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A NEW CLASS OF CLOSED GRAPH

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ABSTRACT

*The purpose of this paper is to define and study a new class of graphs called *weakly generalized closed graphs in Topological spaces. Some more properties of functions with *weakly generalized closed graphs are investigated. And also, we defined some new closed space in order to characterize these graphs by utilizing the notion of weakly generalized closed sets.*

AMS Subject Classification: *54C10.*

Key words: **wg- closed graphs, wg-Urysohn space and *WG- closed space.*

1. INTRODUCTION

The concept of closedness is fundamental with respect to the investigation of general topological spaces. In 1970, Levine initiated the study of the closed sets called generalized closed (briefly g-closed) sets [7] and by doing this, he generalized the concept of closedness. In 1991, Balachandran et al. defined the maps called generalized continuous [1] (briefly g-continuous) maps which contains the class of continuous maps. Likewise, in 1999, Nagaveni. *et al.* introduced and investigated its weaker form of maps called the weakly generalized continuous maps [9].

In 1969, Long studied the properties of closed graphs [8]. In 1993, A. Krawczyk *et al.* [6] studied the topological version of classical result of continuous colouring of closed graph. Many topologists studied the various types of closed graph in topological spaces. We recommend that the reader should refer to the following papers, respectively. ([2] and [14]). Based on that, Noiri *et al.* investigated two other closed graphs called strongly closed graphs [12] and strongly generalized closed graphs [13] in the year 1978 and 2009. Likewise, Bhattacharyya *et al.*, introduced strongly pre closed graph [11] and Caldas et al. defined strongly α closed graph [4]. In 2017, Nagaveni *et al.* investigated the characteristics of weakly generalized Urysohn spaces [10].

In this paper, we introduce the new form of weakly generalized closed graph using weakly generalized closed sets. Also, we studied the new form of weakly generalized closed graph with Urysohn spaces and *WG-Closed space.

Throughout the paper (X,τ) and (Y,σ) (or simply X and Y) are denoted by topological spaces. The interior and the closure of a subset A of (X,τ) are denoted by Int(A) and Cl(A) respectively.

2. PRELIMINARIES

In this section, we list some definitions which are used in this sequel.

Definition 2.1 [9]: A subset A of a space (X,τ) is called a weakly generalized closed (i.e. wg-closed) sets if $Cl(Int(A)) \subset U$ whenever $A \subset U$ and U is open set in X.

The complement of wg-closed set is said to be wg-open set. The family of all wg-open sets are denoted by WGO(X). We set $WGO(X, x) = \{V \in WGO(X) / x \in V\}$ for $x \in X$.

Definition 2.2 [10]: The wg-closure of a subset A of X is, denoted by wg-Cl(A), defined to be the intersection of all wg-closed sets containing A.

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Definition 2.3 [8]: Let $f: (X, \tau) \to (Y, \sigma)$ be any function. Then the subset $\{(x, f(x)) / x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f and is denoted by G(f).

Definition 2.4 [9]: A function $f: (X, \tau) \to (Y, \sigma)$ is called

i. Weakly g- continuous (i.e. wg-continuous) if $f^{-1}(V)$ is wg-open in X for each open subset V of Y.
ii. Weakly g-open (i.e. wg-open) if image of each wg-open set in X is wg-open in Y.
iii. Weakly g-homeomorphism (i. e. wg-homeomorphism) if f is both wg-continuous and wg-open.
iv. Weakly g –irresolute (briefly wgi) if $f^{-1}(V)$ is wg-closed set in X for each wg-closed set in Y.
v. Quasi-weakly generalized irresolute (briefly qwgi) [10] if for eaxh $x \in X$ and for each $V \in WGO(f(x))$ there exists $U \in WGO(x)$ such that $f(U) \subset wg\text{-}cl_y(f(U))$.

Definition 2.5: A space (X, τ) is called

i. wg-T_1 [10] if for every pair of distinct points x, y in X there exists a wg-open set $U \subset X$ containing x but not y and a wg-open set $V \subset X$ containing y but not x.
ii. wg-T_2 [10] if for every pair of distinct points x, y in X there exists disjoint wg-open sets $U \subset X$ and $V \subset X$ containing x and y respectively.
iii. wg- Urysohn space (wg-T_2' space) [10] if every pair of distinct points $x, y \in X$ there exists $U \subseteq GO(X, x)$ and $V \subseteq GO(X, y)$ such that $wg\text{-}Cl(U) \cap wg\text{-}Cl(V) = \emptyset$.

Definition 2.6 [11]: A space (X, τ) will be said to have the property P if the closure is preserved under finite intersection or equivalently, if the closure of intersection of any two subsets equals the intersection of their closures.

3. *WG - CLOSED GRAPH

In this section, we introduce and investigate the properties of functions and some separation axioms using *wg-closed graphs.

Definition 3.1: A function $f: (X, \tau) \to (Y, \sigma)$ is said to have a *wg- closed graph if for each $(x, y) \in X \times Y - G(f)$, there exist a wg-open sets U and V containing x and y respectively, such that $(U \times wg - Cl(V)) \cap G(f) = \emptyset$.

Lemma 3.2: Let $f: (X, \tau) \to (Y, \sigma)$ be a function then the graph $G(f)$ is *wg-closed in $X \times Y$ if and only if for each $(x, y) \in X \times Y - G(f)$, there exist a wg-open set U and V containing x and y respectively, such that $f(U) \cap wg - Cl(V) = \emptyset$.

Proof is obvious from Definition 3.1.

Theorem 3.3: If $f: (X, \tau) \to (Y, \sigma)$ is injection and G(f) is *wg-closed, then X is wg-T_1 space.

Proof: Since f is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. Since G(f) is *wg-closed, by the lemma 3.2 $(x_1, f(x_2)) \in X \times Y - G(f)$, there exist a weakly generalized open sets U and V containing x and y respectively, such that $f(U) \cap wg\text{-}Cl(V) = \Phi$. Therefore, $x_2 \notin U$. Similarly, there exist a wg-open sets W containing $f(x_2)$ such that $x_1 \notin W$. Hence X is wg-T_1.

Theorem 3.4: If $f: (X, \tau) \to (Y, \sigma)$ is surjection with *wg-closed graph, then Y is wg-T_2 and wg-T_1 space.

Proof: let y_1 and $y_2 \in Y$. Since f is surjective, there exist $x_1 \in X$ such that $f(x_1) = y_1$. Since G(f) is *wg-closed, by the lemma 3.2 $(x_1, y_2) \in X \times Y - G(f)$, there exist a wg-open sets U and V containing x_1 and y_2 respectively, such that $f(U) \cap wg\text{-}Cl_Y(V) = \Phi$. Which implies that $y_1 \notin wg\text{-}Cl(V)$. This means there exist $W \in WGO(Y, y_1)$ such that $W \cap V = \Phi$.

So Y is wg-T_2. Hence Y is wg-T_1 space.

Theorem 3.5: If $f: (X, \tau) \to (Y, \sigma)$ is bijective with *wg-closed graphs, then X and Y are wg-T_1 spaces.

The proof is an immediate consequence of Theorem 3.3 and 3.4.

Theorem 3.6: If $f: (X, \tau) \to (Y, \sigma)$ is wg- irresolute and Y is wg-T_2 space, then G(f) is *wg-closed graph.

Proof: Let $(x, y) \in X \times Y - G(f)$. Since Y is wg-$T_2$ space, there exist $V \in WGO(Y, y)$ such that $f(x) \notin wg\text{-}Cl(V)$. Then $Y - wg\text{-}Cl(V) \in WGO(Y, f(x))$. Since f is wg-irresolute, there exist $U \in WGO(X, x)$ such that $f(U) \subseteq Y - wg\text{-}Cl(V)$. Then $f(U) \cap wg\text{-}Cl(V) = \Phi$. Hence G(f) is *wg-closed graph.

Theorem 3.7: A space X is wg-T_2 space if and only if the identity function has *wg-closed graphs.

Proof: Necessity: Let X be wg-T_2. Since the identity function is wg- irresolute by the Theorem 3.6, G(i) is *wg-closed graph.

Sufficiency: Let G(i) be *wg-closed graph. Since i is surjective by Theorem 3.4, X is wg-T_2 space.

Theorem 3.8: If f: $(X, \tau) \rightarrow (Y, \sigma)$ is quasi wg-irresolute, injective function with *wg-closed graph G(f), then X is wg-T_2 space.

Proof: Since f is injective, for any pair of distinct points x_1, $x_2 \in X$, $f(x_1) \neq f(x_2)$. The *wg- closedness of G(f) gives $(x_1, f(x_2)) \in X \times Y - G(f)$, there exists U ∈ WGO(X, x_1) and V ∈ WGO(Y, f(x_2)) such that f(U) ⋂ wg-Cl_Y(V) = Φ, whence one obtains U ⋂ f^{-1}(wg-Cl_Y(V)) = Φ. Consequently, f^{-1}(wg-Cl_Y(V)) ⊂ X – U. Since f is quasi wg- irresolute, it is so at x_2. Then there exists W ∈ WGO(X, x_2) such that f(W) ⊂ wg-Cl_Y(V). It follows that W ⊂ f^{-1}(wg-Cl_Y(V) ⊂ X – U, whence one infers that W ⋂ U = Φ. Hence X is a wg-T_2 space.

Theorem 3.9: If f: $(X, \tau) \rightarrow (Y, \sigma)$ is wg-open with closed graph G(f), then G(f) is *wg-closed graph.

Proof: Let (x, y) $\in X \times Y - G(f)$. Since G(f) is closed, there exist U ∈ ∑(X, x) and V ∈ ∑(Y, y) such that f(U) ∩ V = Φ. Now wg-openness of f yields that f(U) ∈ WGO(Y, y). Hence f(U) ∩ wg-cl(V) = Φ. Then G(f) is *wg-closed graph.

4. *wg-CLOSED GRAPH ON URYSOHN SPACE.

In this section, we investigated the relationship between weakly generalized Urysohn space and *wg-closed graph.

Theorem 4.1: If f: $(X, \tau) \rightarrow (Y, \sigma)$is wg-continuous and Y is an wg-Urysohn space. Then f has a *wg-closed graph.

Proof: Let (x, y) ∉ G(f), then y≠ f(x). Since Y is wg - Urysohn space, there exist two wg-open sets U and V of y and f(x) respectively, such that wg-Cl(U) ∩ wg-Cl(V) = Φ. Since f is wg-continuous, there exists a g-open sets W of x such that f(W) ⊂ U ⊂ Cl(U). So f(W) ∩ Cl(V) = Φ. But wg-Cl(V) ⊂ Cl(V). Then f(W) ∩ wg-Cl(V) = Φ. Hence f has a *wg-closed graph.

Theorem 4.2: If f: $(X, \tau) \rightarrow (Y, \sigma)$is quasi wg-irresolute and Y is wg-Urysohn, then G(f) is *wg-closed.

Proof: Let (x, y) ∈ X ×Y- G(f). Then, y≠ f(x). Since Y is wg -Urysohn, there exists V ∈ GO(Y, y), W ∈ GO(Y, f(x)) such that wg-Cl_Y(V) ∩ wg-Cl_Y(W) = Φ. Since f is quasi wg-irresolute, there exists U ∈ GO(X, x) such that f(U) ⊂ wg-Cl_Y(W). This implies that, f(U) ∩ wg-Cl_Y(V) = Φ. So, by the Lemma 3.2, G(f) is *wg-closed graph.

Theorem 4.3: Let X be a wg-Urysohn space. Then any wg-open bijection f: $(X, \tau) \rightarrow (Y, \sigma)$ has a *wg-closed graph.

Proof: Let (x, y) ∈ X ×Y - G(f). Then y ≠ f(x) and y ≠ f^{-1}(y), where f^{-1}(y) is a singleton. Since X is wg-Urysohn, there exist wg-open sets U_x and U_y such that x ∈U_x and f^{-1}(y) ∈ U_y and wg-Cl(U_x) ∩ wg-Cl(U_y) = Φ. Since f is wg-open, f(U_x) ∈ WGO(Y, f(x)), f(U_y) ∈ WGO(Y, y) and f(U_x) ∩ wg-Cl(f(U_y)) ⊂ wg-Cl(f(U_x) ∩ wg-Cl(f(U_y)) ⊂ f(wg-Cl(U_x)) ∩ f(wg-Cl(U_y)) = Φ. Therefore, by the lemma 3.2, G(f) is *wg-closed graph.

5. *wg-CLOSED GRAPH ON *WG-CLOSED SPACE.

In this section, we introduced new type of space and we characterize this spaces with *WG-closed graph.

Definition 5.1: A space (X, τ) is called *WG-Closed if every wg-open cover of X has a finite subfamily such that the union of their wg-closures cover X.

Definition 5.2: A subset A of X is said to be *WG-Closed relative to X, if every cover of A by wg-open sets of X has a finite subfamily such that the union of their wg-closures cover X.

Definition 5.3: A subset A of X is called wg-clopen if A is both wg-open and wg-closed.

Definition 5.4: A space X is called extremely WG-Disconnected if the wg-closures of every wg-open set is wg-open.

Lemma 5.5: Every wg-clopen subset of a *WG-closed space X is *WG-closed realative to X.

Proof: Let E be any wg-clopen subset of a *WG-closed space X. Let $\{G_i: i \in I\}$ be any cover of E by wg-open sets in X. Then the family $\boldsymbol{\mathcal{G}} = \{G_i\} \cup E^c$ is a cover of X by wg-open sets in X. Because of *WG-Closedness of X there exists a finite subfamily $\boldsymbol{\mathcal{G}}^* = \{G_{i_m}: m = 1, 2, \dots n\} \cup E^c$ of that covers X. So, because of wg-clopenness of E we now infer that the family $\{\text{wg-Cl}(G_{i_m}) : m = 1, 2, \dots n\}$ covers E. Hence E is *WG-closed relative to X.

Theorem 5.6: If Y is *WG-closed, extremely WG-Disconnected, wg-T_2 space, then the function $f: (X, \tau) \to (Y, \sigma)$ with *wg-closed graph G(f) is quasi wg-irresolute.

Proof: Let $x \in X$ and $V \in WGO(Y, f(x))$. Take any $y \in Y$ - Wg-Cl(V). Then $(x, y) \in X \times Y - G(f)$. Now the *WG-closedness of G(f) induces the existence of $U_y(x) \in WGO(X, x)$, $V_y \in WGO(Y, y)$ such that

$$f(U_y(x)) \cap \text{wg-Cl}(V_y) = \Phi. \qquad (1)$$

Wg-T_2 – ness of Y implies the existence of $V_y \in WGO(Y, y)$ such that $f(x) \notin \text{wg-Cl}(V_y)$. Now extremely WG-Disconnectedness of wg-Cl(V_y) and Y – Wg-Cl(V) is also wg-clopen. Now $\{V_y : y \in Y - \text{Wg-Cl}(V)\}$ is a cover of Y – Wg-Cl(V) by wg-open sets in Y. By the Lemma 5.5, there exists a finite subfamily $\{V_{i_m}: m = 1, 2, \dots n\}$ such that Y – Wg-Cl(V) $\subset \bigcup_{m=1}^{n} \text{Wg} - \text{Cl}\,(V_{i_m})$.

Let $W = \bigcap_{m=1}^{n} U_{i_m}(x)$, Where U_{i_m}(x) are wg-open sets in X satisfying (1). Since X enjoys the property P, $W \in WGO(X, x)$. Now,

$$\begin{aligned} f(W) \cap (Y - \text{Wg-Cl}(V)) &\subset f\,(\textstyle\bigcap_{m=1}^{n} U_{i_m}(x)) \cap (\bigcup_{m=1}^{n} \text{Wg} - \text{Cl}\,(V_{i_m})) \\ &= \textstyle\bigcup_{m=1}^{n} (f[U_{i_m}(x)] \cap \text{Wg-Cl}(V_{i_m})) \\ &= \Phi \text{ by (1).} \end{aligned}$$

Therefore, $f(W) \subset$ Wg-Cl(V) and this indicates that f is quasi wg-irresolute.

Theorem 5.7: Let $f: (X, \tau) \to (Y, \sigma)$ have a *wg-closed graph G(f). Then f enjoys the following property that, for every set F *wg-closed relative to Y, $f^{-1}(F)$ is wg-closed in X.

Proof:
If possible let $f^{-1}(F)$ is not wg-closed in X. Then there exists $x \in \text{wg-Cl}(f^{-1}(F)) - f^{-1}(F)$. Let $y \in F$. Then $(x, y) \in X \times Y - G(f)$. *wg-closedness of G(f) gives the existence of $U_y(x) \in WGO(X, x)$, $V_y \in WGO(Y, y)$ such that $f(U_y(x)) \cap \text{wg-Cl}(V_y) = \Phi$.----------(1).

Clearly $\{V_y : y \in F\}$ is a cover of F by wg-open sets in Y. The *wg-closedness of F relative to Y guarantees that the existence of wg-open sets $V_{y_1}, V_{y_2}, \dots, V_{y_n}$ in Y such that

$$F \subset \bigcup_{i=1}^{n} wg - cl(V_{y_i}).$$

Let $\{U_{y_i}(x): i = 1, 2, \dots, n\}$ be the corresponding wg-open sets in X satisfying (1).

Set $U = \bigcap_{i=1}^{n} \{U_{y_i}(x): i = 1, 2, \dots, n\}$. Then $U \in WGO(X, x)$ because of the fact that X enjoys the property P. Now, $[f(U) \cap F] \subset f[\bigcap_{i=1}^{n} U_{y_i}(x)] \cap [\bigcup_{i=1}^{n} wg - cl(V_{y_i})] = \bigcup_{i=1}^{n} [f(U_{y_i}(x)) \cap wg - cl(V_{y_i})] = \Phi$.

But $x \in \text{wg-Cl}(f^{-1}(F))$ implies $U \cap f^{-1}(F) \neq \Phi$, which is a contradiction to the above deduction.

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