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SPECIAL CLASSES OF IDEALS AND FILTERS OF PSEUDO-COMPLEMENTED ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT

In this paper we introduce the concepts of $\alpha-$ideal and $\beta-$filter of a weakly pseudo-complemented ADL in general and of a pseudo- complemented ADL in particular and discuss certain properties of these. Mainly, we prove that the sets of $\alpha-$ideals and $\beta-$filters of a pseudo-complemented ADL form algebraic lattices. Also, we characterize Stone ADLs and Almost Boolean algebras in terms of $\alpha-$ideals and $\beta-$filters.

***Key words**: Almost Distributive Lattice (ADL); weak pseudo-complementation; pseudo-complementation; $\alpha-$ideal; $\beta-$filter; minimal prime ideal; Almost Boolean algebra.*

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1. INTRODUCTION

The concept of an Almost Distributive Lattice (ADL) was introduced by U. M. Swamy and G. C. Rao [3] as a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebra and Boolean ring. Further, U. M. Swamy, G. C. Rao and G. N. Rao [4] have introduced the notion of pseudo-complementation on an ADL and proved that the class of pseudo-complemented ADLs is equationally definable and they exhibited a one-to-one correspondence between maximal elements and pseudo-complementations on an ADL. Later, R. V. Babu, Ch. S. Sundar Raj and B. Venkateswarlu [7] have introduced the notion of weak pseudo-complementation on an ADL and proved several properties of this. In particular, they have proved that an ADL is pseudo-complemented if and only if it is weakly pseudo-complemented, even though a weak pseudo-complementation need not be a pseudo-complementation. In [1], Blyth defined the concepts of $*-$ideals and $*-$filters in pseudo-complemented semi lattices. Here, we extend these concepts to ADLs and we define these in the form of $\alpha-$ideals and $\beta-$filters of weakly pseudo-complemented ADLs in general and of pseudo-complemented ADLs in particular. Mainly, we prove that the $\alpha-$ideals ($\beta-$filters) are independent of the weak pseudo (pseudo)-complementation. The main object of this paper is to study the classes of $\alpha-$ideals and $\beta-$filters of a pseudo-complemented ADL and prove that these classes form algebraic lattices. Also, in this paper we characterize minimal prime ideals in a weakly pseudo-complemented ADL. Mainly, we characterize Stone ADLs and Almost Boolean algebras in terms of their $\alpha-$ideals and $\beta-$filters.

We recall the notion of an Almost Distributive Lattice (abbreviated: ADL) and certain necessary results which will be used in the main text of this paper.

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2. PRELIMINARIES

Definition 2.1 ([3]): An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (ADL), if it satisfies the following conditions for all a, b and c in A.

(1) $0 \wedge a = 0$
(2) $a \vee 0 = a$
(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(4) $(a \vee b) \wedge c = (a \vee c) \vee (b \wedge c)$
(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(6) $(a \vee b) \wedge b = b$

An ADL is $(A, \wedge, \vee, 0)$ is said to be associative if the operation $\vee$ is associative. Throughout this paper by A we mean an associative ADL $(A, \wedge, \vee, 0)$ unless otherwise mentioned.

For any $a, b \in A$, we say that a is less than or equal to b and we write $a \leq b$, if $a \wedge b = a$ (equivalently $a \vee b = b$). It can be easily proved that $\leq$ *is* a partial order on A.

Lemma 1.2 ([3]): The following hold for any a, b and c in an ADL A.

(1) $a \wedge 0 = 0$ and $a = 0 \vee a$
(2) $a \wedge a = a = a \vee a$
(3) $a \wedge b \leq b$ and $a \leq a \vee b$
(4) $a \wedge b = a \Leftrightarrow a \vee b = b$ and $a \wedge b = b \Leftrightarrow a \vee b = a$
(5) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (i.e., $\wedge$ is associative)
(6) $a \vee (b \vee a) = a \vee b$
(7) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
(8) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(9) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$.

An element $m \in A$ is said to be maximal if $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$ for all $x \in A$.

Definition 1.3 ([3]): A non-empty set I of an ADL A is said to be an ideal (filter) of A if $a \vee b$ $(a \wedge b) \in I$ for all a and $b \in I$ and $a \wedge x$ $(x \vee a) \in I$ for all $a \in I$ and $x \in A$.
It follows as a consequence that for any ideal (filter) I of A, $x \wedge a$ $(a \vee x) \in I$ for all $a \in I$ and $x \in A$.
For any $X \subseteq A$, the smallest ideal (filter) of A containing X is called the ideal (filter) generated by X and is denoted by $\langle X]$ $([X\rangle)$. If $X = \{x\}$, we simply write $\langle x]$ $([x\rangle)$ for $\langle \{x\}]$ $([\{x\}\rangle)$. we have the following for any $X \subseteq A$ and $x \in A$

$$\langle X] = \left\{ \left(\bigvee_{i=1}^{n} x_i \right) \wedge a \;/\; n \geq 0,\ x_i \in X \text{ and } a \in A \right\}$$

$$[X\rangle = \left\{ a \vee \left(\bigwedge_{i=1}^{n} x_i \right) \;/\; n \geq 0,\ x_i \in X \text{ and } a \in A \right\}$$

and $\langle x] = \{x \wedge a \mid a \in A\}$ *and* $[x\rangle = \{a \vee x \mid a \in A\}$.

$\langle x]$ $([x\rangle)$ is called the principal ideal (filter) generated by x.

For any subset S of A, let $S^* = \{a \in A : a \wedge s = 0$ for all $s \in S\}$. Then S^* is always an ideal of A for all $S \subseteq A$. It can be proved that $S^* = \langle S]^*$ in particular for any $a \in A$,

$\langle a]^* = \{a\}^* = \{x \in A \mid a \wedge x = 0\}$.

Definition 1.4 ([7]): Let A be an ADL. A mapping $a \mapsto a^*$ of A into itself is called a weak pseudo-complementation on A if $a \wedge b = 0 \Leftrightarrow a^* \wedge b = b$ for any a and $b \in A$. An ADL A is said to be weakly pseudo-complemented if there is a weak pseudo-complementation $a \mapsto a^*$ on A.

Theorem 1.5 ([7]): The following are equivalent to each other for any mapping $a \mapsto a^*$ of an ADL A into itself.

(1) $a \mapsto a^*$ is a weak pseudo-complementation on A
(2) $\{a\}^* = \langle a^*]$ for any $a \in A$
(3) For any $a \in A$, $a \wedge a^* = 0$ and $a \wedge b = 0 \Rightarrow a^* \wedge b = b$ for any $b \in A$.

Let us recall from [8], that two elements a and b in an ADL A are said to be associates to each other if $a \wedge b = b$ and $b \wedge a = a$ (equivalently $\langle a] = \langle b]$); in this case we write $a \sim b$. Also, in this case for any ideal (filter) I of A, $a \in I \Leftrightarrow b \in I$.

Theorem 1.6 ([7]): Let $a \mapsto a^*$ and $a \mapsto a^+$ be two weak pseudo-complementations on an ADL A. Then the following hold for any a and $b \in A$.

(1) $a^* \sim a^+$
(2) $a^{**} \sim a^{++}$
(3) $a^* \sim b^* \Leftrightarrow a^+ \sim a^+$
(4) $a^* = 0 \Leftrightarrow a^+ = 0$
(5) $a^* \wedge 0^+ \sim a^+$
(6) $a^* \vee a^{**} \sim 0^* \Leftrightarrow a^+ \vee a^{++} \sim 0^+$

Theorem 1.7 ([7]): Let $a \mapsto a^*$ be a weak pseudo-complementation on an ADL A. Then the following hold for any a and $b \in A$.

(1) 0^* is a maximal element $in\ A$
(2) m is maximal $in\ A \Rightarrow m^* = o$
(3) $0^{**} = 0$
(4) $a^* \wedge a = 0$
(5) $a^{**} \wedge a = a$
(6) $a \wedge b = 0 \Leftrightarrow a^{**} \wedge b = 0 \Leftrightarrow a \wedge b^{**} = 0 \Leftrightarrow a^{**} \wedge b^{**} = 0$
(7) $a^* \sim a^{***}$
(8) $a^* = 0 \Leftrightarrow a^{**}$ is maximal
(9) $a = 0 \Leftrightarrow a^{**} = 0$
(10) $(a \vee b)^* \sim a^* \wedge b^*$

Theorem 1.8 ([7]): Let A be an ADL and $a \mapsto a^*$ be a weak pseudo-complementation on A. Then the following hold for any a and $b \in A$.

(1) $a \sim b \Rightarrow a^* \sim b^*$
(2) $(a \wedge b)^* \sim (b \wedge a)^*$
(3) $(a \vee b)^* \sim (b \vee a)^*$
(4) $(a \wedge b)^{**} \sim a^{**} \wedge b^{**}$.

Definition 1.9 ([4]): Let $(A, \wedge, \vee, 0)$ be an ADL. Then a unary operation $a \mapsto a^*$ on A is called a pseudo-complementation on A if, for any $a, b \in A$, the following independent axioms are satisfied

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$
(2) $a \wedge a^* = 0$
(3) $(a \vee b)^* = a^* \wedge b^*$

3. α −IDEALS AND β −FILTERS

In this section, we define an $\alpha-$ideal and a $\beta-$filter of a weakly pseudo-complemented ADL in general and of a pseudo-complemented ADL in particular and provide certain examples of these.

For any non-empty subset X of an ADL A, let us denote $\alpha(X)$ by

$$\alpha(X) = \{y \in A : x \wedge y = 0 \text{ for some } x \in X\}.$$

Clearly $\alpha(X) = \bigcup_{x \in X} \{x\}^*$.

Lemma 2.1: Let A be an ADL and X a non-empty subset of A such that $x \wedge y \in X$ for all $x, y \in X$. Then $\alpha(X)$ is an ideal of A.

Corollary 2.2: Let A be an ADL and F be a filter of A. Then $\alpha(F)$ is an ideal of A.

Lemma 2.3: Let A be an ADL and $*$ a weak pseudo-complementation on A. Then for any filter F of A,
$\alpha(F) = \{x \in A : x^* \in F\}$ and $\alpha(F)$ is independent of the weak pseudo-complementation $*$ on A.

Proof: Let $x \in \alpha(F)$. Then $x \wedge y = 0$ for some $y \in F$, therefore $x^* \wedge y = y$, and hence $x^* \vee y = x^*$, $y \in F$. Therefore, $x^* \in F$. On the other hand if $x^* \in F$, then $x \wedge x^* = 0$ and $x^* \in F$, it implies that $x \in \alpha(F)$. Let $+$ be a weak pseudo-complementation on A.

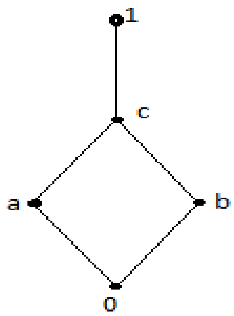
Then, $x^* \in F \Rightarrow x^+ \sim x^* \wedge 0^+ \in F \Rightarrow x^+ \in F$
and $x^+ \in F \Rightarrow x^* \sim x^+ \wedge 0^* \in F \Rightarrow x^* \in F$

Since 0^* and 0^+ are maximal and hence are in F. Therefore $\alpha(F)$ is independent of the weak pseudo complementation $*$ on A.

For any non-empty subset X of an ADL A, let us define $\beta(X)$ by
$$\beta(X) = \{x \in A : \{x\}^* \subseteq X\}.$$

Remark 2.4: $\beta(I)$ need not be a filter even though I is an ideal of A. For, consider the following example.

Example 2.5: Let L be a lattice represented by the Hasse diagram given below,



Then $I = \{0, a, b, c\}$ is an ideal of L. Clearly, $\{0\}^* = L$, $\{a\}^* = \{0, b\}, \{b\}^* = \{0, a\}$, $\{c\}^* = \{0\}$ and $\{1\}^* = \{0\}$. Therefore $\beta(I) = \{a, b, c, 1\}$ which is not a filter of L, since $a \wedge b = 0 \notin \beta(I)$.

Lemma 2.6: Let A be an ADL and $*$ a weak pseudo-complementation on A. Then for any ideal I of A, $\beta(I) = \{x \in A : x^* \in I\}$ and $\beta(I)$ is independent of the weak pseudo-complementation $*$ on A.

Lemma 2.7: For any ideal I of a weakly pseudo-complemented ADL A, $\alpha(\beta(I)) \subseteq I$.

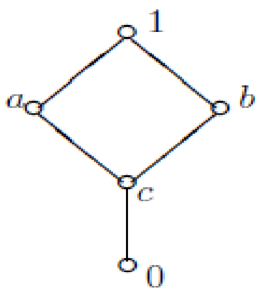
Proof: Let $*$ be a weak pseudo-complementation on A and $x \in \alpha(\beta(I))$. Then $x \wedge y = 0 = y \wedge x$, for some $y \in \beta(I)$. This implies, $x = y^* \wedge x$ and $y^* \in I$, therefore $x \in I$.

Lemma 2.8: For any filter F of a weakly pseudo-complemented ADL A, $F \subseteq \beta(\alpha(F))$.

Proof: Let $*$ be a weak pseudo-complementation on A and $x \in F$. Then, $x^{**} = x^{**} \vee x \in F$ (since $x^* \wedge x = 0$ and hence $x^{**} \wedge x = x$) and hence $x^* \in \alpha(F)$, it follows that $x \in \beta(\alpha(F))$.

Remark 2.9: The following examples shows that the equality may not hold in 2.7 and 2.8.

Example 2.9(1): Let L be the lattice represented by the Hasse diagram given below



Define the map $*$ on L by $0^* = 1$ and $a^* = b^* = c^* = 1^* = 0$. Then $*$ is a pseudo-complementation on L and $I = \{0, c, a\}$ is an ideal of L. Now $\beta(I) = \{x \in L : x^* \in I\} = \{c, a, b, 1\}$ and $\alpha(\beta(I)) = \{x \in L : x^* \in \beta(I)\} = \{0\}$. This shows $\alpha(\beta(I)) \neq I$.

Example 2.9(2): Let $A = \{a,\ b,\ c,\ 0\}$. Define $\vee$ and $\wedge$ on A as follows

∨	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

∧	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then $(A,\ \wedge,\ \vee,\ 0)$ is an ADL. Define the map $*$ on A by $x^* = 0$ for all $x \neq 0$ and $0^* = a$. Then $*$ is a pseudo - complementation on A. Let $F = \{a,\ b\}$, then F is a filter of A and $\alpha(F) = \{0\}$ and $\beta(\alpha(F)) = \{a,\ b,\ c\}$. This shows, $F \neq \beta(\alpha(F))$.

Theorem 2.10: Let $*$ be a weak pseudo-complementation on an ADL A and I an ideal of A. Then the following are equivalent to each other.
(1) $x \in I \Rightarrow x^{**} \in I$
(2) $I \subseteq \alpha(\beta(I))$
(3) $I = \alpha(\beta(I))$
(4) $I = \alpha(F)$ for some filter F of A.

Proof:
$(\mathbf{1}) \Rightarrow (\mathbf{2})$: Assume (1). Let $x \in I$. Then $x^{**} \in I$ and hence $x^* \in \beta(I)$. Since $x \wedge x^* = 0$ and $x^* \in \beta(I)$, we get $x \in \alpha(\beta(I))$. Therefore $I \subseteq \alpha(\beta(I))$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$ is clear by the lemma 2.7.

$(\mathbf{3}) \Rightarrow (\mathbf{4})$: Assume that $I = \alpha(\beta(I))$. It sufficies to prove that $\beta(I)$ is a filter of A. Let $x, y \in \beta(I)$.
Then $x, y^* \in I$ and hence $x^* \vee y^* \in I$. Therefore $x^* \vee y^* \in \alpha(\beta(I))$, implies $(x^* \vee y^*) \wedge a = 0$ for some $a \in \beta(I)$ and hence $(x^* \vee y^*)^{**} \wedge a = 0,\ a \in \beta(I)$. This shows that $(x^* \vee y^*)^{**} \in \alpha(\beta(I))$.
Now, $(x \wedge y)^* \sim (x \wedge y)^{***} \sim (x^{**} \wedge y^{**})^* \sim (x^* \vee y^*)^{**} \in I$. Therefore $(x \wedge y)^* \in I$ and hence $x \wedge y \in \beta(I)$. Further, let $x \in \beta(I)$ and $a \in A$. Then $x^* \in I$ and $(a \vee x)^* \sim (x \vee a)^* \sim x^* \wedge a^* I$. This implies $(a \vee x)^* \in I$ and hence $a \vee x \in \beta(I)$. Therefore $\beta(I)$ is a filter of A.

$(\mathbf{5}) \Rightarrow (\mathbf{1})$: Assume that $I = \alpha(F)$ for some filter F of A. Then,

$$\begin{aligned} x \in I \Rightarrow x \in \alpha(F) &\Rightarrow x^* \in F \\ &\Rightarrow x^{***} \sim x^* \in F \\ &\Rightarrow x^{***} \in F \\ &\Rightarrow x^{**} \in \alpha(F) \\ &\Rightarrow x^{**} \in I. \end{aligned}$$

Theorem 2.11: Let $*$ be a weak pseudo-complementation on an ADL A and F a filter of A. Then the following are equivalent to each other.
(1) $x^{**} \in F \Rightarrow x \in F$
(2) $\beta(\alpha(F)) \subseteq F$
(3) $F = \beta(\alpha(F))$
(4) $F = \beta(I)$ for some ideal I of A

Now, we introduce $\alpha-$ideals and $\beta-$filters in weakly pseudo-complemented ADLs.

Definition 2.12: Let A be an ADL and $*$ a weak pseudo-complementation on A. Then
(1) an ideal I of A is said to be an α $-$ideal of A if any one (and hence all) of the conditions in theorem 2.10 holds
(2) a filter F of A is said to be a β $-$filter of A if any one (and hence all) of the conditions in theorem 2.11 holds.

Example 2.12: Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ADLs.
Write $L = A \times B = \{(0,0), (a,0), (0,b_1), (0,b_2), (a,b_1), (a,b_2)\}$. Then $(L,\ \wedge,\ \vee,\ 0)$ is an ADL under point-wise operations. Consider the ideals $I_1 = \{(0,0), (a,0)\}$ and $I_2 = \{(0,0), (0,b_1), (0,b_2)\}$.
Then I_1 and I_2 are α $-$ideals since $I_1 = \alpha(F_1)$, $I_2 = \alpha(F_2)$
where $F_1 = \{(0,b_1),\ (0,b_2),\ (a,b_1),\ (a,b_2)\}$ and $F_2 = \{(a,0),\ (a,b_1),\ (a,b_2)\}$ are filters. Also, F_1 and F_2 are β $-$filters, since $F_1 = \beta(I_1)$ and $F_2 = \beta(I_2)$.

3. PROPERTIES OF α −IDEALS AND β –FILTERS

Let us recall from [5, 4] that, an element a of an ADL A is said to be dense if $\{a\}^* = \{0\}$. It can be verified that a is dense if and only if $a^* = 0$ where $*$ is a weak pseudo (pseudo)-complementation on A. Also, the set $D(A)$ of dense elements of A is a filter of A.

Proposition 3.1: Let A be a weakly pseudo-complemented ADL. Then $A, \{0\}$ are α −ideals and $A, D(A)$ are β −filters.

Proof: It is by the fact that $A = \alpha(A)$, $\{0\} = \alpha(D(A))$ and $A = \beta(A)$, $D(A) = \beta(\{0\})$. It can be easily observed that, for any subset S of A, S^* is an α −ideal of A.

Lemma 3.2: Let A be a weakly pseudo-complemented ADL and F be a filter of A. Then F is a β −filter of A if and only if $D(A) \subseteq F$.

Proof: Let $*$ be a weak pseudo-complementation on A. Suppose that F is a β −filter of A and $x \in D(A)$. Then $x^* = 0$ and hence $x^{**} = 0^*$ which is maximal. Therefore $x^{**} \in F$ and hence $x \in F$. Thus $D(A) \subseteq F$. Conversely, let $x^{**} \in F$ then $x \vee x^* \in F$, since $x \vee x^*$ is dense. Therefore $x^{**} \wedge (x \vee x^*) \in F$, implies $(x^{**} \wedge x) \vee (x^{**} \wedge x^*) \in F$ which implies $x \vee 0 \in F$ and hence $x \in F$. Therefore F is a β −filter of A.

Corollary 3.3: $D(A)$ is the smallest β −filter of A.

Now, we shall discuss certain properties of α −ideals and β −filters in pseudo-complemented ADLs.

In general the lattice $\mathcal{I}(A)$ of ideals of an ADL A is a complete lattice since it is closed under arbitrary intersections. However, the lattice $\mathcal{F}(A)$ of filters of A is not necessary complete; infact $\mathcal{F}(A)$ is complete if and only if A has a maximal element. If $*$ is a pseudo-complementation on A, then 0^* is necessarily a maximal element in A and hence $0^* \in F$ for all $F \in \mathcal{F}(A)$. Therefore, every class $\{F_i\}_{i\in\Delta}$ of filters of a pseudo complemented ADL A has infimum $\bigcap_{i\in\Delta} F_i$ and supremum $\bigvee_{i\in\Delta} F_i = \left[\bigcup_{i\in\Delta} F_i\right\rangle$ in $\mathcal{F}(A)$.

Theorem 3.4: Let A be a pseudo-complemented ADL. Then the set $\mathcal{I}_\alpha(A)$ of α −ideals of A is a complete distributive lattice ordered by set inclusion, in which the lattice operations are as follows:
If $\{I_i\}_{i\in\Delta} \subseteq \mathcal{I}_\alpha(A)$ then

$$glb\{I_i : i \in \Delta\} = \cap_{i\in\Delta} I_i \text{ and } lub\{I_i : i \in \Delta\} = \alpha\left(\bigvee_{i\in\Delta} F_i\right)$$

where to each $i \in \Delta$, $I_i = \alpha(F_i)$, for some filter F_i of A.

Proof: Straight forward.

Theorem 3.5: Let A be a pseudo-complemented ADL. Then the lattice $\mathcal{I}_\alpha(A)$ of α −ideals of A is an algebraic lattice.

Proof: Let $*$ be a pseudo-complementation on A and $I \in \mathcal{I}_\alpha(A)$. Then

$$I = \alpha(F) = \bigcup_{a\in F}\{a\}^* = sup\{(a^*] : a \in F\}$$

for some filter F of A. Further, it can be verified that, for any $a \in A$, $(a^*]$ is a compact element in $\mathcal{I}_\alpha(A)$ and it follows that $\mathcal{I}_\alpha(A)$ is an algebraic lattice.

Theorem 3.6: Let A be a weakly pseudo (pseudo)-complemented ADL and I be an α −ideal of A. Then $\beta(I)$ is a filter of A (and hence β −filter).

Proof: Straight forward.
Form the theorem 3.6, β and α induce isotone mappings $\hat{\beta} : \mathcal{I}_\alpha(A) \to \mathcal{F}_\beta(A)$ and $\hat{\alpha} : \mathcal{F}_\beta(A) \to \mathcal{I}_\alpha(A)$ where $\mathcal{F}_\beta(A)$ denote the set of β −filters of a pseudo-complemented ADL A. Also, from 2.10 and 2.11, $\hat{\alpha}$ and $\hat{\beta}$ are isomorphisms which are inverses to each other. Therefore we have the following.

Theorem 3.7: $\mathcal{I}_{\alpha}(A) \cong \mathcal{F}_{\beta}(A)$.

The following are immediate consequences.

Theorem 3.8: The set $\mathcal{F}_{\beta}(A)$ of $\beta-$filters of a pseudo-complemented ADL A, ordered by set inclusion, is a complete distributive lattice in which the lattice operations are as follows: If $\{F_i\}_{i\in\Delta} \subseteq \mathcal{F}_{\beta}(A))$, then

$$inf\{F_i : i \in \Delta\} = \bigcap_{i\in\Delta} F_i \text{ and}$$

$$sup\{F_i : i \in \Delta\} = \left\{x \in A : x^{**} \in \bigvee_{i\in\Delta} F_i\right\} = \beta\left(\alpha\left(\bigvee_{i\in\Delta} F_i\right)\right)$$

Theorem 3.9: Let A be a pseudo-complemented ADL. Then the lattice $\mathcal{F}_{\beta}(A)$ is an algebraic lattice.
Recall from [5], that an ADL A with a pseudo-complementation $*$ is said to be a Stone ADL, if $x^* \vee x^{**} = 0^*$ for all $x \in A$ or equivalently $(x \wedge y)^* = x^* \vee y^*$ for all $x, y \in A$.

Here we characterize Stone ADLs interms of $\alpha-$ideals and $\beta-$filters.

Theorem 3.10: Let A be an ADL and $*$ a pseudo-complementation on A. Then A is a Stone ADL if and only if $\mathcal{I}_{\alpha}(A)$ is a sublattice of $\mathcal{I}(A)$.

Proof: Suppose that A is a Stone ADL. Let $I, J \in \mathcal{I}_{\alpha}(A)$ and $x \in I \vee J$. Then $x = a \vee b$ for some $a \in I$ and $b \in J$ and hence $a^{**} \in I$ and $b^{**} \in J$.
Now, $x^{**} = (a \vee b)^{**} = (a^* \wedge b^*)^* = a^{**} \vee b^{**} \in I \vee J$. Therefore $I \vee J$ is an $\alpha-$ideal and hence $I \vee J \in \mathcal{I}_{\alpha}(A)$ and clearly $I \cap J \in \mathcal{I}_{\alpha}(A)$. Thus $\mathcal{I}_{\alpha}(A)$ is a sublattice of $\mathcal{I}(A)$.
Conversely, we suppose that $\mathcal{I}_{\alpha}(A)$ is a sublattice of $\mathcal{I}(A)$. Let $x \in A$. Then, $(x^*]$ and $(x^{**}]$ are $\alpha-$ideals of A, by assumption $< x^*] \vee (x^{**}] = < x^* \vee x^{**}]$ is an $\alpha-$ideal of A. This implies $0^* = (x^{**} \wedge x^*)^* = (x^{**} \wedge x^{***})^* = (x^* \vee x^{**})^{**} \in < x^* \vee x^{**}]$ and since 0^* is maximal, we get that $< x^* \vee x^{**}] = A = < 0^*]$. This implies $x^* \vee x^{**} = 0^*$ since x^* and $x^{**} \leq 0^*$. Thus A is a Stone ADL.

Theorem 3.11: Let A be a Stone ADL. Then $\beta(I)$ is a filter of A, for all ideals I of A.

4. PRIME IDEALS AND FILTERS

Recall from [2], a proper ideal (filter) P of an ADL A is said to be prime, if for any $a, b \in A, a \wedge b(a \vee b) \in P \Rightarrow$ either $a \in P$ or $b \in P$. A prime ideal P of an ADL A is called minimal if there is no prime ideal Q of A such that $Q \subset P$.

Remark 4.1: In general, a prime ideal may not be an $\alpha-$ideal and $\alpha-$ideal need not be prime. For example, in 2.9(1) the prime ideals $\{0, c, a\}$ and $\{0, c, b\}$ are not $\alpha-$ideals, since $c^{**} = 0^* = 1$ and in 2.5 $\{0\}$ is an $\alpha-$ideal but not prime since $a \wedge b = 0$. Also, a prime filter may not be a $\beta-$filter. For, in 2.9(1), $\{b, 1\}$ is a prime filter but not a $\beta-$filter since $a^{**} = 0^* = 1$.

Theorem 4.2 ([2]): Let P be a prime ideal of an ADL A. Then P is minimal prime ideal if and only if $\{a\}^* \not\subseteq P$ for all $a \in P$.

In the following, minimal prime ideals of a weakly pseudo-complemented ADL are characterized in terms of their $\alpha-$ideals.

Theorem 4.3: Let A be an ADL and $*$ a weak pseudo-complementation on A and P a prime ideal of A. Then the following conditions are equivalent.

(1) P is minimal
(2) $x \in P$ implies that $x^* \notin P$
(3) $x \in P$ implies that $x^{**} \in P$
(4) $P \cap D(A) = \phi$.

Proof:
$(\mathbf{1}) \Rightarrow (\mathbf{2})$: Let P be minimal and $x \in P$. Then by theorems 4.2 and 1.5(2), $(x^*] = \{x\}^* \nsubseteq P$. This implies $x^* \notin P$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$: Assume (2). Let $x \in P$. Then $x^* \notin P$. Since $x^* \wedge x^{**} = 0 \in P$ and P is prime, we get $x^{**} \in P$.

$(\mathbf{3}) \Rightarrow (\mathbf{4})$: Assume (3). If $x \in P \cap D(A)$ for some $x \in A$, then $x \in P$ and $x^* = 0$ and hence $x^{**} = 0^* \notin P$, since 0^* is maximal, a contradiction to (3).

$(\mathbf{4}) \Rightarrow (\mathbf{1})$: If P is not minimal, then $Q \subset P$ for some prime ideal Q of A. Let $x \in P - Q$. Then $x \wedge x^* = 0 \in Q$ and $x \notin Q$; therefore $x^* \in Q \subset P$, which implies that $x \vee x^* \in P$. Also, $x \vee x^*$ is dense, thus we obtain $x \vee x^* \in P \cap D(A)$, a contradiction to (4). Hence P is minimal.

Theorem 4.4: Let A be an ADL and $*$ a weak pseudo-complementation on A and let P be a prime ideal of A. Then the following conditions are equivalent.

(1) $A - P$ is maximal filter
(2) $A - P$ is prime filter and $a \vee a^* \in A - P$ for each $a \in A$
(3) P is a minimal prime ideal
(4) P is an $\alpha-$ideal
(5) $P \cap D(A) = \phi$.

Proof:
$(\mathbf{1}) \Rightarrow (\mathbf{2})$: Clearly $A - P$ is a prime filter since every maximal filter is prime filter. Let $a \in A$ such that $a \notin A - P$ then $A - P \neq [(A - P) \cup \{a\}) = (A - P) \vee [a\rangle$, so by the maximality, we get $(A - P) \vee [a\rangle = A$. In particular, $0 = x \wedge a$ for some $x \in A - P$ and hence $a^* \wedge x = x$. But $x \in A - P$ implies $a^* \in A - P$ and hence $a \vee a^* \in A - P$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$: Since $A - P$ is a prime filter, P is a prime ideal. Let Q be a prime ideal and $Q \subset P$ with $a \in P - Q$. Then $a \wedge a^* = 0 \in Q$ and hence $a^* \in Q \subset P$, this implies $a \vee a^* \in P$, which is a contradiction to hypothesis. Thus P is minimal prime ideal.

$(\mathbf{3}) \Rightarrow (\mathbf{4})$ and $(4) \Rightarrow (5)$ follow from the above theorem.

$(\mathbf{5}) \Rightarrow (\mathbf{1})$: Since P is a prime ideal, $A - P$ is a prime filter. Let F be a filter of A and $A - P \subset F$ with $a \in F - (A - P)$. Since $a \vee a^*$ is dense, $a \vee a^* \in D(A)$ and hence $a \vee a^* \notin P$. But $a \in P$ and therefore $a^* \in A - P \subset F$. Also $a \in F$ and thus $a \wedge a^* = 0 \in F$ which implies that $F = A$.

For any a and $b \in A$ with $a \leq b$, the interval $[a, b] = \{x \in A : a \leq x \leq b\}$ is bounded distributive lattice with respect to the operations induced by those in the ADL A.

Recall from [3, 6] that, an ADL with a maximal element is said to be an Almost Boolean algebra if for any $a, b \in A$ with $a < b$, the interval $[a, b]$ is a complemented lattice.

Theorem 4.5 ([6]): Let A be an ADL with a maximal element. Then, the following are equivalent.

(1) A is an Almost Boolean algebra
(2) For any $a \in A$, there exist $b \in A$ such that $a \wedge b = 0$ and $a \vee b$ is maximal
(3) $[0, m]$ is a Boolean algebra for all maximal elements m
(4) There exists a maximal element m such that $[0, m]$ is a Boolean algebra.

In general, an Almost Boolean algebra is pseudo-complemented but converse is not true. However, in the following we characterize an Almost Boolean algebra in terms of $\alpha-$ideals and $\beta-$filters.

Theorem 4.6: Let A be a pseudo-complemented ADL. Then, A is an Almost Boolean algebra if and only if every ideal of A is an $\alpha-$ideal.

Proof: Let $*$ be a pseudo-complementation on A. We assume that A is an Almost Boolean algebra and let I be an ideal of A. Let $x \in I$. Then $x \vee 0^*$ is maximal since 0^* is maximal. By assumption, the interval $[0, x \vee 0^*]$ is a Boolean algebra and $x \leq x \vee 0^*$. Therefore, there exists $y \in [0, x \vee 0^*]$ such that $x \wedge y = 0$ and $x \vee y = x \vee 0^*$, hence $x \vee y$ is maximal and $y \wedge x^{**} = 0$.
Now, $x^{**} = (x \vee y) \wedge x^{**} = (x \wedge x^{**}) \vee (y \wedge x^{**}) = x \wedge x^{**} \in I$ (since $x \in I$ and I is an ideal). Thus, I is an $\alpha-$ideal.

Conversely suppose that every ideal of A is an $\alpha-$ideal. Then, for any $x \in A$, $(x]$ is an $\alpha-$ideal and hence $x^{**} \in (x]$. This implies $x^{**} = x \wedge x^{**}$.

Now, $x \wedge 0^* = x^{**} \wedge x \wedge 0^* = x \wedge x^{**} \wedge 0^* = x^{**} \wedge 0^* = x^{**}$. (since $x^* \wedge x = 0$ and hence $x^{**} \wedge x = x$ and $x^{**} \leq 0^*$). Let $x \in [0, 0^*]$. Then $x = x \wedge 0^* = x^{**}$. This shows that $[0, 0^*] = A^*$, which is a Boolean algebra (refer [4]). By theorem 4.5, A is an Almost Boolean algebra.

The following is similar to above theorem.

Theorem 4.7: A is an Almost Boolean algebra if and only if every filter of A is a $\beta-$filter.

REFERENCES

1. Blyth T. S.: *Ideals and Filters of Pseudo-complemented semilattices*, Proc. Edinburgh Math. Soc. 23 (1980) 301-316.
2. Rao G. C. and Ravikumar S.: *Minimal prime ideals in Almost Distributive Lattices*, Int. j. Contemp. Math.Sci 4(10) (2000) 95-104.
3. Swamy U. M. and Rao G. C.: *Almost Distributive Lattices*, J. Australian Math. Soc., (Series A), Vol.31 (1981), 77- 91.
4. Swamy U. M., Rao G. C. and Rao G. N.: *Pseudo complementation on Almost Distributive Lattices*, Southeast Asian Bull. Math., Vol. 24 (2000), 95-104.
5. Swamy U. M., Rao G. C. and Rao G. N.: *Stone Almost Distributive lattices*, Southeast Asian Bullettin of Mathematics, 27 (2003), 513-526.
6. Swamy U. M., Santhi Sundar Raj Ch. and Chudamani R.: *On Almost Boolean algebras and Rings*, International journal of mathematical archive, 7(12), 2016, 1-7.
7. Vasu Babu R., Santhi Sundar Raj Ch. and Venkateswarlu B.: *Weak Pseudo-complementations on ADLs*, Archivum Mathematicum, Tomus 50 (2014) 151-159.
8. Venkateswarlu B. and Vasu Babu R.: *Associate elements in ADLs*, Asian-European Journal of Mathematics, Vol. 7, No. 4(2014) 1450066 (7 pages).

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