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CONVOLUTION PROPERTY OF MULTIVALENT FUNCTIONS WITH COEFFICIENT OF ALTERNATING TYPE USING q-DERIVATIVE

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ABSTRACT

By applying the concept of fractional q-calculus, we investigate coefficient bounds and convolution results of multivalent functions with coefficients of alternating type

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1. INTRODUCTION

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$.

For $-1 \le A < B \le 1$, let $P(A, B)$ [3] denote the class of functions which are of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where ω is a bounded analytic function satisfying the conditions $\omega(0) = 0 \ \ and |\omega(z)| < 1.$

We consider another subclass M_p which consists of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}, \ a_{k+1} \ge 0.$$

The q-shifted factorial is defined for $\alpha, q \in C$ as a product of n factors by

$$(\alpha, q)_n = \begin{cases} 1, & n = 0; \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & n \in N \end{cases} \tag{1.2}$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, (n > 0), \tag{1.3}$$

where the q-gamma functions [1, 2] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty} (0 < q < 1). \tag{1.4}$$

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Note that, if $|q|<1,$ the q-shifted factorial (1.2) remains meaningful for $n=\infty$ as a convergent infinite product

$$(\alpha;q)_\infty = \prod_{m=0}^{\infty}(1-\alpha q^m).$$

Now recall the following q-analogue definitions given by Gasper and Rahman [1]. The recurrence relation for q-gamma function is given by

$$\Gamma_p(x+1)=[x]_q\Gamma_p(x), where, [x]_q = \frac{(1-q^x)}{(1-q)}, \tag{1.5}$$

and called q-analogue of $x.$

Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [1])

$$D_q f(z) = \frac{f(z)-f(zq)}{z(1-q)}, (z\neq 0, q\neq 0). \tag{1.6}$$

$$\int_0^z f(t)d_q(t) = z(1-q)\sum_{m=0}^{\infty} q^m f(zq^m). \tag{1.7}$$

In view of the relation

$$\lim_{q\to 1^-}\frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n, \tag{1.8}$$

we observe that the q-shifted fractional (1.1) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n+1).$

Consider the following definitions.

$$S_q^*(A,B) = \left\{ f \mid f\in A_p \quad and \quad \frac{zD_q(f(z))}{f(z)} \in P(A,B)\right\}$$

$$H_q(A,B) = \left\{ f \mid f\in A_p \quad and \quad \frac{D_q(zD_q(f(z)))}{D_q(f(z))} \in P(A,B)\right\}$$

$$M_q^*(A,B) = \left\{ f \mid f\in M_p \quad and \quad \frac{zD_q(f(z))}{f(z)} \in P(A,B)\right\}$$

$$C_q(A,B) = \left\{ f \mid f\in M_p \quad and \quad \frac{D_q(zD_q(f(z)))}{D_q(f(z))} \in P(A,B)\right\}.$$

Note that these classes generalize the classes of Padamanabhan and Ganeshan [5], Silverman [6], Khairnar and Meena More [4].

If $f(z) = z^p + \sum_{k=p+1}^{\infty}(-1)^{k+1}a_{k+1}z^{k+1}$ and $g(z) = z^p + \sum_{k=p+1}^{\infty}(-1)^{k+1}b_{k+1}z^{k+1}$, then their modified hadamard product is defined by

$$h(z) = f(z)*g(z) = z^p + \sum_{k=p+1}^{\infty}(-1)^{k+1}a_{k+1}b_{k+1}z^{k+1}.$$

In this paper we discuss some properties of convolution for the class $M_q^*(A,B)$ and $C_q(A,B).$

2. MAIN RESULTS

Lemma 2.1 *A function* $f(z) = z^p + \sum_{k=p+1}^{\infty}(-1)^{k+1}a_{k+1}z^{k+1}$, $a_{k+1}\geq 0$ *is in* $M_q^*(A,B)$ *if and only if*

$$\sum_{k=p+1}^{\infty}\left\{\frac{[k+1]_q(B+1)-(A+1)}{(A+1)-(B+1)[p]_q}\right\}a_{k+1} \leq 1. \tag{2.1}$$

Proof: Consider

$$f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}$$

$$\frac{zD_q f(z)}{f(z)} = \frac{[p]_q z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} [k+1]_q z^{k+1}}{z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}}.$$

Now $\frac{zD_q f(z)}{f(z)} \in P(A,B)$ if and only if

$$\frac{[p]_q z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} [k+1]_q z^{k+1}}{z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}} = \frac{1+A\omega(z)}{1+B\omega(z)}$$

$$\omega(z)\left[(B[p]_q - A)z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} (B[k+1]_q - A) a_{k+1} z^{k+1}\right] = (1-[p]_q) z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} (1-[k+1]_q) a_{k+1} z^{k+1}$$

by using the condition $|\omega(z)| \le 1$, we get

$$\left| \frac{(1-[p]_q) z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} (1-[k+1]_q) a_{k+1} z^{k+1}}{(B[p]_q - A) z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} (B[k+1]_q - A) a_{k+1} z^{k+1}} \right| \le 1.$$

Allowing $|z| = r \to 1$

$$\left\{ \frac{(1-[p]_q) + \sum_{k=p+1}^{\infty} (1-[k+1]_q) a_{k+1}}{(B[p]_q - A) + \sum_{k=p+1}^{\infty} (B[k+1]_q - A) a_k} \right\} \le 1$$

$$\sum_{k=p+1}^{\infty} [1-(B+1)[k+1]_q + A] a_{k+1} \le (B+1)[p]_q - (A+1)$$

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q (B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} \right\} a_{k+1} \le 1.$$

and the result follows.

As a consequence we have the following result.

Lemma 2.2: *A function* $f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}$, $a_{k+1} \ge 0$ *is in* $C_q(A,B)$ *if and only if*

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[k+1]_q \{[k+1]_q (B+1) - (A+1)\}}{[p]_q \{(A+1) - (B+1)[p]_q\}} \right\} a_{k+1} \le 1. \quad (2.2)$$

Theorem 2.3: *If* $f(z)=z^p+\sum_{k=p+1}^{\infty}(-1)^{k+1}a_{k+1}z^{k+1}$ *and* $g(z)=z^p+\sum_{k=p+1}^{\infty}(-1)^{k+1}b_{k+1}z^{k+1}$, $a_{k+1},b_{k+1}\geq 0$ *are elements of classes* $M_q^*(A,B)$ *then* $h(z)=f(z)*g(z)=z^p+\sum_{k=p+1}^{\infty}(-1)^{k+1}a_{k+1}b_{k+1}z^k$ is an element of $M_q^*(A_1,B_1)$ with $-1\leq A_1<B_1\leq 1$ where $A_1\geq -1$, $B_1\leq \dfrac{A_1+1-s}{s}$.

Proof: Since $f,g\in M_q^*(A,B)$, by Lemma 2.1 we have,

$$\sum_{k=p+1}^{\infty}\left\{\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}\right\}a_{k+1}\leq 1$$

$$\sum_{k=p+1}^{\infty}\left\{\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}\right\}b_{k+1}\leq 1.$$

We need A_1,B_1 such that $-1\leq A_1<B_1\leq 1$ and

$$h(z)=f(z)*g(z)\in M_q^*(A_1,B_1).$$

Now $h(z)\in M_q^*(A_1,B_1)$ if

$$\sum_{k=p+1}^{\infty}\left\{\frac{[[k+1]_q(B_1+1)-(A_1+1)]}{(A_1+1)-(B_1+1)[p]_q}\right\}a_{k+1}b_{k+1}\leq 1. \tag{2.3}$$

i.e., $\sum_{k=p+1}^{\infty}u_1a_{k+1}b_{k+1}\leq 1$, where $u_1=\dfrac{[[k+1]_q(B_1+1)-(A_1+1)]}{(A_1+1)-(B_1+1)[p]_q}$.

Using Cauchy-Schwarz inequality we have,

$$\sum_{k=p+1}^{\infty}\sqrt{ua_{k+1}b_{k+1}}\leq\left\{\sum_{k=p+1}^{\infty}ua_{k+1}\right\}^{\frac{1}{2}}\left\{\sum_{k=p+1}^{\infty}ub_{k+1}\right\}^{\frac{1}{2}}\leq 1,$$

where $u=\dfrac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}$.

(2.3) is true if

$$u_1a_{k+1}b_{k+1}\leq u\sqrt{a_{k+1}b_{k+1}}$$

using (2.3) we have,

$$u_1\sqrt{a_{k+1}b_{k+1}}\leq 1 \quad \text{for} \quad k=2,3\cdots$$

Therefore it is enough to find u_1 such that

$$\frac{1}{u}\leq\frac{u}{u_1}.$$

i.e., $\dfrac{[[k+1]_q(B_1+1)-(A_1+1)]}{(A_1+1)-(B_1+1)[p]_q}\leq u^2$

$$A_1\geq -1+\frac{(B_1+1)([k+1]_q+[p]_qu^2)}{u^2+1}.$$

Consider $B_1=1$ and $k=2$ to obtain,

$$A_1 \geq -1 + \frac{2([3]_q + [p]_q u^2)}{u^2+1}$$
$$= -1 + 2\frac{\left[[3]_q\left((A+1)-(B+1)[p]_q\right)^2 + [p]_q\left([3]_q(B+1)-(A+1)\right)^2\right]}{\left((A+1)-(B+1)[p]_q\right)^2 + \left([3]_q(B+1)-(A+1)\right)^2}$$
$$= -1 + 2s$$

where $s = \frac{\left[[3]_q\left((A+1)-(B+1)[p]_q\right)^2 + [p]_q\left([3]_q(B+1)-(A+1)\right)^2\right]}{\left((A+1)-(B+1)[p]_q\right)^2 + \left([3]_q(B+1)-(A+1)\right)^2}.$

Theorem 2.4: *If* $f(z) \in M_q^*(A,B)$ *and* $g(z) \in M_q^*(A',B')$ *then* $f(z)*g(z) \in M_q^*(A_1,B_1)$ *where* $A_1 \geq -1, B_1 \leq \frac{A_1+1-x}{x}$, *with*

$$x = \frac{[3]_q\left((A+1)-(B+1)[p]_q\right)\left((A'+1)-(B'+1)[p]_q\right) + [p]_q\left([3]_q(B+1)-(A+1)\right)\left([3]_q(B'+1)-(A'+1)\right)}{\left((A+1)-(B+1)[p]_q\right)\left((A'+1)-(B'+1)[p]_q\right) + \left([3]_q(B+1)-(A+1)\right)\left([3]_q(B'+1)-(A'+1)\right)}.$$

Proof: Analogously proceeding as developed in Theorem 2.3, we require

$$\left\{\frac{[[k+1]_q(B_1+1)-(A_1+1)]}{(A_1+1)-(B_1+1)[p]_q}\right\} \leq \left\{\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}\right\}\left\{\frac{[[k+1]_q(B'+1)-(A'+1)]}{(A'+1)-(B'+1)[p]_q}\right\} = \delta.$$

i.e.,
$$\frac{B_1+1}{A_1+1} \leq \frac{\delta+1}{[k+1]_q + \delta[p]_q}$$
$$\frac{A_1+1}{B_1+1} \geq \frac{[k+1]_q + \delta[p]_q}{\delta+1}.$$
$$\frac{A_1+1}{B_1+1} \geq$$
$$\frac{[k+1]_q\left((A+1)-(B+1)[p]_q\right)\left((A'+1)-(B'+1)[p]_q\right) + [p]_q\left([k+1]_q(B+1)-(A+1)\right)\left([k+1]_q(B'+1)-(A'+1)\right)}{\left((A+1)-(B+1)[p]_q\right)\left((A'+1)-(B'+1)[p]_q\right) + \left([k+1]_q(B+1)-(A+1)\right)\left([k+1]_q(B'+1)-(A'+1)\right)}.$$

Taking $k=2$, we get $\frac{A_1+1}{B_1+1} \geq$

$$\frac{[3]_q\left((A+1)-(B+1)[p]_q\right)\left((A'+1)-(B'+1)[p]_q\right) + [p]_q\left([3]_q(B+1)-(A+1)\right)\left([3]_q(B'+1)-(A'+1)\right)}{\left((A+1)-(B+1)[p]_q\right)\left((A'+1)-(B'+1)[p]_q\right) + \left([3]_q(B+1)-(A+1)\right)\left([3]_q(B'+1)-(A'+1)\right)} = x.$$

i.e.,
$$\frac{A_1+1}{B_1+1} \geq x,$$
$$B_1 \leq \frac{A_1+1-x}{x}.$$ But $B_1 \geq -1$, we get $A_1 > -1.$

Theorem 2.5: *If* $f(z) \in C_q(A,B)$ *and* $g(z) \in C_q(A',B')$ *then* $f(z)*g(z) \in C_q(A_1,B_1)$ *where* $A_1 \geq -1$, $B_1 \leq \frac{A_1+1-y}{y}$ with

$$y = \frac{[3]_q[p]_q\left[\begin{array}{l}[3]_q\left((A+1)-(B+1)[p]_q\right)\left((A'+1)-(B'+1)[p]_q\right)\\ +[p]_q\left([3]_q(B+1)-(A+1)\right)\left([3]_q(B'+1)-(A'+1)\right)\end{array}\right]}{[p]_q\left((A+1)-(B+1)[p]_q\right)\left((A'+1)-(B'+1)[p]_q\right) + [k+1]_q\left([3]_q(B+1)-(A+1)\right)\left([3]_q(B'+1)-(A'+1)\right)}.$$

Proof: The proof of the theorem follows the patteren of that in Theorem 2.4

$$\frac{A_1+1}{B_1+1} \geq$$

$$\frac{[3]_q[p]_q\big[[3]_q\big((A+1)-(B+1)[p]_q\big)\big((A'+1)-(B'+1)[p]_q\big)+[p]_q\big([3]_q(B+1)-(A+1)\big)\big([3]_q(B'+1)-(A'+1)\big)\big]}{[p]_q\big((A+1)-(B+1)[p]_q\big)\big((A'+1)-(B'+1)[p]_q\big)+[k+1]_q\big([3]_q(B+1)-(A+1)\big)\big([3]_q(B'+1)-(A'+1)\big)} = y.$$

$$\frac{\gamma_1 - B_1 p}{B_1+1} \geq y$$

$$B_1 \leq \frac{A_1+1-y}{y}. \tag{2.4}$$

$B_1 \geq -1,$ using (2.4) we get $A_1 \geq -1.$

Theorem 2.6: *If* $f(z) = z^p + \sum_{k=p+1}^{\infty}(-1)^{k+1}a_{k+1}z^{k+1}, a_{k+1} \geq 0 \in M_q^*(A,B)$ *and* $g(z) = z^p + \sum_{k=p+1}^{\infty}(-1)^{k+1}b_{k+1}z^{k+1}$ *with* $|b_{k+1}| \leq 1$ *for* $k \geq 2$ *then* $f(z)*g(z) \in S_q^*(A,B).$

Proof: Since $f \in M_q^*(A,B)$ we have,

$$\sum_{k=p+1}^{\infty}\left\{\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}\right\}a_{k+1}b_{k+1} \leq \sum_{k=p+1}^{\infty}\left\{\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}\right\}a_{k+1}|b_{k+1}| \leq 1.$$

This shows that

$$f(z)*g(z) = z^p + \sum_{k=p+1}^{\infty}(-1)^{k+1}a_{k+1}b_{k+1}z^{k+1} \in S_q^*(A,B).$$

The proof of Theorem 2.7 below follows the patteren of that in Theorem 2.6.

Theorem 2.7: *If* $f(z) = z^p + \sum_{k=p+1}^{\infty}(-1)^{k+1}a_{k+1}z^{k+1}, a_{k+1} \geq 0 \in C_q(A,B)$ *and* $g(z) = z^p + \sum_{k=p+1}^{\infty}(-1)^{k+1}b_{k+1}z^{k+1}$ *with* $|b_{k+1}| \leq 1$ *for* $k \geq 2$ *then* $f(z)*g(z) \in H_q(A,B).$

Theorem 2.8: *If* $f,g \in M_q^*(A,B)$ *then* $h(z) = z^p + \sum_{k=p+1}^{\infty}(-1)^{k+1}(a_{k+1}^2 + b_{k+1}^2)z^{k+1} \in M_q^*(A_1,B_1,p,\alpha)$

where $A_1 \geq -1, B_1 \leq \dfrac{A_1+1-s}{s}$, with $s = \dfrac{2[3]_q\big((A+1)-(B+1)[p]_q\big)^2 + [p]_q\big([3]_q(B+1)-(A+1)\big)^2}{2\big((A+1)-(B+1)[p]_q\big)^2 + \big([3]_q(B+1)-(A+1)\big)^2}.$

Proof: Since $f(z), g(z) \in M_q^*(A,B)$ we have,

$$\sum_{k=p+1}^{\infty}\left\{\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}a_{k+1}\right\} \leq 1$$

$$\sum_{k=p+1}^{\infty}\left\{\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}\right\}b_{k+1} \leq 1.$$

We see that

$$\sum_{k=p+1}^{\infty}\left\{\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}\right\}^2 a_{k+1}^2 \leq \left\{\sum_{k=p+1}^{\infty}\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}a_{k+1}\right\}^2 \leq 1. \tag{2.5}$$

$$\left\{\sum_{k=p+1}^{\infty}\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}b_{k+1}\right\}^2 \leq 1 \tag{2.6}$$

Adding (2.5) and (2.6) we get

$$\sum_{k=p+1}^{\infty}\left\{\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}\right\}^2(a_{k+1}^2+b_{k+1}^2)\leq 2. \tag{2.7}$$

Now $f(z),g(z)\in M_q^*(A_1,B_1,p,\alpha)$

$$\sum_{k=p+1}^{\infty}\left\{\frac{[[k+1]_q(B_1+1)-(A_1+1)]}{(A_1+1)-(B_1+1)[p]_q}\right\}(a_{k+1}^2+b_{k+1}^2)\leq 1$$

(2.7) implies that it is enough show that

$$\frac{[[k+1]_q(B_1+1)-(A_1+1)]}{(A_1+1)-(B_1+1)[p]_q}\leq\frac{1}{2}\left\{\frac{[[k+1]_q(B+1)-(A+1)]}{(A+1)-(B+1)[p]_q}\right\}^2=\frac{u^2}{2}.$$

i.e.,
$$\frac{k(B_1+1)}{A_1+1}\leq\frac{u^2+2}{2[k+1]_q+[p]_q u^2}$$

$$\frac{A_1+1}{k(B_1+1)}\geq\frac{2[k+1]_q+[p]_q u^2}{u^2+2}=\beta(k).$$

Notice that $\beta(k)$ decreases as k increases and replacing k by 2 and simplifying we obtain, $\gamma_1\geq -p, B_1\leq\frac{A_1+1-s}{s}$, with $s=\frac{2[3]_q\left((A+1)-(B+1)[p]_q\right)^2+[p]_q\left([3]_q(B+1)-(A+1)\right)^2}{2\left((A+1)-(B+1)[p]_q\right)^2+\left([3]_q(B+1)-(A+1)\right)^2}$.

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