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COMMON FIXED POINT THEOREMS USING E.A. LIKE PROPERTY IN FUZZY METRIC SPACES

KAMAL WADHWA, ASHLEKHA DUBEY*

Government P.G. College Pipariya - (M.P.), India.

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ABSTRACT

In this paper we are proving a common fixed point theorems for mappings satisfying Common E.A like property in fuzzy metric space .Our results generalize the main results of R.K. Sharma and Sonal Bharti [6].

Keywords: *Fuzzy Metric Space, Common E.A like property and weakly compatible mapping.*

Mathematics Subject Classification*: 52H25, 47H10.*

1. INTRODUCTION

The fundamental work for the fuzzy theory was first given by Zadeh [9] in 1965, who introduced the concept of fuzzy set. Kramosil and Michalek [6] developed the fuzzy metric space and later George and Veeramani [2] modified the notion of fuzzy metric spaces by introducing the concept of continuous t–norm. Many researchers have extremely developed the theory by defining different concepts and amalgamation of many properties. Fuzzy set theory has its significance in various fields such as communication, gaming, signal processing, modelling theory, image processing, etc.

E. A like property in fuzzy metric space was defined by Kamal Wadhwa. *et al.* [8]

In this paper we prove a common fixed point theorems for six self-maps satisfying contractive type implicit relation by using Common E.A like property in fuzzy metric space.

2. PRELIMINARY NOTES

Definition 2.1 [6]: A mapping $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-norm if $*$ is satisfying the following conditions:

1) $*$ is commutative and associative;
2) The mapping $*: [0, 1]\times [0, 1]\rightarrow [0, 1]$ is continuous.
3) $a * 1= a$ for all $a\in[0, 1]$;
4) $a* b \leq c *d$ whenever $a \leq c$ and $b \leq d$ for all $a ,b ,c ,d \in[0, 1]$.

Definition 2.2 (Kramosil and Michalek[4]): A Fuzzy Metric Space is a triple $(X,M, *)$ where X is a nonempty set , $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, 1]$ such that the followings axioms hold:

(KM-1) $M(x, y, 0) = 0$ for all $x, y \in X$;
(KM-2) $M(x, y, t) = 1$ for all $x, y \in X$ where $t > 0 \Leftrightarrow x = y$;
(KM-3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$
(KM-4) $M(x, y, .): [0, \infty)\rightarrow[0, 1]$ is left continuous for all $x, y \in X$;
(KM-5) $M(x, z, t+s) \geq M(x, y, t) * M(y, z, s)$ for all $x, y, z \in X$ and for all $s, t > 0$.

*Corresponding Author: Ashlekha Dubey**
Government P.G. College Pipariya - (M.P.), India.

We will refer to these spaces as KM-fuzzy metric Spaces.

Example 2.3[4]: Let (X, d) be a metric space $a*b = T_M(a, b)$, and for all x, y ∈ X and t > 0
$M(x, y, t) = \frac{t}{t+d(x,y)}$ for all t > 0, M(x, y, 0) = 0, for all x, y ∈ X then (X, M, ∗) is a fuzzy metric space, it is called the fuzzy metric space induced by (X, d).

Definition 2.4 (George and Veeramani [2, 3]): A Fuzzy Metric Space is a triple (X, M, ∗) where X is a nonempty set , ∗ is a continous t-norm and M is a fuzzy set on $X^2 \times [0, 1]$ such that the following axioms hold:
(GV-1) M(x, y, t) > 0;
(GV-2) M(x, y, t) =1 ⇔ x=y;
(GV-3) M(x, y, t) = M(y, x, t) ;
(GV-4) M(x, y, .): [0, ∞)→[0, 1] is continuous;
(GV-5) M(x, z, t+s) ≥ M(x, y, t) ∗ M(y, z, s) for all x, y, z ∈ X and for all s, t > 0.

From (GV-1) to (GV-2), it follows that x ≠ y, then 0 < M(x, y, t) < 1for all t > 0.

Fuzzy metric spaces in the sense of George and Veeramani will be called GV- fuzzy metric spaces.

Lemma 2.5 Grabiec [5]: For every x, y ∈ X, the mappings M(x, y, .) is a non decreasing function.

Definition 2.6 [6]: Let (X, M, ∗) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent to x∈ X if
$\lim_{n\to\infty} M(x_n, x, t) = 1$ for all t>0

Further, the sequence $\{ x_n \}$ said to be Cauchy sequence in X if
$\lim_{n\to\infty} M(x_n, x_{n+m}, t) = 1$
for all t>0 and m ∈ N.

A fuzzy metric space (X, M, ∗) is called complete if every Cauchy sequence converges to a point in X

Definition 2.7 [6]: Two self mappings A and S on a fuzzy metric spaces (X, M, ∗) are said to be compatible if $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ for all t > 0, . Whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$.

Definition 2.8 [6]: Two self mappings A and S on a fuzzy metric spaces (X, M, ∗) are said to be weakly compatible if they commute at their coincidence points, that is if for x∈ X, Ax = Sx implies that M(ASx, SAx, t)=1 for all t > 0.

Definition 2.9 [6]: Two self mappings A and S on a fuzzy metric spaces (X, M, ∗) are said to be semi weakly compatible if M(ASz, SAz, t) = 1 for all t>0,where z is a fixed point of either A or S.

Definition 2.10 (E.A like property): Let A and B be two self maps of a fuzzy metric space (X, M, ∗) .We say that A and B satisfy the"E.A like property" if there exists a sequence $\{x_n\}$ such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some z ∈ A(X) or z ∈ B(X), i.e., z ∈ A(X) ∪ B(X).

[7] Common E.A like property

Let A, B, S, T : X→ X where X is a fuzzy metric space then the pair (A,S) and (B,T) said to satisfy "Common E.A like property" if there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$
for some z ∈ S(X)∩T(X) or z ∈ A(X)∩B(X).

Example 1: Let X = [-1, 1] and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all x, y ∈ X then (X, M, *) is a fuzzy metric space where T(a, b) = min {a, b}.

Define the self mappings A, B, S and T as
$A(X) = \left(\frac{x}{2} - \frac{1}{6}\right)$, $B(X) = \left(\frac{x}{4}\right)$, $S(X) = \left(\frac{x+1}{2} - \frac{2}{3}\right)$,$T(X) = x^3$

Define the sequences $\{x_n\}$ and $\{y_n\}$ where $x_n = \left(\frac{1}{3} + \frac{1}{n}\right)$ and $y_n = \frac{1}{2n}$

We have $A(X) = [\frac{-2}{3}, \frac{1}{3}]$, $B(X) = [\frac{-1}{4}, \frac{1}{4}]$, $S(X) = [\frac{-2}{3}, \frac{1}{3}]$, $T(X) = [-1, 1]$,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty}\left(\frac{\left(\frac{1}{3}+\frac{1}{n}\right)}{2} - \frac{1}{6}\right) = 0 \in S(X)$$

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty}\left(\frac{\left(\frac{1}{3}+\frac{1}{n}\right)+1}{2} - \frac{2}{3}\right) = 0 \in A(X)$$

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty}\left(\frac{1}{2n}\right)^3 = 0 \in B(X)$$

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty}\left(\frac{\frac{1}{2n}}{4}\right) = 0 \in T(X)$$

Thus $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = 0$
Where $0 \in S(X)\cap T(X)$ or $z \in A(X)\cap B(X)$.

Hence the pairs (A, S) and (B, T) satisfies Common E.A. Like Property.

Definition 2.11 (A class of implicit relation): Let $\emptyset$ be the set of all real continuous function $\emptyset: (R^+)^4 \to R$, non decreasing in first argument and satisfying the following conditions .

1) For $u, v \geq 0$, $\varphi\{u, 1, v, 1\} \geq 0$ or $\varphi\{u, 1, 1, v\} \geq 0$ implies that $u \geq v$.
2) $\varphi\{u, u, 1, 1\} \geq 0$ implies that $u \geq 1$.

Lemma 2.12 [1]: Let $(X, M, *)$ be a fuzzy metric space. If there exists a number $k \in (0, 1)$
$M(x, y, kt) \geq M(x, y, t)$ for all $x, y \in X$ & $t > 0$ then $x = y$.

3. MAIN RESULTS

R.K. Sharma and Sonal Bharti [6] proved the following result-

Theorem: Let A, B, S, T, P and Q be self mappings of a complete fuzzy metric space $(X, M, *)$ satisfying

1. $A(X) \subseteq QT(X)$, $B(X) \subseteq PS(X)$;
2. the pairs (A, PS) is semi–compatible and (B, QT) is weakly compatible;
3. One of A or PS is continuous;
4. For some $\varphi \in \emptyset$,there exist $k \in (0,1)$ such that for every $x,y \in X$ and $t > 0$
   $\varphi\{M(Ax, By, Kt), M(PSx, QTy, Kt), M(Ax, PSx, t), M(By, QTy, t)\} \geq 0$
   $\varphi\{M(Ax, By, Kt), M(PSx, QTy, t), M(Ax, PSx, kt), M(By, QTy, t)\} \geq 0$.
5. The pairs (P, S) and (Q,T) are commuting mappings.
6. The pairs (P, A), (S, A), (Q, B) (T, B) are semi weakly compatible mappings.
   Then A, B, S, T, P and Q have unique common fixed point in X.

Now we prove our main results for weakly compatible maps under Common E.A like property for four & six self mappings as follows:

Theorem 3.1: Let A, B, P and Q be self mappings of a complete fuzzy metric space $(X, M, *)$ satisfying

1. Pairs (A, P) and (B,Q) satisfy common E.A like property.
2. Pairs (A, P) and (B,Q) are weakly compatible.
3. For some $\varphi \in \emptyset$,there exist $k \in (0,1)$ such that for every $x,y \in X$ and $t > 0$
   $\varphi\{M(Ax, By, Kt), M(Px, Qy, Kt), M(Ax, Px, t), M(By, Qy, t)\} \geq 0$ (3.1.1)
   then A,B,P and Q have unique common fixed point in X.

Proof: Since (A,P) and (B,Q) satisfy common E.A like property therefore there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z$$

Where $z \in P(X)\cap Q(X)$ or $z \in A(X)\cap B(X)$

Suppose $z \in P(X) \cap Q(X)$ now we have $\lim_{n\to\infty} Ax_n = z \in P(X)$ then $z=Pu$ for some $u \in X$

Now we claim that $Au=Pu$, from (3.1.1) we have,

$$\varphi\{M(Au, By_n, Kt), M(Pu, Qy_n, Kt), M(Au, Pu, t), M(By_n, Qy_n, t)\} \geq 0$$

Taking limit $n\to\infty$, we get

$$\varphi\{M(Au, z, Kt), M(z, z, Kt), M(Au, z, t), M(z, z, t)\} \geq 0$$
$$\varphi\{M(Au, z, Kt), 1, M(Au, z, t), 1\} \geq 0$$

From 2.11 (1) we have,

$$M(Au, z, Kt) \geq M(Au, z, t)$$

Hence from the lemma 2.12 we have Au = z implies that Au = z = Pu,

Since the pairs (A, P) is weakly compatible therefore Az = APu = Pau = Pz.

Again $\lim_{n\to\infty} By_n = z \in Q(X)$ then z = Qv for some v ∈ X

Now we claim that Qv = Bv, from (3.1.1) we have,

$$\varphi\{M(Ax_n, Bv, Kt), M(Px_n, Qv, Kt), M(Ax_n, Px_n, t), M(Bv, Qv, Kt)\} \geq = 0$$

Taking limit n→∞, we get

$$\varphi\{M(z, Bv, Kt), M(z, z, Kt), M(z, z, t), M(Bv, z, t)\} \geq 0$$
$$\varphi\{M(z, Bv, Kt), 1,1, M(Bv, z, t)\} \geq 0$$

From 2.11 (1) we have,

$$M(z, Bv, Kt) \geq M(Bv, z, t)$$

Hence from the lemma 2.12 we have z = Bv implies that z = Qv = Bv.

Since the pairs (B, Q) is weak compatible, therefore Qz = QBv = BQv = Bz.

Now we show that Az=z, from (3.1.1) we have

$$\varphi\{M(Az, By_n, Kt), M(Pz, Qy_n, Kt), M(Az, Pz, t), M(By_n, Qy_n, t)\} \geq 0$$

Taking limit n→∞, we get

$$\varphi\{M(Az, z, Kt), M(z, z, Kt), M(Az, z, t), M(z, z, t)\} \geq 0$$
$$\varphi\{M(Az, z, Kt), 1, M(Az, z, t), 1\} \geq 0$$

From 2.11 (1) we have,

$$M(Az, z, Kt) \geq M(Az, z, t)$$

Hence from lemma 2.12, we have Az = z

Now we show that Bz = z, from (3.1.1) we have

$$\varphi\{M(Ax_n, Bz, Kt), M(Px_n, Qz, Kt), M(Ax_n, Px_n, t), M(Bz, Qz, t)\} \geq 0$$

Taking limit n→∞, we get

$$\varphi\{M(z, Bz, Kt), M(z, z, Kt), M(z, z, t), M(Bz, z, t)\} \geq 0$$
$$\varphi\{M(z, Bz, Kt), 1,1, M(Bz, z, t)\} \geq 0$$

From 2.11 (1) we have,

$$M(Bz, z, Kt) \geq M(Bz, z, t)$$

Hence from the lemma 2.12 we have Bz = z also Qz = z therefore Az = Pz = Bz = Qz = z.

Uniqueness: To prove the uniqueness we suppose that z and w are the two common fixed point of A, B, P & Q.

Putting x = z and y = w in (3.1.1), we get

$$\varphi\{M(Az, Bw, Kt), M(Pz, Qw, Kt), M(Az, Pz, t), M(Bw, Qw, t)\} \geq 0$$
$$\varphi\{M(z, w, Kt), M(z, w, Kt), M(z, z, t), M(w, w, t)\} \geq 0$$
$$\varphi\{M(z, w, Kt), M(z, w, Kt), 1,1\} \geq 0$$

From 2.11 (2) we have,

$M(z, w, Kt) \geq 1$ Yields that z = w

Hence Aw = Pw = Bw = Qw = w. w is the unique common fixed point of the self maps A, B, P and Q .

Theorem 3.2: Let A, B, S, T, P and Q be self mappings of a complete fuzzy metric space (X, M, $*$) satisfying

1. Pairs (A, PS) and (B, Q T) satisfy Common E.A like property.
2. Pairs (A, PS) and (B, QT) are weakly compatible.
3. Pairs (P, S) and (Q, T) are commuting mappings.
4. For some $\varphi \in \emptyset$, there exist k ∈ (0,1) such that for every x,y ∈ X and t > 0
   $\varphi\{M(Ax, By, Kt), M(PSx, QTy, Kt), M(Ax, PSx, t), M(By, QTy, t)\} \geq 0$ (3.2.1)
5. Pairs (P, S) and (Q, T) are commuting mappings.
6. Pairs (P, A), (S, A),(Q, B) (T,B) are semi weakly compatible mappings.
   then A, B, S, T, P and Q have unique common fixed point in X.

Proof. Since (A, PS) and (B, QT) satisfy Common E.A like property therefore there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} PSx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} QTy_n = z$$

Where $z \in PS(X) \cap QT(X)$ or $z \in A(X) \cap B(X)$

Suppose $z \in PS(X) \cap QT(X)$ now we have $\lim_{n\to\infty} Ax_n = z \in PS(X)$ then z = PSu for some u ∈ X

Now we claim that Au=PSu, from (3.2.1) we have,

$$\varphi\{M(Au, By_n, Kt), M(PSu, QTy_n, Kt), M(Au, PSu, t), M(By_n, QTy_n, t)\} \geq 0$$

Taking limit n→∞, we get

$$\varphi\{M(Au, z, Kt), M(z, z, Kt), M(Au, z, t), M(z, z, t)\} \geq 0$$
$$\varphi\{M(Au, z, Kt), 1, M(Au, z, t), 1\} \geq 0$$

From 2.11 (1) we have,

$$M(Au, z, Kt) \geq M(Au, z, t)$$

Hence from the lemma 2.12, we have Au = z implies that Au = z = PS.

Since the pairs (A, PS) is weakly compatible therefore Az = APSu = PSAu = PSz.

Again $\lim_{n\to\infty} By_n = z \in QT(X)$ then z=Tv for some v ∈ X

Now we claim that QTv = Bv, from (3.2.1) we have,

$$\varphi\{M(Ax_n, Bv, Kt), M(PSx_n, QTv, Kt), M(Ax_n, PSx_n, t), M(Bv, QTv, Kt)\} \geq 0$$

Taking limit n→∞, we get

$$\varphi\{M(z, Bv, Kt), M(z, z, Kt), M(z, z, t), M(Bv, z, t)\} \geq 0$$
$$\varphi\{M(z, Bv, Kt), 1, 1, M(Bv, z, t)\} \geq 0$$

From 2.11 (1) we have,

$$M(z, Bv, Kt) \geq M(Bv, z, t)$$

Hence from the lemma 2.12 we have z = QTv = Bv

Since the pairs (B, QT) is weakly compatible, therefore QTz = QTBv = BQTv = Bz.

Now we show that Az = z, from (3.2.1) we have

$$\varphi\{M(Az, By_n, Kt), M(PSz, QTy_n, Kt), M(Az, PSz, t), M(By_n, QTy_n, t)\} \geq 0$$

Taking limit n→∞, we get

$$\varphi\{M(Az, z, Kt), M(z, z, Kt), M(Az, z, t), M(z, z, t)\} \geq 0$$
$$\varphi\{M(Az, z, Kt), 1, M(Az, z, t), 1\} \geq 0$$

From 2.11 (1) we have,

$$M(Az, z, Kt) \geq M(Az, z, t)$$

Hence from lemma 2.12 Az = z

Now we show that Bz = z, from (3.2.1) we have

$$\varphi\{M(Ax_n, Bz, Kt), M(PSx_n, QTz, Kt), M(Ax_n, PSx_n, t), M(Bz, QTz, t)\} \geq 0$$

Taking limit $n\to\infty$, we get

$$\varphi\{M(z, Bz, Kt), M(z, z, Kt), M(z, z, t), M(Bz, z, t)\} \geq 0$$
$$\varphi\{M(z, Bz, Kt), 1, 1, M(Bz, z, t)\} \geq 0$$

From 2.11 (1) we have,

$$M(Bz, z, Kt) \geq M(Bz, z, t)$$

Now it is clear from the lemma 2.12 that Bz = z hence QTz = z.

Therefore Az = PSz = Bz = QTz = z .

Uniqueness: To prove the uniqueness we suppose that z and w are the two common fixed point of A, B, PS & QT.

Putting x = z and y = w in (3.2.1), we get

$$\varphi\{M(Az, Bw, Kt), M(PSz, QTw, Kt), M(Az, PSz, t), M(Bw, QTw, t)\} \geq 0$$
$$\varphi\{M(z, w, Kt), M(z, w, Kt), M(z, z, t), M(w, w, t)\} \geq 0$$
$$\varphi\{M(z, w, Kt), M(z, w, Kt), 1, 1\} \geq 0$$

From 2.11 (2) we have,

$M(z, w, Kt) \geq 1$ Yields that z = w.

Hence z is the unique common fixed point of the self maps A, B, PS and QT Az = PSz = Bz = QTz = z.

By using condition 5 and 6 we have

Pz = P(PSz) = P(SPz) = PS(Pz);
Pz = P(Az) = A(Pz);
Sz = S(PSz) = SP(Sz) = PS(Sz);
Sz = SAz = ASz;

Pz and Sz are common fixed points of the maps PS and A

Therefore z = Pz = Sz = Az = PSz (3.2.2)

Qz = Q(QTz) = Q(TQz) = QT(Qz);
Qz = Q(Bz) = B(Qz);
Tz = T(QTz) = TQ(Tz) = QT(Tz);
Tz = TBz = BTz;

Qz and Tz are common fixed points of the maps QT and B

Therefore z = Qz = Tz = Bz = QTz (3.2.3)

From (3.2.2) and (3.2.3) we have z = Az = Bz = Sz = Tz = Pz = Q that's why z is the common fixed point of the maps A, B, PS and QT Consequently it is the unique common fixed point of the maps A, B, S, T, P, Q.

4. CONCLUSION

Our results improve the results of R.K. Sharma and Sonal Bharti [6] in the following senses:

i. Containment of ranges has been removed in theorem 3.2.
ii. Continuity of mappings is not needed in theorem 3.2.

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