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TOTAL COLORING OF STAR, WHEEL AND HELM GRAPH FAMILY

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ABSTRACT

A total coloring of a graph G is an assignment of colors to both the vertices and edges of G such that adjacent or incident elements of G are not colored with the same color. The total chromatic number of a graph G is a smallest positive integer for which G admits a total coloring. In this paper, we obtain the total chromatic number of Star, Wheel and helm graph.

Keywords: *Total coloring, total chromatic number Star, Wheel, and Helm graph.*

AMS Classification Number: *05C15.*

1. INTRODUCTION

A proper k-coloring of a graph G is a function C: V(G) → {1, 2, … k) such that C(u) ≠ C(v), for all uv∈E(G). The chromatic number denoted by $\chi(G)$, is the minimum number k for which the graph G admits a proper coloring. A total coloring of a graph G is a coloring of all elements of G, i.e. vertices and edges, so that no two adjacent or incident elements receive the same color. The minimum number of colors is called the total chromatic number $\chi_T(G)$ of G. Total coloring conjecture posed independently by Behzad [2] and Vizing [10].It states that every graph of maximum degree Δ admits a $(\Delta+2)$ total coloring Molloy and Reed [7] established a best bound for total coloring as $(\Delta+10^{26})$ for each graph of maximum. degree Δ. The Conjecture has been proved for graphs with maximum. degree 3 by Rosenfielf [8] and Vijayaditya [11] and with $\Delta\in\{4,5\}$ by Kostochka [6].The survey of total colorings of graphs has been given in a paper by Behazed [3]. Behzad *et al.* [4] has also proved TCC for complete graphs. The TCC for complete multipartite graphs have been proved by yap [15], Anderson [1], Sanders and Zheo [9], Borodin [5] have proved the TCC for planar graphs G with $\Delta(G)\neq 5$. The Concept of total coloring is explored by Xie and Yang [14], Wang [12] and Wang *et.al.* [13].

Conjecture 1.1: $\Delta(G) + 1 \leq \chi_T G \leq \Delta(G) + 2$.

Proposition 1.2: A graph G is said to be of type I if $\chi_T(G) = \Delta(G) + 1$ and is of type II if $\chi_T(G) = \Delta(G) + 2$.

2. TOTAL COLORING OF STAR GRAPH FAMILY

In this section, we discuss the total chromatic number of Star graph

2.1 Definition: In **graph** theory, a **star** S_k is the complete bipartite **graph** $K_{1,k}$: a tree with one internal node and k leaves. Alternatively, some authors define S_k to be the tree of order k with maximum diameter 2; in which case a **star** of k > 2 has k -1 leaves.

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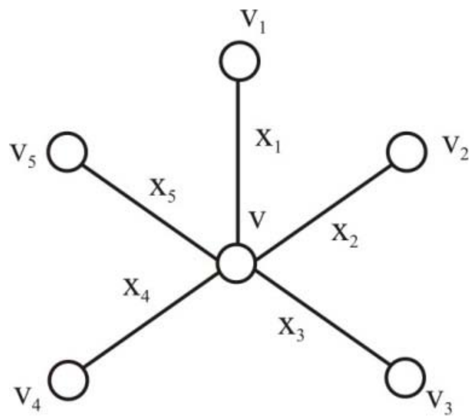

Figure-1: Star Graph $K_{1,5}$

2.2 Coloring Algorithm

Input; $K_{1,n}$, $n \geq 1$
$V \leftarrow \{v_1, v_2, v_3 \ldots\ldots v_n)$
$E \leftarrow \{x_k \leftarrow vv_k ; k = 1 \text{ to } n\}$
$v \leftarrow 1;$
for k = 1 to n
{
$x_k \leftarrow k+1;$
}
end for
for k = 1 to n-1
{
$v_k \leftarrow k + 2;$
}
end for
$v_n \leftarrow 2;$
end procedure

Output: vertex and edge colored $K_{1,n}$.

2.3 Theorem: The total chromatic number of star graph $K_{1,n}$ ($n \geqq 4$) is n + 1
(i-e) $\chi_T (K_{1,n}) = n + 1, n \geqq 4$

Proof: Since $\Delta (K_{1,n}) = n$, we need minimum n + 1 colors for proper coloring. Therefore, $\chi_T (K_{1,n}) \geqq n+1$.Consider the color classes of $K_{1,n}$. The color class of 1 and 2 are {v} and $\{x_1, v_n\}$ respectively. The color class of k (k =3 to n+1) is $(x_{k-1}, v_{k-2}\}$. The elements of each of these color classes are neither incident nor adjacent. Therefore, the above coloring is a total coloring.
$\therefore \chi_T (K_{1,n}) = n + 1, n \geq 4$

Remark: The total chromatic number of $K_{1,4}$ is $\chi_T (K_{1,4}) = 5$

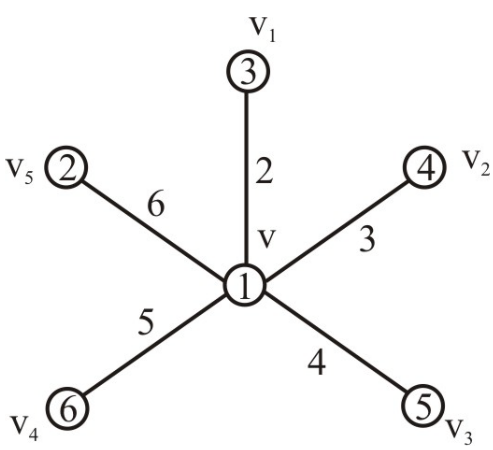

Figure-2: $\chi_T (K_{1,4}) = 5$

3. TOTAL COLORING OF WHEEL GRAPH FAMILY

In this section, we discuss the total chromatic number of Wheel graph

3.1 Definition: A wheel graph is a graph formed by connecting a single vertex to all vertices of a cycle. A Wheel graph with one internal vertex & n leaves is denoted by $W_{n.}$ A wheel graph with n vertices can also be defined as the 1–Skeleton of an (n-1) – gonal pyramid.

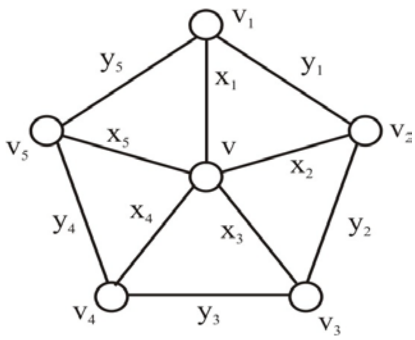

Figure-3: Wheel Graph W_5

3.2 Coloring algorithm

Input; W_n, $n \geq 4$
$V \leftarrow \{v, v_1, v_2, \ldots\ldots v_n)$
$E \leftarrow \{x_k \leftarrow vv_k (k = 1 \text{ to } n\};$
$y_k \leftarrow v_k v_{k+1}$ (k = 1 to n-1);
$y_n \leftarrow v_n v_1$ }
$v \leftarrow 1;$
for k = 1 to n
{
$x_k \leftarrow k+1;$
}
end for
for k = 1 to n – 1
{
$v_k \leftarrow k+2;$
}
end for
$v_n \leftarrow 2$
for k = 1 to n
{
$t \leftarrow k + 4;$
if $t \leq n + 1;$
$y_k \leftarrow t;$
else
$y_k \leftarrow t - n;$
}
end for
end procedure
Output: vertex and edge colored W_n

3.3 Theorem: The total chromatic number of wheel Graph W_n is n+1, $n \geq 4$, (i-e) $\chi_T(W_n) = n+1$, $n \geq 4$,

Proof: Since $\Delta(W_n) = n$, We need minimum (n+1) colors for proper coloring.
$\therefore \chi_T(W_n) \geq n + 1$.Consider the color classes of W_n.

The color class of 1 is $\{v\}$.The color class of k ($2 \leq k \leq n + 1$) is $\{v_t, x_{k-1}, y_s$; t = k -2 if k > 2 and t= n if k =2, s = k- 4 if k > 4 and s = n+k- 4 if $2 \leq k \leq 4\}$. The elements of each of these color classes are neither adjacent nor incident. Therefore, the coloring given in 3.1 is a total coloring.
$\therefore \chi_T(W_n) = n + 1$, $n \geq 4$,

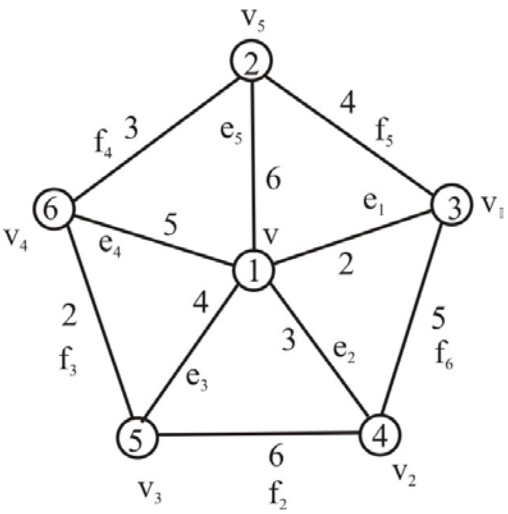

Figure-4: $\chi_\tau[W_5] = 6$

4. TOTAL COLORING OF HELM GRAPH FAMILY

In this section, we discuss the total chromatic number of Helm graph.

4.1 Definition: A **helm graph**, denoted by $\mathbf{H_n}$ is a graph obtained by attaching a single edge and node to each node of the outer circuit of a wheel graph $\mathbf{W_n}$.

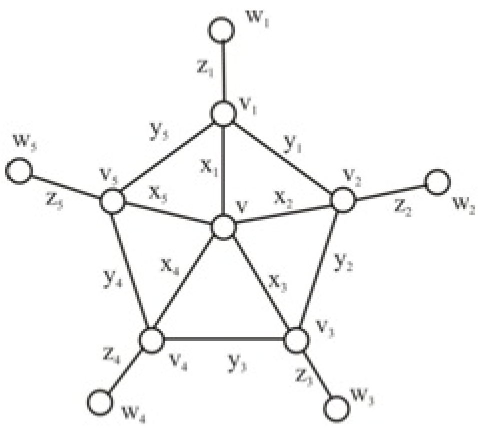

Figure-5: Helm Graph H_5

4.2 Coloring Algorithm

Input: H_n, $n \geq 4$

$v \leftarrow \{v, v_1, v_2, \ldots\ldots v_n, w_1, w_2, \ldots.. w_n\}$
$E \leftarrow \{x_k \leftarrow vv_k$ (k =1 to n),
$y_k \leftarrow v_k v_{k+1}$ (k =1 to n-1), $y_n \leftarrow v_n v_1$,
$z_k \leftarrow v_k w_k$(k=1 to n)}
$v \leftarrow 1$;
for k = 1 to n
{
$x_k \leftarrow k + 1$;
}
end for.
for k =1 to n – 1
{
$v_k \leftarrow k + 2$;
}
end for
$v_n \leftarrow 2$;
for k =1 to n

```
        {
         t ← k + 4;
         if  t ≤ n + 1;
        y_k ← t;
        else
        y_k ← t - n;
       }
end for
for k =1 to n
      {
       z_k  ← 1;
       }
end for
for k =1 to n
        {
         s←k + 1;
        if s ≤ n + 1;
        w_k ← s;
     else
    w_k ← s- n
   }
end for
end procedure
```

Output: vertex and edge colored H_n.

4.3 Theorem: The total chromatic number of Helm Graph H_n is n+1, $n \geq 4$

(i.e) $\chi_T(H_n) = n+1, n \geq 4$

Proof: Since $\Delta(H_n) = n$, (n+1) colors are required for proper coloring and hence $\chi_T(H_n) \geq n+1$ Now, the color class of 1 is{v, z_k (k =1to n)}. The color class of k. $(2 \leq k \leq n + 1)$ is $\{v_t, x_{k-1}, y_s, w_{k-1}$; t = k-2 if k > 2 and t = n if k = 2, s = k – 4, if k > 4 and s = n + k – 4, $2 \leq k \leq 4\}$.The elements in each color classes are neither incident nor adjacent.

Therefore, the coloring given in the algorithm 4.1 is a total coloring of H_n.

$\chi_T(H_n) = n + 1, n \geq 4$

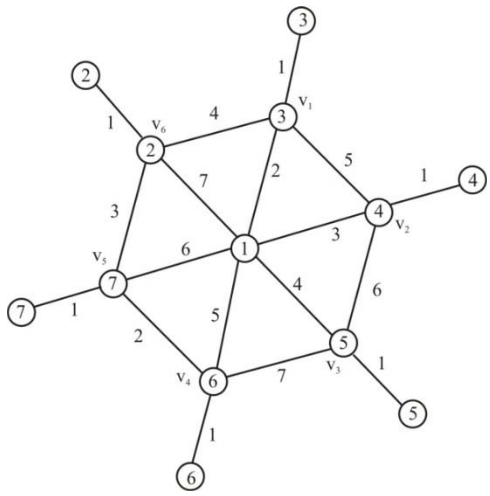

Figure-6: $\chi_T(H_6) = 7$

CONCLUSION

In this paper we have obtained the following results.

1. The total chromatic number of $K_{1,n}$ is $\chi_T(K_1, n) = n + 1$ $n \geq 4$
2. The total chromatic number of W_n is $\chi_T(W_n) = n + 1, n \geq 4$
3. The total chromatic number of H_n is $\chi_T(H_n) = n + 1, n \geq 4$

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