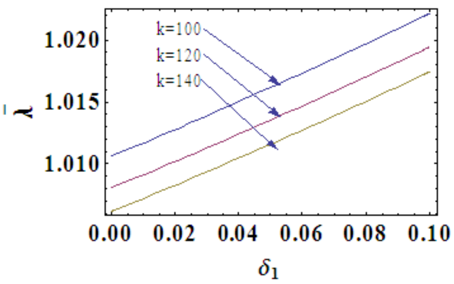

Figure-8: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different k $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1,$ $Q = 0.1, \tau = 0.02, \delta_2 = -0.02)$

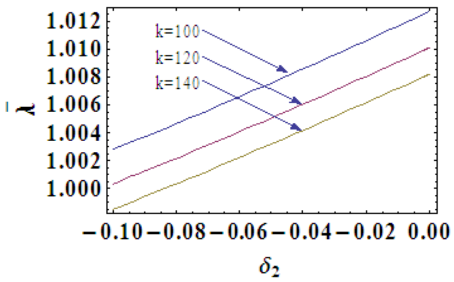

Figure-9: Variation of flow resistance $\bar{\lambda}$ with δ_2 for different k $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1,$ $Q = 0.1, \tau = 0.02, \delta_1 = 0.0)$

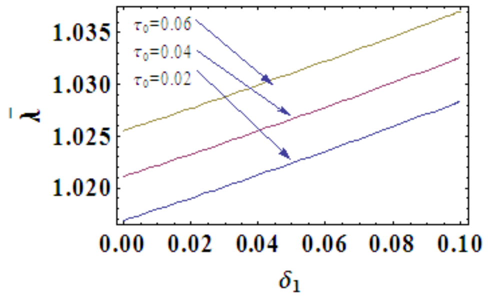

Figure-10: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different τ_0 $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, \mathrm{k} = 2000, \delta_2 = 0.0)$

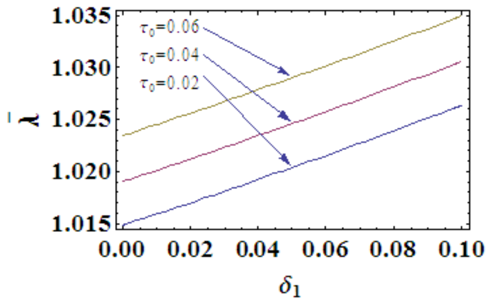

Figure-11: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different τ_0 $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, \mathrm{k} = 2000, \delta_2 = -0.02)$

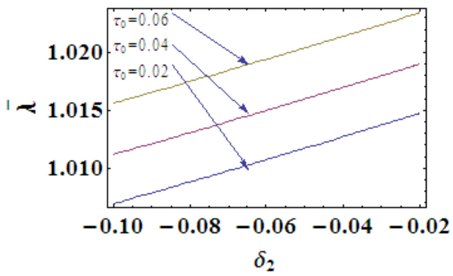

Figure-12: Variation of flow resistance $\bar{\lambda}$ with δ_2 for different τ_0 $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, \mathrm{k} = 2000, \delta_1 = 0.0)$

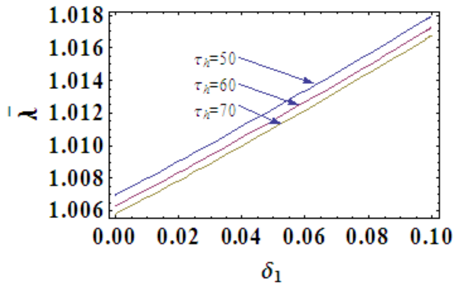

Figure-13: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different τ_h $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, \mathrm{k} = 2000, \delta_2 = 0.0, \tau_0 = 1)$

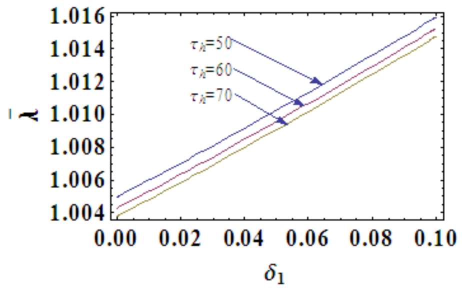

Figure-14: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different τ_h $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, \mathrm{k} = 2000, \delta_2 = -0.02, \tau_0 = 1)$

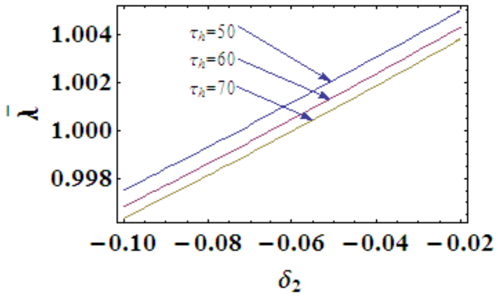

Figure-15: Variation of flow resistance $\bar{\lambda}$ with δ_2 for different τ_h $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, \mathrm{k} = 2000, \delta_1 = 0.0, \tau_0 = 1)$

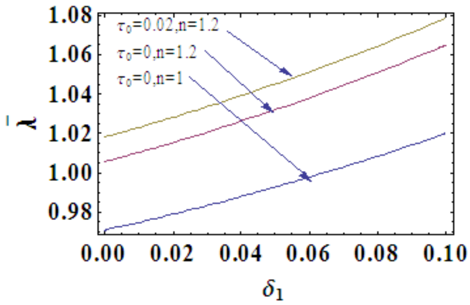

Figure-16: Variation of flow resistance $\bar{\lambda}$ with δ_1 for different fluids $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, \delta_2 = 0.0)$

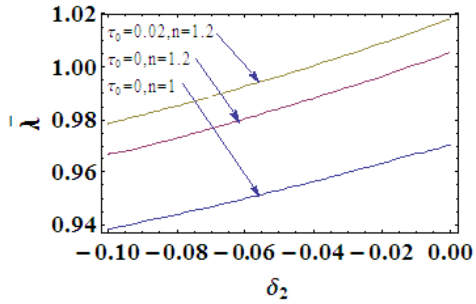

Figure-17: Variation of flow resistance $\bar{\lambda}$ with δ_2 for different fluids $(d_1 = 0.2, d_2 = 0.2, L_1 = L_2 = 0.2, L = 1, Q = 0.1, \delta_1 = 0.0)$

CONCLUSION

The steady flow of H-B fluid flow through an artery with both stenosis and dilatations have been presented. The results have been obtained for H-B fluid and observed that the resistance to the flow increase with the height of stenosis, length, power law index, yield stress but decrease with wall shear stress, stenotic dilatation. The flow resistance of the Herschel-Bulkley fluid is more than the Newtonian fluid.

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