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RELATIVE L-RITT ORDER OF ENTIRE DIRICHLET SERIES

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ABSTRACT

We introduce the idea of relative L-Ritt order of entire Dirichlet series with respect to a meromorphic function and prove sum and product theorems and a theorem on derivative.

Keywords: *Entire Dirichlet series, Relative order, L-Ritt order.*

AMS Subject Classification: *30B50, 30D99.*

1. INTRODUCTION AND DEFINITIONS

For entire function f let $F(r)=\max\{|f(z)|:|z|=r\}$. If f is non constant then $F(r)$ is strictly increasing and a continuous function of r and its inverse

$$F^{-1}:(|f(0)|,\infty)\to(0,\infty) \text{ exists and } \lim_{R\to\infty}F^{-1}(R)=\infty.$$

In 1988, Bernal [1] introduced the definition of relative order of f with respect to g denoted by $\rho_g(f)$, as

$$\rho_g(f)=\inf\{\mu>0:F(r)<G(r^{\mu}) \text{ for all } r>r_0(\mu)>0\}.$$

Let $f(s)$ be an entire function of the complex variable $s=\sigma+it$ defined by everywhere absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty}a_n e^{s\lambda_n} \tag{1.1}$$

where $0<\lambda_n<\lambda_{n+1}(n\ge 1)$, $\lambda_n\to\infty$ as $n\to\infty$ and $a_n^{'}s$ are complex constants.

Let $F(\sigma)=l.u.b\{|f(\sigma+it)|,-\infty<t<\infty\}$. Then the Ritt order [7] of $f(s)$, denoted by $\rho(f)$ is given by

$$\rho(f)=\limsup_{\sigma\to\infty}\frac{\log\log F(\sigma)}{\sigma}$$
$$=\inf\{\mu>0:\log F(\sigma)<\exp(\sigma\mu) \text{ for all } \sigma>\sigma(\mu)\}.$$

In [5] Lahiri and Banerjee introduced relative Ritt order as follows.

The relative Ritt order of $f(s)$ with respect to an entire $g(s)$ is defined by

$$\rho_g(f)=\inf\{\mu>0:\log F(\sigma)<G(\sigma\mu) \text{ for all large } \sigma\}$$

where $G(r)=\max\{|g(s)|:|s|=r\}$.

Let $L=L(\sigma)$ be a positive continuous function increasing slowly i.e., $L(a\sigma)\approx L(\sigma)$ as $\sigma\to\infty$ for every constant a.

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Then L-Ritt order [4] of $f(s)$ is defined as follows:

$$\rho^L(f)=\limsup_{\sigma\to\infty}\frac{\log\log F(\sigma)}{\sigma L(\sigma)}.$$

In 2014, A. Kumar and A. Rastogi [4] introduced relative L-Ritt order of an entire Dirichlet series as follows.

The relative L-Ritt order $\rho_g^L(f)$ of $f(s)$ with respect to $g(s)$ is defined as

$$\rho_g^L(f)=\limsup_{\sigma\to\infty}\frac{G^{-1}\log F(\sigma)}{\sigma L(\sigma)}.$$

At this stage it therefore seems reasonable to define suitably the relative L-Ritt order of entire Dirichlet series (1.1) with respect to a meromorphic function and to enquire its basic properties.

First we define characteristic function of an entire function $f(s)$ defined by everywhere absolutely convergent Dirichlet series (1.1) by

$$T_f(\sigma)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(\sigma e^{i\theta})|\,d\theta.$$

Clearly $T_f(\sigma)\le\log^+F(\sigma).$

The following definition is now introduced.

Definition 1.1: The relative L-Ritt order $\rho_g^L(f)$ of $f(s)$ with respect to a meromorphic $g(s)$ is defined as

$$\rho_g^L(f)=\inf\{\mu>0: T_f(\sigma)<\left[T_g(\sigma)L(\sigma)\right]^{\mu} \text{ for all large } \sigma\}$$

where $T_g(\sigma)$ is the Nevanlinna Characteristic function of $g(s)$.

Note 1.1: It is clear that $\rho_g^L(f)=\limsup_{\sigma\to\infty}\dfrac{\log T_f(\sigma)}{\log\left[T_g(\sigma)L(\sigma)\right]}.$

Definition 1.2: A non constant meromorphic function $g(s)$ is said to have the property (B) if for any $n>1$ and large $\sigma,\ T_g(n\sigma)=O(T_g(\sigma)).$

2. SUM AND PRODUCT THEOREMS

In this section we assume that f_1, f_2 etc., are entire functions of s defined by everywhere absolutely convergent ordinary Dirichlet series $\sum_{n=1}^{\infty}\frac{a_n}{n^s},\ \sum_{n=1}^{\infty}\frac{b_n}{n^s}$ etc. The product of two such series is considered by Dirichlet product method which is also everywhere absolutely convergent {see [3], pp 66}.

Theorem 2.1: Let f_1 and f_2 be entire functions having respective relative L-Ritt orders $\rho_g^L(f_1)$ and $\rho_g^L(f_2)$ with respect to meromorphic g.
Then (i) $\rho_g^L(f_1\pm f_2)\le\max\{\rho_g^L(f_1),\rho_g^L(f_2)\}$
and (ii) $\rho_g^L(f_1f_2)\le\max\{\rho_g^L(f_1),\rho_g^L(f_2)\}.$

Proof: We may suppose that $\rho_g^L(f_1)$ and $\rho_g^L(f_2)$ are both finite, because if one of $\rho_g^L(f_1)$, $\rho_g^L(f_2)$ or both are infinite, the inequality are evident.

Let $\rho_i=\rho_g^L(f_i)$, $i=1,2$ and $\rho_1\le\rho_2$.

For arbitrary $\varepsilon > 0$ and for all large σ, we have

$$T_{f_1}(\sigma) < \left[T_g(\sigma)L(\sigma)\right]^{\rho_1+\varepsilon} \le \left[T_g(\sigma)L(\sigma)\right]^{\rho_2+\varepsilon} \text{ and } T_{f_2}(\sigma) < \left[T_g(\sigma)L(\sigma)\right]^{\rho_2+\varepsilon}.$$

Now for all large σ,

$$\begin{aligned} T_{f_1 \pm f_2}(\sigma) &\le T_{f_1}(\sigma) + T_{f_2}(\sigma) + \log 2 \\ &< 3\left[T_g(\sigma)L(\sigma)\right]^{\rho_2+\varepsilon} \\ &< \left[T_g(\sigma)L(\sigma)\right]^{\rho_2+3\varepsilon}. \end{aligned}$$

So $\quad \rho_g^L(f_1 \pm f_2) \le \rho_2 + 3\varepsilon.$

Since $\varepsilon > 0$ is arbitrary,

$$\rho_g^L(f_1 \pm f_2) \le \rho_2 = \max\{\rho_g^L(f_1), \rho_g^L(f_2)\}$$

which proves (i).

For (ii), since

$$\begin{aligned} T_{f_1 f_2}(\sigma) &\le T_{f_1}(\sigma) + T_{f_2}(\sigma) \\ &< 2\left[T_g(\sigma)L(\sigma)\right]^{\rho_2+\varepsilon} \\ &< \left[T_g(\sigma)L(\sigma)\right]^{\rho_2+2\varepsilon}. \end{aligned}$$

So $\quad \rho_g^L(f_1 f_2) \le \rho_2 + 2\varepsilon.$

Since $\varepsilon > 0$ is arbitrary,

$$\rho_g^L(f_1 f_2) \le \rho_2 = \max\{\rho_g^L(f_1), \rho_g^L(f_2)\}.$$

3. RELATIVE L-RITT ORDER ON DERIVATIVE

Theorem 3.1: Let f be an entire function defined by (1.1) and g is transcendental such that g and g' satisfy property (B). Then $\rho_{g'}^L(f) = \rho_g^L(f)$.

We need the following lemmas.

Lemma 3.2 [6]: Let g be a transcendental meromorphic function. Then

$$T_{g'}(\sigma) \le 2T_g(2\sigma) + o\{T_g(2\sigma)\} \text{ for all large values of } \sigma.$$

Lemma 3.3 [2]: Let g be a meromorphic function. Then for all large σ,

$$T_g(\sigma) < C\{T_{g'}(2\sigma) + \log \sigma\}$$

where C is a constant which is only dependent on $g(0)$.

Proof of the theorem: From Lemmas (3.2) and (3.3) we have for all large σ,

$$T_{g'}(\sigma) < K_1 T_g(2\sigma) \tag{3.1}$$

and

$$T_g(\sigma) < K_2 T_{g'}(2\sigma) \tag{3.2}$$

where K_1 and K_2 are positive constants.

For arbitrary $\varepsilon > 0$ and for all large σ,

$$T_f(\sigma) \le \left[T_{g'}(\sigma)L(\sigma)\right]^{\rho_{g'}^L(f)+\varepsilon}.$$

So, $\log T_f(\sigma) < \left(\rho_{g'}^L(f)+\varepsilon\right)\left[\log K_1 + \log T_g(2\sigma) + \log L(\sigma)\right]$, using (3.1)

$= \left(\rho_{g'}^L(f)+\varepsilon\right)\left[\log(O(T_g(\sigma))) + \log L(\sigma) + O(1)\right]$, since g has the property (B)

$= \left(\rho_{g'}^L(f)+\varepsilon\right)[\log(T_g(\sigma)L(\sigma)) + O(1)]$

Therefore, $\limsup\limits_{\sigma\to\infty} \dfrac{\log T_f(\sigma)}{\log[T_g(\sigma)L(\sigma)]} \le \rho_{g'}^L(f)+\varepsilon.$

Since $\varepsilon > 0$ is arbitrary,

$$\rho_g^L(f) \le \rho_{g'}^L(f). \tag{3.3}$$

Also for arbitrary $\varepsilon > 0$ and for all large σ

$$T_f(\sigma) < [T_g(\sigma)L(\sigma)]^{\rho_g^L(f)+\varepsilon}.$$

So using (3.2) and since g' has property (B), we have

$$\log T_f(\sigma) < \left(\rho_g^L(f)+\varepsilon\right)\left[\log(T_{g'}(\sigma)L(\sigma)) + O(1)\right].$$

Therefore, $\limsup\limits_{\sigma\to\infty} \dfrac{\log T_f(\sigma)}{\log[T_{g'}(\sigma)L(\sigma)]} \le \rho_g^L(f)+\varepsilon.$

Since $\varepsilon > 0$ is arbitrary,

$$\rho_{g'}^L(f) \le \rho_g^L(f). \tag{3.4}$$

Hence from (3.3) and (3.4)

$$\rho_{g'}^L(f) = \rho_g^L(f).$$

4. FINITENESS OF $\rho_g^L(f)$

Definition 4.1: Let f be entire and g be a meromorphic function which is not transcendental.

Let $\alpha = \inf \mu$ such that $\displaystyle\int_{\sigma_0}^{\infty} \frac{T_f(\sigma)L(\sigma)}{[T_g(\sigma)L(\sigma)]^{\mu+1}} d\sigma$, $\sigma_0 \ge \sigma_0' > 0$ converges.

Lemma 4.1: If $\displaystyle\int_{\sigma_0}^{\infty} \frac{T_f(\sigma)L(\sigma)}{[T_g(\sigma)L(\sigma)]^{\mu+1}} d\sigma$ is convergent then $\displaystyle\lim_{\sigma\to\infty} \frac{T_f(\sigma)}{[T_g(\sigma)L(\sigma)]^{\mu}} = 0$ where $0 < \mu < \infty.$

Proof: Given $\varepsilon > 0,$ there is a number $\sigma'(\varepsilon) \ge \sigma_0$ such that

$$\int_{\sigma}^{\infty} \frac{T_f(t)L(t)}{[T_g(t)L(t)]^{\mu+1}} dt < \varepsilon \text{ whenever } \sigma > \sigma'(\varepsilon)$$

and so $\displaystyle\int_{\sigma}^{2\sigma} \frac{T_f(t)L(t)}{[T_g(t)L(t)]^{\mu+1}} dt < \varepsilon$ for $\sigma > \sigma'(\varepsilon)$.

Since $T_f(\sigma), T_g(\sigma)$ and $L(\sigma)$ are non-decreasing, we have for $\sigma > \sigma'(\varepsilon)$

$$\varepsilon > \frac{T_f(\sigma)L(\sigma)}{[T_g(2\sigma)L(2\sigma)]^{\mu+1}} \int_{\sigma}^{2\sigma} dt$$

$$= \sigma \frac{T_f(\sigma)L(\sigma)}{\left[T_g(2\sigma)L(2\sigma)\right]^{\mu+1}}.$$

So, $$\frac{T_f(\sigma)L(\sigma)}{\left[T_g(\sigma)L(\sigma)\right]^{\mu}} < \frac{\varepsilon}{\sigma}.\frac{\left[T_g(2\sigma)L(2\sigma)\right]^{\mu+1}}{\left[T_g(\sigma)L(\sigma)\right]^{\mu}}$$

i.e., $$\frac{T_f(\sigma)}{\left[T_g(\sigma)L(\sigma)\right]^{\mu}} < \frac{\log 2\sigma}{\sigma}\frac{T_g(2\sigma)}{\log 2\sigma}\frac{L(2\sigma)}{L(\sigma)}\left[\frac{T_g(2\sigma)L(2\sigma)}{T_g(\sigma)L(\sigma)}\right]^{\mu}\varepsilon.$$

Since g is not transcendental, so $T_g(\sigma) = O(\log\sigma)$ and hence $T_g(2\sigma) = O(T_g(\sigma))$ and also $L(2\sigma) \approx L(\sigma)$ as $\sigma \to \infty$.

So $$\lim_{\sigma\to\infty}\frac{T_f(\sigma)}{\left[T_g(\sigma)L(\sigma)\right]^{\mu}} = 0.$$

This proves the lemma.

Theorem 4.2: If $\rho_g^L(f)$ be the relative L-Ritt order of f with respect to g and α is defined by Definition (4.1), then $\rho_g^L(f)$ is finite if α is finite.

Proof: Suppose α is given.
Then for arbitrary $\varepsilon > 0$, the integral

$$\int_{\sigma_0}^{\infty}\frac{T_f(\sigma)L(\sigma)}{\left[T_g(\sigma)L(\sigma)\right]^{\alpha+\varepsilon+1}}d\sigma$$ converges.

So by Lemma (4.1) $$\lim_{\sigma\to\infty}\frac{T_f(\sigma)}{\left[T_g(\sigma)L(\sigma)\right]^{\alpha+\varepsilon}} = 0.$$

Thus for all sufficiently large values of σ

$$\frac{T_f(\sigma)}{\left[T_g(\sigma)L(\sigma)\right]^{\alpha+\varepsilon}} < \varepsilon$$

i.e., $$T_f(\sigma) < \varepsilon\left[T_g(\sigma)L(\sigma)\right]^{\alpha+\varepsilon}$$

i.e., $$\log T_f(\sigma) < \log\varepsilon + (\alpha+\varepsilon)\log\left[T_g(\sigma)L(\sigma)\right].$$

So, $$\limsup_{\sigma\to\infty}\frac{\log T_f(\sigma)}{\log\left[T_g(\sigma)L(\sigma)\right]} \le \alpha + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\rho_g^L(f) \le \alpha$ and this proves the theorem.

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