International Journal of Mathematical Archive-8(7), 2017, 253-261
Available online through www.ijma.info ISSN 2229 – 5046

STABILITY OF PICARD ITERATION PROCEDURE USING CIRIC MULTI-VALUED CONTRACTION CONDITION

ANIL RAJPUT[1]
Professor and HOD, Department of Mathematics
Chandra Shekhar Azad Govt. P. G. College, Sehore, (M.P.), India.

ABHA TENGURIA[2]
Professor and HOD, Department of Mathematics,
Govt. M. L. B. Girls College, Bhopal, (M.P.), India.

ANJALI OJHA[3*]
Research Scholar, Department of Mathematics,
Chandra Shekhar Azad Govt. P. G. College, Sehore, (M.P.), India.

(Received On: 18-06-17; Revised & Accepted On: 27-07-17)

ABSTRACT

Stability of iterative procedure plays very important role in various fields like computer programming. game theory etc. The aim of this paper is to establish the stability results using Ciric multi-valued contraction for iterative procedure on a metric space.

Keywords: *Picard iterative procedure, fixed point, stability of iterative procedure, Ciric multi-valued contraction.*

1. INTRODUCTION

According to Rhoades [21] the concept of stability of a fixed point iteration procedure was due to Ostrowski, It has been systematically studied by Harder in her thesis and published in the papers of Harder and Hicks ([6] and [7]). Let $T: X \rightarrow X$ be a mapping on complete metric space (X, d). Let T has atleast one fixed point and there exists a sequence $\{x_n\}$ which converges to a fixed point $q \in$ X. Let $\{y_n\}$ be an arbitrary sequence in X and $x_{n+1} = f(T, x_n)$ be an iteration procedure, now set $\in_n = d(y_{n+1}, f(T, y_n))$ .The iteration procedure $x_{n+1} = f(T, x_n)$ is called T-stable if $\lim_{n\to\infty} \in_n = 0$ implies that $\lim_{n\to\infty} y_n = q$ and, is said to be almost T-stable, if the convergence of the series $\sum_{i=1}^{\infty} \in_i < \infty$ implies that $\lim_{n\to\infty} y_n = q$.

First time Harder and Hicks [6, 7] defined the concept of the stability of general iterative procedures, after that many authors have studied various special cases of the general iterative procedure. Some of them are Berinde [2], Imoru and Olatinwo [9], Jachymski [10, 11], Matkowski and Singh [13], Osilike [18] and Rhoades [21]. In the year 2005, Singh et al [25] introduced the stability of Jungck and Jungck-Mann iterative procedures for a pair of Jungck-Osilike-type maps on an arbitrary set with values in a metric space.

Non linear equation and approximating fixed points of a corresponding contractive type operator have very close relationship between them. There are so many methods for approximating fixed points .It is very interesting to know whether these methods are numerically stable or not. Many authors done remarkable work on the role of stability of iterative procedure , some of them are czerwik *et al* [4,5], Harder and Hicks [7,8], Lim [12], Matkowski and Singh [13], Ortega and Rheinboldt [16], Osilike [17,19], Ostrowski [20], Rhoades [22,23], Rus *et al* [24] and Singh *et al* [26].

All these papers applied the concept of stability which was introduced by Harder [8] and some of them used the concept of almost stability introduced by Osilike [18].

Corresponding Author: Anjali Ojha[3*]
Research Scholar, Department of Mathematics,
Chandra Shekhar Azad Govt. P. G. College, Sehore, (M.P.), India.

2. PRELIMINARIES

Here we defined various definitions which are used in this paper.

Let (X, d) be a metric space and

CB(X) = {A: A is a nonempty closed bounded subset of X},

CL(X) = {A: A is a nonempty closed subset of X}.

For A, B ∈ CL(X) and ε >0,

$N(\varepsilon, A) = \{x \in X : d(x, a) < \varepsilon \textit{ for some } a \in A\}$,

$E_{A,B} = \{\varepsilon > 0 : A \subseteq N(\varepsilon, B),\ B \subseteq N(\varepsilon, A)\}$,

$H(A, B) = \{\inf E_{A,B} \ \ if\ E_{A,B} \neq \emptyset,\ +\infty \ \ if\ E_{A,B} = \emptyset\}$

H is called the generalized Hausdorff metric (resp. Hausdorff metric) for CL(X) induced by d. For any nonempty subsets A, B of X, d(A, B) will denote the distance between the subsets A and B, while we write d(a, B) for d(A, B) when A = {a}.

Definition 2.1[14, 15]: Let $T: X \to CL(X)$ be a mapping on a complete metric space. If there exist a constant q such that $0 \leq q < 1$ and $H(Tx, Ty) \leq q\, d(x, y)$ for all $x, y \in X$, then the map $T: X \to CL(X)$ is called a Nadler multi-valued contraction.

Definition 2.2 [30]: If there exist real numbers α, β *and* γ such that $0 \leq \alpha < 1$, $0 \leq \beta < \frac{1}{2}$ and $0 \leq \gamma < \frac{1}{2}$ and atleast one of the following condition holds:

(i) $H(Tx, Ty) \leq \alpha\, d(x, y)$,
(ii) $H(Tx, Ty) \leq \beta[\, d(x, Tx) + d(y, Ty)]$,
(iii) $H(Tx, Ty) \leq \gamma[\, d(x, Ty) + d(y, Tx)]$ where $x, y \in X$

Then the map $T: X \to CL(X)$ is called a Zamfirescu multi-valued contraction.

Definition 2.3 [3]: If there exists a nonnegative number q such that

$$H(Tx, Ty) \leq q \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \left(\frac{1}{2}\right)\left(d(x, Ty) + d(y, Tx)\right)\right\}$$

for all $x, y \in X$ then the map $T: X \to CL(X)$ is called a Ciric generalized multi-valued contraction.

The inequality given by Timis [29] in his paper is as follows:

$$H(Tx, Ty) \leq q \max\left\{ d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \left(\frac{1}{2}\right)\left(d(x, Tx) + d(y, Ty)\right)\right\}$$

Where q is any nonnegative number.

Definition 2.4: Definition of Stability

Let $T: X \to CL(X)$ be a mapping on a metric space X. Let the sequence $\{x_n\}$ converges to a fixed point p of T and the iteration procedure is $x_{n+1} \in f(T, x_n)$

Let $\{y_n\}$ be any arbitrary sequence in X and set

$\epsilon_n = H(y_{n+1}, f(T, y_n))$, $n = 0, 1, 2...$

Then the iteration procedure $x_{n+1} \in f(T, x_n)$ is called T-stable if $\lim_{n\to\infty} \in_n = 0$ implies that $\lim_{n\to\infty} y_n = p$.

Ostrowski [20] proved that the picard iterative procedure is stable for single valued Banach contraction. While Singh and chadha [27] extended this theorem for multivalued contraction which is as follows

Theorem 2.1 [27]: Let (X, d) be a complete metric space and $T: X \to CL(X)$ a multi-valued contraction with constant q defined as

$H(Tx, Ty) \leq q d(x, y)$

for all $x, y \in X$ where $0 \leq q < 1$

Let p be the fixed point of T. Let $x_0 \in X$ and $x_{n+1} \in Tx_n, n = 0,1,2 \ldots$ Suppose that $\{y_n\}_{n=1}^{\infty}$ be a sequence in X and

$\in_n = H(y_{n+1}, Ty_n), \quad n = 0,1,2 \ldots$

Then

$$d(p, y_{n+1}) \le d\,(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j} \in_j$$

Further, if Tp is singleton then

$\lim_n y_n = p$ if and only if $\lim_n \in_n = 0$

After this Singh, Jain and Mishra [28] extended this theorem for Zamfirescu multi-valued contraction which is given as

Theorem 2.2 [28]: Let (X, d) be a complete metric space and $T: X \to CL(X)$ a multi-valued contraction which is defined in definition (2.2).

Let p be the fixed point of T. Let $x_0 \in X$ and $x_{n+1} \in Tx_n$, $n = 0,1,2 \ldots$ Suppose that $\{y_n\}_{n=1}^{\infty}$ be a sequence in X and

$\in_n = H(y_{n+1}, Ty_n)$

Then,

$$d(p, y_{n+1}) \le d\,(p, x_{n+1}) + \sum_{k=0}^{n} 2\delta^{n+1-k} H(x_k, Tx_k) + \delta^{n+1} d(x_0, y_0) + \sum_{k=0}^{n} \delta^{n-k} \in_k$$

Where $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$ *and* $n = 0,1,2 \ldots$

Further, if Tp is singleton then,

$\lim\limits_{n\to\infty} y_n = p$ ifand only if $\lim\limits_{n\to\infty} \in_n = 0$

We will use the following lemma

Lemma 2.1 [7]: If c is a real number such that $0 < |\,c\,| < 1$ and $\{b_k\}_{k=0}^{\infty}$ is a sequence of real numbers such that $\lim_{k\to\infty} b_k = 0$, then $\lim_{n\to\infty}\left(\sum_{k=0}^{n} c^{n-k} b_k\right) = 0$

3. MAIN RESULT

Theorem 3.1: Let (X, d) be a complete metric space and $T: X \to CL(X)$ a multi-valued contraction with constant q defined as

$$H(Tx, Ty) \le q \max\left\{d(x,y), d(x,Tx), d(y,Ty), \left(\tfrac{1}{2}\right)\left(d(x,Ty) + d(y,Tx)\right)\right\}$$

for all $x, y \in X$ where $0 \le q < 1$.

Let p be the fixed point of T. Let $x_0 \in X$ and $x_{n+1} \in Tx_n, n = 0,1,2 \ldots$ Suppose that $\{y_n\}_{n=1}^{\infty}$ be a sequence in X and

$$\in_n = H(y_{n+1}, Ty_n) \tag{A}$$

Then,

$$d(p, y_{n+1}) \le d(p, x_{n+1}) + \sum_{k=0}^{n} \delta^{n+1-k} H(x_k, Tx_k) + \delta^{n+1} d(x_o, y_0) + \sum_{k=0}^{n} \delta^{n-k} \epsilon_k \tag{I}$$

Where $\delta = \max\left(\frac{q}{1-q}, \frac{q}{2-q}, \frac{2q}{2-q}\right)$ and $n = 0,1,2 \ldots$

Further, if Tp is singleton then

$$\lim_{n\to\infty} y_n = p \text{ if and only if } \lim_{n\to\infty} \in_n = 0 \tag{II}$$

Proof: Let $x, y \in X$. Since T is a ciric generalised multi-valued contraction then,

$$\begin{aligned}
H(Tx, Ty) &\le q \max\left\{d(x,y), d(x,Tx), d(y,Ty), \left(\frac{1}{2}\right)\left(d(x,Ty) + d(y,Tx)\right)\right\} \\
&\le q \max\left\{H(x,y), H(x,Tx), H(y,Ty), \left(\frac{1}{2}\right)\left(H(x,Ty) + H(y,Tx)\right)\right\} \\
&\le q \max\left\{H(x,y), H(x,Tx), H(y,x) + H(x,Tx) + H(Tx,Ty), \left(\tfrac{1}{2}\right)\begin{pmatrix} H(x,Tx) + H(Tx,Ty) + H(y,x) \\ +H(x,Tx) \end{pmatrix}\right\} \quad (1) \\
&\le q \max\left\{H(x,y), H(x,Tx), H(y,x) + H(x,Tx) + H(Tx,Ty), \left(\tfrac{1}{2}\right)\left(2H(x,Tx) + H(Tx,Ty) + H(x,y)\right)\right\}
\end{aligned}$$

$$H(Tx, Ty) \le qH(x, y)$$

$$H(Tx, Ty) \le qH(x, Tx)$$

Now,

$$H(Tx,Ty) \leq qH(y,Ty)$$
$$H(Tx,Ty) \leq q \max\{H(y,x) + H(x,Tx) + H(Tx,Ty)\}$$
$$(1-q)H(Tx,Ty) \leq q \max\{H(x,y) + qH(x,Tx)\}$$
$$H(Tx,Ty) \leq \frac{q}{1-q} H(x,y) + \left(\frac{q}{1-q}\right) H(x,Tx) \tag{2}$$
$$H(Tx,Ty) \leq q\left\{\left(\frac{1}{2}\right)\left(2H(x,Tx) + H(Tx,Ty) + H(x,y)\right)\right\}$$
$$H(Tx,Ty) \leq q\left\{\left(H(x,Tx) + \left(\frac{1}{2}\right)H(Tx,Ty) + \left(\frac{1}{2}\right)H(x,y)\right)\right\}$$
$$\left(1-\frac{q}{2}\right)H(Tx,Ty) \leq q\left\{\left(H(x,Tx) + \left(\frac{1}{2}\right)H(x,y)\right)\right\}$$
$$\left(\frac{2-q}{2}\right)H(Tx,Ty) \leq q\left\{\left(H(x,Tx) + \left(\frac{1}{2}\right)H(x,y)\right)\right\}$$
$$H(Tx,Ty) \leq \left(\frac{2q}{(2-q)}\right)H(x,Tx) + \left(\frac{q}{2-q}\right)H(x,y) \tag{3}$$

This yields,

$$H(Tx,Ty) \leq \delta H(x,Tx) + \delta H(x,y)$$
$$H(Tx,Ty) \leq \delta H(x,Tx) + \delta d(x,y) \tag{B}$$

Where $\delta = \max\left(\frac{q}{1-q}, \frac{q}{2-q}, \frac{2q}{2-q}\right)$

Since,

$$d(p,y_{n+1}) \leq d(p,x_{n+1}) + d(x_{n+1},y_{n+1}) \tag{4}$$

We have,

$$\begin{aligned} d(x_{n+1},y_{n+1}) &\leq H(Tx_n,y_{n+1}) \\ &\leq H(Tx_n,Ty_n) + H(Ty_n,y_{n+1}) \\ &\leq \delta H(x_n,Tx_n) + \delta d(x_n,y_n) + \epsilon_n \qquad \text{(From (A) and (B),)} \end{aligned} \tag{5}$$

Consequently,

$$d(x_n,y_n) \leq \delta H(x_{n-1},Tx_{n-1}) + \delta d(x_{n-1},y_{n-1}) + \epsilon_{n-1} \tag{6}$$

Using (5) and (6) in (4) we get,

$$\begin{aligned} d(p,y_{n+1}) &\leq d(p,x_{n+1}) + \delta H(x_n,Tx_n) + \epsilon_n + \delta[\delta H(x_{n-1},Tx_{n-1}) + \delta d(x_{n-1},y_{n-1}) + \epsilon_{n-1}] \\ &\leq [d(p,x_{n+1}) + \delta H(x_n,Tx_n) + \delta^2 H(x_{n-1},Tx_{n-1}) + \delta^2 d(x_{n-1},y_{n-1}) + (\epsilon_n + \delta\epsilon_{n-1})] \end{aligned}$$

On repeating this process $(n-1)$ times, we get,

$$d(p,y_{n+1}) \leq d(p,x_{n+1}) + \sum_{k=0}^{n} \delta^{n+1-k} H(x_k,Tx_k) + \delta^{n+1} d(x_o,y_0) + \sum_{k=0}^{n} \delta^{n-k}\epsilon_k$$

This proves (I).

From (A) we have

$$\begin{aligned} \epsilon_n &= H(y_{n+1},Ty_n) \\ &\leq d(y_{n+1},p) + H(p,Tp) + H(Tp,Ty_n) \\ &\leq d\left(y_{n+1},p) + H(p,Tp) + \delta H(p,Tp) + \delta d(p,y_n)\right) \qquad \text{(From (B))} \end{aligned}$$

This concludes that $\epsilon_n \to 0$ as $n \to \infty$ because $Tp = \{p\}$ by hypothesis.

Conversly, suppose that $\epsilon_n \to 0$ as $n \to \infty$

First we assume that $\lim_{k\to\infty} H(x_k, Tx_k) = 0$, if $Tp = \{p\}$

For,

$$\begin{aligned} H(x_k,Tx_k) &\leq H(x_k,Tp) + H(Tp,Tx_k) \\ &\leq d(x_k,\{p\}) + H(Tp,Tx_k) \end{aligned} \tag{7}$$

As we know that T is a ciric multivalued contraction and $\{Tx_k\}$ is a Cauchy sequence. Also $Tx_k \to Tp$ as $k \to \infty$. So putting $k \to \infty$ in (7) we get the required result.

Now $0 \leq \delta < 1$, If $\delta = 0$ then (I) gives,

$$\lim_{n\to\infty} y_n = p$$

So we consider that $0 \leq \delta < 1$, Then,

$$\delta^{n+1} d(x_0, y_0) \to 0 \; as \; n \to \infty$$

Since, $\lim_{k\to\infty} H(x_k, Tx_k) = 0, \lim_{k\to\infty} \epsilon_k = 0$, therefore by lemma 2.1,

$$\sum_{k=0}^{n} 2\delta^{n+1-k} H(x_k, Tx_k) \to 0 \; and \; \sum_{k=0}^{n} \delta^{n-k} \epsilon_k \to 0 \; as \; n \to \infty$$

Hence from (I)

$$\lim_{n\to\infty} y_n = p.$$

Corollary 3.1: Let (X, d)be a complete metric space and $T: X \to X$ be a Ciric contraction with constant q defined as $d(Tx, Ty) \leq q \max\left\{d(x, y), d(x, Tx), d(y, Ty), \left(\frac{1}{2}\right)\left(d(x, Ty) + d(y, Tx)\right)\right\}$ for all $x, y \in X$ where $0 \leq q < 1$.

Let p be the fixed point of T. Let $x_0 \in X$ and $x_{n+1} = Tx_n, n = 0,1,2 \ldots$ Suppose that $\{y_n\}_{n=1}^{\infty}$ be a sequence in X and $\in_n = d(y_{n+1}, Ty_n)$

Then,

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} \delta^{n+1-k} d(x_k, x_{k+1}) + \delta^{n+1} d(x_o, y_0) + \sum_{k=0}^{n} \delta^{n-k} \epsilon_k$$

where $\delta = \max\left(\frac{q}{1-q}, \frac{q}{2-q}, \frac{2q}{2-q}\right)$ and $n = 0,1,2 \ldots$

Further, if Tp is singleton then,

$\lim_{n\to\infty} y_n = p$ if and only if $\lim_{n\to\infty} \in_n = 0$

Proof: We know that if T is single valued mapping then $\in_n = H(y_{n+1}, Ty_n)$ becomes $\in_n = d(y_{n+1}, Ty_n)$ and $H(x_n, Tx_n)$ becomes $d(x_n, x_{n+1})$.

As we know that $p \in X$ in second part of Theorem 3.1, is not necessarily a unique fixed point of T. This shows that Tp contains just one point.

Theorem 3.2: In case of theorem 3.1, if we replace $\in_n = H(y_{n+1}, Ty_n)$ by $\in_n = d(y_{n+1}, p_n)$, $p_n \in Ty_n$, $n = 0,1,2 \ldots$ Then, $d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} \delta^{(n+1-k)} H(x_k, Tx_k) + \delta^{(n+1)} d(x_o, y_0) + \sum_{k=0}^{n} \delta^{n-k}(H_k + \epsilon_k)$
Where

$$H_k = H(x_{k+1}, Tx_k) \tag{III}$$

Further, if Tp is singleton then

$$\lim_{n\to\infty} y_n = p \text{ ifand only if } \lim_{n\to\infty} \in_n = 0 \tag{IV}$$

If T is continuous and $\lim_{n\to\infty} \in_n = 0$ then $\lim_{n\to\infty} y_n = p$

Proof: Let T be a Ciric multi-valued contraction, from (B) we have,
$H(Tx_n, Ty_n) \leq \delta H(x_n, Tx_n) + \delta d(x_n, y_n)$ for any $x_n, y_n \epsilon X$. Thus, if n is any non negative integer then,

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq d(x_{n+1}, p_n) + d(p_n, y_{n+1}) \\ &\leq H(x_{n+1}, Ty_n) + \in_n \\ &\leq H(x_{n+1}, Tx_n) + H(Tx_n, Ty_n) + \in_n \\ &\leq H_n + \delta H(x_n, Tx_n) + \delta d(x_n, y_n) + \in_n \\ &\leq H_n + \delta H(x_n, Tx_n) + \delta\{H_{n-1} + \delta H(x_{n-1}, Tx_{n-1}) + \delta d(x_{n-1}, y_{n-1}) + \in_{n-1}\} + \in_n \\ &\leq \delta H(x_n, Tx_n) + \delta^2 H(x_{n-1}, Tx_{n-1}) + \delta^2 d(x_{n-1}, y_{n-1}) + \delta(H_{n-1} + \in_{n-1}) + (H_n + \in_n) \end{aligned}$$

Respectively,

$$d(x_{n+1}, y_{n+1}) \leq \sum_{k=0}^{n} \delta^{(n+1-k)} H(x_k, Tx_k) + \delta^{(n+1)} d(x_o, y_0) + \sum_{k=0}^{n} \delta^{n-k}(H_k + \epsilon_k) \quad (8)$$

Thus,

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + d(x_{n+1}, y_{n+1})$$

From (8),

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} \delta^{(n+1-k)} H(x_k, Tx_k) + \delta^{(n+1)} d(x_o, y_0) + \sum_{k=0}^{n} \delta^{n-k}(H_k + \epsilon_k)$$

Now if we assume that $y_n \to p \; as \; n \to \infty$

Then,

$$\in_n = d(y_{n+1}, p_n) \leq H(y_{n+1}, Ty_n)$$

Thus, theorem 3.1 shows that $\lim_{n\to\infty} \in_n = 0$

Now if we consider that T is continuous and $\lim_{n\to\infty} \in_n = 0$ then from (III)

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} \delta^{(n+1-k)} H(x_k, Tx_k) + \delta^{(n+1)} d(x_o, y_0) + \sum_{k=0}^{n} \delta^{n-k} t_k,$$

where $t_k = (H_k + \epsilon_k)$. According to the proof of theorem 3.1 we need to show that the sequence $\{t_k\}$ is convergent to 0.As we assumed that, the sequence $\{\epsilon_k\}$ is convergent to 0, it is sufficient to show that $\{H_n \}$ is also convergent to 0. Since T is continuous,

$$\lim_{n\to\infty} H_n = \lim_{n\to\infty} H(x_{n+1}, Tx_n) = H(p, Tp) = 0$$

Thus the proof is complete.

Theorem 3.3: Let (X, d) be a complete metric space and $T: X \to CL(X)$ a multi-valued contraction with constant q defined as

$$H(Tx, Ty) \leq q \max\left\{ d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \left(\tfrac{1}{2}\right)\left(d(x, Tx) + d(y, Ty)\right)\right\}.$$

For all x, y∈ X where $0 \leq q < 1$

Let p be the fixed point of T. Let $x_0 \in X$ and $x_{n+1} \in Tx_n, n = 0,1,2 \ldots$ Suppose that $\{y_n\}_{n=1}^{\infty}$ be a sequence in X and

$$\in_n = H(y_{n+1}, Ty_n) \tag{C}$$

Then,

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} \delta^{n+1-k} H(x_k, Tx_k) + \delta^{\,n+1} d(x_o, y_0) + \sum_{k=0}^{n} \delta^{n-k} \epsilon_k \tag{V}$$

Where $\delta = \max\left(q, \frac{q}{1-q}, \frac{q}{2-q}, \frac{2q}{2-q}\right)$

Further, if Tp is singleton then

$$\lim_{n\to\infty} y_n = p \text{ ifand only if } \lim_{n\to\infty} \in_n = 0 \tag{VI}$$

Proof: Let $x, y \in X$. Since T is a generalised multi-valued contraction given by Timis [29] then,

$$H(Tx, Ty) \leq q \max\left\{ d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \left(\tfrac{1}{2}\right)\left(d(x, Tx) + d(y, Ty)\right)\right\}$$

$$H(Tx, Ty) \leq q\, H(x, y) \tag{9}$$

$$H(Tx, Ty) \leq q\, H(x, Ty)$$
$$\leq q\, [H(x, Tx) + H(Tx, Ty)]$$
$$(1 - q) H(Tx, Ty) \leq q\, [H(x, Tx)]$$
$$H(Tx, Ty) \leq \frac{q}{1-q} H(x, Tx) \tag{10}$$

$$H(Tx, Ty) \leq q\, H(y, Tx)$$
$$H(Tx, Ty) \leq q[\, H(y, x) + H(x, Tx)] \tag{11}$$

$$H(Tx, Ty) \leq \frac{q}{2}[H(x, Tx) + H(y, Ty)$$
$$H(Tx, Ty) \leq \frac{q}{2}[H(x, Tx) + H(y, x) + H(x, Tx) + H(Tx, Ty)]$$
$$(1 - \frac{q}{2}) H(Tx, Ty) \leq \frac{q}{2}[2H(x, Tx) + H(y, x)]$$
$$H(Tx, Ty) \leq \frac{q}{2-q}[2H(x, Tx) + H(y, x)] \tag{12}$$

This yields,

$$H(Tx, Ty) \leq \delta H(x, Tx) + \delta H(x, y)$$
$$H(Tx, Ty) \leq \delta H(x, Tx) + \delta d(x, y) \tag{D}$$

Where,

$$\delta = \max\left(q, \frac{q}{1-q}, \frac{q}{2-q}, \frac{2q}{2-q}\right)$$

Since,

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + d(x_{n+1}, y_{n+1}) \tag{13}$$

We have,

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq H(Tx_n, y_{n+1}) \\ &\leq H(Tx_n, Ty_n) + H(Ty_n, y_{n+1}) \end{aligned}$$

From (C) and (D), we have,

$$\leq \delta H(x_n, Tx_n) + \delta d(x_n, y_n) + \epsilon_n \tag{14}$$

Consequently,

$$d(x_n, y_n) \leq \delta H(x_{n-1}, Tx_{n-1}) + \delta d(x_{n-1}, y_{n-1}) + \epsilon_{n-1} \tag{15}$$

Using (14) and (15) in (13) we get,

$$\begin{aligned} d(p, y_{n+1}) &\leq d(p, x_{n+1}) + \delta H(x_n, Tx_n) + \epsilon_n + \delta[\delta H(x_{n-1}, Tx_{n-1}) + \delta d(x_{n-1}, y_{n-1}) + \epsilon_{n-1}] \\ &\leq [d(p, x_{n+1}) + \delta H(x_n, Tx_n) + \delta^2 H(x_{n-1}, Tx_{n-1}) + \delta^2 d(x_{n-1}, y_{n-1})\ (\epsilon_n + \delta\epsilon_{n-1})] \end{aligned}$$

On repeating this process $(\text{n} - 1)$times, we get,

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} \delta^{n+1-k} H(x_k, Tx_k) + \delta^{n+1} d(x_o, y_0) + \sum_{k=0}^{n} \delta^{n-k} \epsilon_k$$

This proves (V)

By (C)

$$\begin{aligned} \in_n &= H(y_{n+1}, Ty_n) \\ &\leq d(y_{n+1}, p) + H(p, Tp) + H(Tp, Ty_n) \end{aligned}$$

From (B),

$$\leq d\big(y_{n+1}, p) + H(p, Tp) + \delta H(p, Tp) + \delta d(p, y_n)\big)$$

This concludes that $\in_n \to 0\ as\ n \to \infty$ because $Tp = \{p\}$ by hypothesis.

Conversely, suppose that $\in_n \to 0\ as\ n \to \infty$

First we assume that $\lim_{k\to\infty} H\big(x_{k,}Tx_k\big) = 0, if\ \ Tp = \{p\}$

For,

$$\begin{aligned} H\big(x_{k,}Tx_k\big) &\leq H\big(x_{k,}Tp\big) + H(Tp, Tx_k) \\ &\leq d\big(x_{k,}\{p\}\big) + H(Tp, Tx_k) \end{aligned} \tag{16}$$

As we know that T is a multi-valued contraction and $\{Tx_k\}$ is a Cauchy sequence. Also $Tx_k \to Tp$ as $k \to \infty$. So putting $k \to \infty$ in (16) we get the required result.

Now $0 \leq \delta < 1$, If $\delta = 0$ then (V) gives,

$$\lim_{n\to\infty} y_n = p$$

So we consider that $0 \leq \delta < 1$,

Then,

$$\delta^{n+1} d\big(x_{0,}y_0\big) \to 0\ as\ n \to \infty$$

Since $\lim_{k\to\infty} H\big(x_{k,}Tx_k\big) = 0, \lim_{k\to\infty}\epsilon_k = 0$, therefore by lemma 2.1, $\sum_{k=0}^{n} 2\delta^{n+1-k} H(x_k, Tx_k) \to 0\ and\ \ \sum_{k=0}^{n} \delta^{n-k}\epsilon_k \to 0\ \ as\ n \to \infty$

Hence from (V),

$$\lim_{n\to\infty} y_n = p.$$

Corollary 3.2: Let (X, d) be a complete metric space and $T: X \to X$ is a single valued mapping with constant q defined as

$$d(Tx, Ty) \leq\ q \max\Big\{ d(Tx, Ty), d(x, y), d(x, Ty), d(y, Tx), \Big(\tfrac{1}{2}\Big)\big(d(x, Tx) + d(y, Ty)\big)\Big\}.$$

For all x, y∈ X where $0 \leq q < 1$

Let p be the fixed point of T. Let $x_0 \in X$ and $x_{n+1} = Tx_n, n = 0,1,2$ ... Suppose that $\{y_n\}_{n=1}^{\infty}$ be a sequence in X and

$$\in_n = d(y_{n+1}, Ty_n)$$

Then,

$$d(p,y_{n+1}) \leq d(p,x_{n+1}) + \sum_{k=0}^{n} \delta^{n+1-k} d(x_k,Tx_k) + \delta^{n+1} d(x_o,y_0) + \sum_{k=0}^{n} \delta^{n-k} \epsilon_k$$

Where $\delta = \max\left(q, \frac{q}{1-q}, \frac{q}{2-q}, \frac{2q}{2-q}\right)$

Further, if Tp is singleton then

$\lim_{n\to\infty} y_n = p$ ifand only if $\lim_{n\to\infty} \in_n = 0$

Proof: We know that if T is single valued mapping then $\in_n = H(y_{n+1}, Ty_n)$ becomes $\in_n = d(y_{n+1}, Ty_n)$ and $H(x_n, Tx_n)$ becomes $d(x_n, x_{n+1})$.

As we know that $p \in X$ in second part of Theorem 3.3, is not necessarily a unique fixed point of T. This shows that Tp contains just one point.

Theorem 3.4: In case of theorem 3.3, if we replace $\in_n = H(y_{n+1}, Ty_n)$ by $\in_n = d(y_{n+1}, p_n)$, $p_n \in Ty_n$, $n = 0,1,2 \ldots$ Then,

$$d(p,y_{n+1}) \leq d(p,x_{n+1}) + \sum_{k=0}^{n} \delta^{(n+1-k)} H(x_k,Tx_k) + \delta^{(n+1)} d(x_o,y_0) + \sum_{k=0}^{n} \delta^{n-k} (H_k + \epsilon_k)$$

Where

$$H_k = H(x_{k+1}, Tx_k)$$

Further, if Tp is singleton then

$\lim_{n\to\infty} y_n = p$ if and only if $\lim_{n\to\infty} \in_n = 0$

If T is continuous and $\lim_{n\to\infty} \in_n = 0$ then $\lim_{n\to\infty} y_n = p$.

Proof: Proof directly follows from theorem 3.2

REFERENCES

1. Berinde V, Iterative Approximation of Fixed Points, Springer-Verlag 2007.
2. Berinde V, Iterative Approximation of Fixed Points, Editura Efemeride, Baia Mare, (2002).
3. Ciric, L.B, Fixed points for generalized multi-valued contractions. Mat Vesnik. 9(24):265–272 (1972).
4. Czerwik S, Dlutek K, Singh S.L, Round-off stability of iteration procedures for operators in b-metric spaces. J Natur Phys Sci.11, 87–94 (1997).
5. Czerwik S, Dlutek K, Singh S.L, Round-off stability of iteration procedures for set-valued operators in b-metric spaces. J Natural Phys Sci. 15(1-2):1–8 (2001).
6. Harder A. M and Hicks T.I, A stable iteration procedure for non-expansive mappings, Math. Japonica, 33, No. 5(1988), 687-692.
7. Harder A. M and Hicks T.I, Stability results for fixed point iteration procedures, Math. Japonica, 33, No. 5(1988), 693-706.
8. Harder A. M, Fixed point theory and stability results for fixed point iterative procedures. Ph.D. Thesis, Univ of Missouri-Rolla. (1987).
9. Imoru C.O and Olatinwo M.O, Some stability theorems for some iteration procedures, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica 45(2006), 81-88.
10. Jachymski J. R, Common fixed point theorems for families of maps, Indian J. Pure Appl. Math. 25, No. 9(1994), 925-937.
11. Jachymski J. R, An extension of A. Ostrowski's theorem on the round-off stability of iterations, Aequationes Math. 53, No. 3(1997), 242-253.
12. Lim T.C, On fixed point stability for set-valued contractive mappings with applications to generalized differential equations. J Math Anal Appl. 110, 436–441 (1985). doi:10.1016/0022-247X(85)90306-3.
13. Matkowski J and Singh S.L, Round off stability of functional iterations on product spaces, Indian J. Math. 39, No. 3(1997), 275-286.
14. Nadler, SB Jr, Multi-valued contraction mappings. Pacific J Math. 30, 475–488 (1969).
15. Nadler, SB Jr, Hyperspaces of Sets. Marcel-Dekker, New York (1978).
16. Ortega J.M and Rheinboldt W.C, Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York (1970).
17. Osilike M.O, Stability results for fixed point iteration procedures, J. Nigerian Math. Soc. 14/15(1995/96), 17-29.
18. Osilike M.O, A note on the stability of iteration procedures for strong pseudocontractions and strongly accretive type equations J. Math. Anal. Appl.250 (2000), no.2, 726-730.
19. Osilike M.O, Stability results for the Ishikawa fixed point iteration procedure. Indian J Pure Appl Math. 26, 937–945(1995).

20. Ostrowski A.M, The round-off stability of iterations. Z Angew Math Mech. 47, 77–81 (1967). doi:10.1002/zamm.19670470202.
21. Rhoades B. E, Fixed point theorems and stability results for fixed point iteration procedures II Indian J. Pure Appl. Math. 24, No. 11(1993), 691-703.
22. Rhoades B.E, Fixed point theorems and stability results for fixed point iteration procedures. Indian J Pure Appl Math. 21, 1–9 (1990)
23. Rhoades B.E, Fixed point theorems and stability results for fixed point iteration procedures-II. Indian J Pure Appl Math.24, 697–703 (1993)
24. Rus I.A, Petrusel A. and Petrusel G, Fixed Point Theory 1950-2000: Romanian Contributions. House of the Book of Science,Cluj-Napoca (2002).
25. Singh S.L, Bhatnagar C and Mishra S.N, Stability of Jungck-type iteration procedures, Int. J. Math. Math. Sci. 19(2005) 3035-3043.
26. Singh S.L, Mishra S.N and Chadha V, Round-off stability of iterations on product spaces. C R Math Rep Acad Sci Canada. 16(3):105–109 (1994).
27. Singh S.L and Chadha V, Round-off stability of iterations for multivalued operators. C R Math Rep Acad Sci Canada. 17(5):187–192 (1995).
28. Singh S.L, Mishra S.N and Jain S, Round-off stability for multi-valued maps, Springer 2012.
29. Timis Ioana, Stability of Jungck-Type iterative procedure for some contractive type mappings via implicit relations, Miskole Mathematical Notes, Vol. 13 (2012), No. 2, pp. 555-567.
30. Zamfirescu T, Fixed point theorems in metric spaces. Arch Math (Basel). 23, 292–298 (1972). doi:10.1007/BF01304884.

Source of support: Nil, Conflict of interest: None Declared.