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APPROXIMATE CONTROLLABILITY RESULTS FOR IMPULSIVE NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS OF SOBOLEV TYPE WITH UNBOUNDED DELAY IN HILBERT SPACES

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ABSTRACT

In this paper, we discuss the approximate controllability of the impulsive neutral stochastic differential equations of Sobolev type with unbounded delay in Hilbert Spaces. A set of sufficient conditions are established for the existence and approximate controllability of the mild solutions using Krasnoselskii-Schaefer type fixed point theorem and stochastic analysis theory. An application involving partial differential equations with unbounded delay is addressed.

Keywords: *Approximate Controllability, Fixed point theorem, Stochastic differential equation, Mild solution, Nonlocal conditions.*

2010 Mathematics Subject Classification: *65C30, 34K40, 34K45.*

1. INTRODUCTION

Impulsive dynamical systems are characterized by the occurrence of an abrupt change in the state of the system, which occur at certain time instants over a period of negligible duration. The dynamical behavior of such systems is much more complex than the behavior of dynamical systems without impulse effects. The presence of impulse means that the state trajectory does not preserve the basic properties which are associated with non-impulsive dynamical systems. In fact, the theory of impulsive differential equations has found extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. Recently, several works reported the existence results for impulsive functional differential systems of first order and second order in [5, 12, 16, 17, 42 - 45, 52, 55]

The study of stochastic differential equations has attracted great interest due to its applications in characterizing many problems in physics, biology, chemistry, mechanics and so an. The deterministic models often fluctuate due to noise, so we must move from deterministic control to stochastic control problems. In the present literature, there is only a limited number of papers that deal with the approximate controllability of stochastic system. The stochastic differential equations (SDEs) can be used to characterize a response of such a model [2, 14, 26, 37, 46]. SDEs naturally refer to the time dynamics of the evolution of a state vector, based on the (approximate) physics of the real system, together with a driving noise process. The noise process can be assumed in several ways. It often symbolizes processes not included in the model, but present in the real system. The qualitative properties of SDEs such as existence, controllability and stability for the first-order stochastic differential equations have been investigated by several authors [13, 15, 28, 30, 33, 38 – 41, 43].

Neutral differential equations arise in many areas of applied mathematics such as electronics, fluid dynamics, biological models and chemical kinetics and for this reason, these type of equations have received much attention in recent years. The literature relative to ordinary neutral differential equations is very extensive, thus we suggest [19] concerning this matter. For theory and applications on neutral partial differential equations with nonlocal and classical conditions, refer [1, 4, 6, 10, 21 - 24, 47, 48, 50, 55]. Partial neutral differential equations with finite delay arise, for instance, from transmission line theory. Wu and Xia [54] have shown that a ring array of identical coupled lossless transmission lines leads to a system of neutral functional differential equations with discrete diffusive coupling which exhibits various types of discrete wave. By taking a natural limit, they obtain from this system of neutral equations a scalar partial neutral functional differential equation with finite delay defined on the unit circle. Such a partial neutral functional differential equation is also investigated by Hale [20].

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On the other hand, partial neutral differential equation with unbounded delay arises in [18, 36] for the description of heat conduction in materials with fading memory. In the classical theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly on the temperature $u(\cdot)$ and on its gradient $\Delta u(\cdot)$. Under these conditions, the classic heat equation describes the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [18, 36], the internal energy and the heat flux are described as functional of u and u_x. The next system, see for instance [29], has been frequently used to describe the phenomena,

$$\frac{d}{dt}\left[c_0 u(t,x) + \int_{-\infty}^{t} k_1(t-s)u(s,x)ds\right] = c_1\Delta u(t,x) + \int_{-\infty}^{t} k_2(t-s)\Delta u(s,x)ds + f(t,x), t \geq 0$$
$$u(t,x) = 0, x \in \partial\Omega, t \in \mathbb{R}.$$

In this system, $\Omega \in \mathbb{R}^n$ is open, bounded and with smooth boundary; $(t,x) \in [0,\infty)\times\Omega; u(t,x)$ represents the temperature in x at the time; $c_{0,}c_1$ are physical constants and $k_i: \mathbb{R} \to \mathbb{R}, i = 1,2.$ are the internal energy and the heat flux relaxation, respectively. By assuming the solution $u(\cdot)$ is known on $(-\infty, 0]$ and that $k_2 \equiv 0$ we can transform this system into an abstract neutral system with unbounded delay.

The idea of controllability is of enormous influence in mathematical control theory and engineering because they have closely related to pole assignment, structural decomposition, observer design etc. There is a variety of controllability of systems represented by semilinear evolution equations, integrodifferential evolution equations, neutral functional evolution inclusions and impulsive evolution inclusions. There are two basic theories of controllability can be identified which are approximate controllability and exact controllability. Exact controllability allows to govern the system to arbitrary final state while approximate controllability means that system can be governed to arbitrary small neighborhood of final state. In other words approximate controllability gives the possibility of governing the system to states which form the dense subspace in the state space. Controllability for first order, second order and fractional order differential systems have been studied by many authors, [15, 34 -35, 42, 44, 49 - 53, 56]

On the other hand, the Sobolev - type differential equations arise naturally in the mathematical modeling of various physical phenomena such as in the fluid flow through fissured rocks, thermodynamics, shear in second order fluids and so on see [1, 3, 7, 11, 25, 27, 31, 32, 41] and the references therein. Inspired by the above mentioned papers based on Sobolev type, we are establishing a set of sufficient conditions for the approximate controllability of neutral impulsive stochastic differential equations of Sobolev type with unbounded delay in Hilbert Spaces.

The Paper is organized as follows: In Section 2, we introduce some preliminaries such as definitions and some useful lemmas. In section 3, the main result is established and we extend our result to nonlocal conditions. In section 4, an application is given to demonstrate obtained results.

2. PRELIMINARIES

In this section, the basic preliminaries, definitions, lemmas, notations and some results which are needed to establish our main results are discussed.

Let $(H, \|\cdot\|_H)$ and $(K, \|\cdot\|_K)$ be two real separable Hilbert spaces and for convenience, we use the same notation $\|\cdot\|$ to denote the norms in H and K and $\langle\cdot,\cdot\rangle$ to denote the inner product space without any confusion. Let $\mathcal{L}(K,H)$ be space of bounded linear operators from K into H. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. Let $\{w(t), t \geq 0\}$ represents a $Q-$ Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with the co-variance operator Q such that $Tr(Q) < \infty$. Further, we assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in K, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, k=1,2,... and sequence of independent Wiener processes such that $\{\beta_k\}_{k\geq 1}$ such that

$$\langle w(t), e\rangle_K = \sum_{k=1}^{\infty}\sqrt{\lambda_k}\langle e_k, e\rangle_K\, \beta_k(t), t \geq 0.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}, H)$ be the space of all Hilbert- Schmidt operators from $Q^{\frac{1}{2}}K\ to\ H$ with the inner product $\langle\varphi,\psi\rangle_{\mathcal{L}_2^0} = Tr[\varphi Q\psi^*]$.

In this paper, we investigate the approximate controllability of stochastic impulsive neutral functional differential equations of Sobolev - type with unbounded delay in the form

$$\frac{d}{dt}[Lx(t) - g(t,x_t)] = Ax(t) + Bu(t) + f(t,x_t) + \sigma(s,x_s)dw(s), \quad t \in J := [0,b], t \neq t_k, k = 1,2,..m \,. \tag{2.1}$$

$$x(t) = \phi(t) \in \mathcal{B}_h, t \in (-\infty, 0] \tag{2.2}$$

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1,2,\ldots m. \tag{2.3}$$

where the state $x(\cdot)$ takes the values in the separable real Hilbert spaces H, A and L are linear operators on H. The histories $x_t \in (-\infty,0] \to \mathcal{B}_h, x_t(\theta) = x(t+\theta)$ for $t \geq 0$ belongs to the phase space $\mathcal{B}_h$, which will be defined later. The initial data $\phi = \{\phi(t), t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued stochastic process independent of W with finite second moments. Further $f, g: J \times \mathcal{B}_h \to$ H and $\sigma: J \times \mathcal{B}_h \to \mathcal{L}_2^0(K, H)$ are appropriate mappings specified later and the control function $u(.)$ is given in $\mathcal{L}(J, U)$, a Hilbert space of admissible control functions with U as Hilbert space. B is a bounded linear operator from U into H. And also $I_k: H \to H$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, for all $k = 1,2, \dots m$. $0 = t_0 < t_1 < t_2 \dots . < t_m < t_{m+1} = b$. Here $x(t_k^+)$ and $x(t_k^-)$ represents right and left limits of $x(t)$ at $t = t_k$, respectively.

The operators $A: D(A) \subset H \to H$ and $L: D(L) \subset H \to H$ satisfy the following conditions:
(A1) A and L are closed linear operators.

(A2) $D(L) \subset D(A)$ and L is bijective.

(A3) $L^{-1}: H \to D(L)$ is continuous.

Further, from **(A1)** and **(A2),** L^{-1} is closed and with (A3) by using the closed graph theorem, we obtain the boundedness of the linear operator $AL^{-1}: H \to H$. Further AL^{-1}generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in H. Let us denote $max_{t\in J}\|T(t)\|^2 = M$, $\|L^{-1}\|^2 = M_L$.

Definition 2.1 (Phase space): Assume that $h: (-\infty, 0] \to (0, \infty)$ is a continuous function with $l = \int_{-\infty}^0 h(t)dt < +\infty$ and ϕ is a $\mathcal{F}_0$- measurable functions mappings from $(-\infty, 0]$ into H. Define the phase space $\mathcal{B}_h$ by

$$\mathcal{B}_h = \{\phi: (-\infty, 0] \to H, \text{for any } a > 0, (E\|\phi(\theta)\|^2)^{\frac{1}{2}}$$

is a bounded and measurable function on $[-a, 0]$ with $\phi(0) = 0$ and

$$\int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0} \left((E\|\phi(\theta)\|^2)^{\frac{1}{2}}\right) ds\} < \infty.$$

If $\mathcal{B}_h$ is endowed with the norm

$$||\phi||_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0} \left((E\|\phi(\theta)\|^2)^{\frac{1}{2}}\right) ds, \phi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, ||\,.\,||_{\mathcal{B}_h})$ is a Banach space.

Now we consider the space of

$$\mathcal{B}_h' = \{x: x \in \mathcal{C}(-\infty, b] \to H) \text{ such that } x_k \in \mathcal{C}(J_k, H) \text{ and there exists } x(t_k^+)$$

and $x(t_k^-)$ with $x(t_k), x_0 = \phi \in \mathcal{B}_h, k = 1,2, \dots m\}$ where x_k is the restriction of x to $J_k = (t_k, t_{k+1}], k = 0,1, \dots m$ and $\mathcal{C}((-\infty, b], H)$ denote the space of all continuous H-Valued stochastic process $\{\xi(t), t \in (-\infty, b]\}$.
Set $||\,.\,||_b$ be a seminorm defined by

$$||\,x||_b = ||\,\phi||_{\mathcal{B}_h} + sup_{s\in[0,b]}(E\|x(s)\|^2)^{\frac{1}{2}}, x \in \mathcal{B}_h'.$$

Lemma 2.2: Assume that $x \in \mathcal{B}_h'$, then for all $t \in J$, $x_t \in \mathcal{B}_h$. Moreover

$$l\ (E\|\phi(\theta)\|^2)^{\frac{1}{2}} \leq l\ sup_{s\in[0,t]}(E\|x(s)\|^2)^{\frac{1}{2}} + ||\,\phi||_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^0 h(s)ds < \infty$.

Definition 2.3: A continuous H- valued process x is said to be a mild solution of (2.1)-(2.3) if
(i) $x(t)$ is $\mathcal{F}_t$- adapted and $\{x_t: t \in [0, b]\}$ is $\mathcal{B}_h$-valued.
(ii) for each $t \in J$, $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) = {} & L^{-1}T(t)[L\phi(0) - g(0,\phi)] + L^{-1}g(t, x_t) + \int_0^t L^{-1}AL^{-1}T(t-s)g(s, x_s)ds \\ & + \int_0^t L^{-1}T(t-s)f(s, x_s)ds + \int_0^t L^{-1}T(t-s)Bu(s)ds \\ & + \int_0^t L^{-1}T(t-s)\sigma(s, x_s)dw(s) + \sum_{0<t_k<t} L^{-1}T(t-t_k)I_k(x_{t_k}), \quad t \in J \end{aligned}$$

(iii) $x(t) = \phi(t)$ on $(-\infty, 0]$ satisfying $\|\phi\|_{\mathcal{B}_h}^2 < \infty$.

It is convienent at this point to introduce the controllability and relevant operators associated with basic assumptions

$$\gamma_0^b = \int_0^b L^{-1}T(b-s)BB^*L^{-1}T^*(b-s)ds: H \to H,$$

$$R(\alpha, \gamma_0^b) = (\alpha I + \gamma_0^b)^{-1}: H \to H$$

where B^* denotes the adjoint of B and $T^*(t)$ is the adjoint of $T(t)$. It is straight forward that the operator γ_0^b is a linear bounded operator.

(H_0) $\alpha R(\alpha, \gamma_0^b) \to 0$ as $\alpha \to 0^+$ in the strong operator topology.

By Hypothesis **(H_0)** holds if and only if the linear system

$$\frac{d[Lx(t)]}{dt} = Ax(t) + Bu(t)\,, t \in [0, b] \tag{2.5}$$

$$x(0) = \phi \in \mathcal{B}_h \tag{2.6}$$

is approximately controllable on [0,b].

Lemma 2.4 (Krasnoselskii's fixed point theorem): Let N be a Hilbert space, let $\widehat{N}$ be a bounded, closed and convex subset of N and let F_1, F_2 be maps of $\widehat{N}$ into Nsuch that $F_1x + F_2y \in \widehat{N}$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1x + F_2y = x$ has a solution on $\widehat{N}$.

3. APPROXIMATE CONTROLLABILITY RESULTS

In this section, we shall formulate and prove sufficient conditions for the approximate controllability results for impulsive neutral stochastic differential equation of Sobolev type with unbounded delay of the form (2.1)-(2.3) by using Krasnoselskii-Schaefer-type fixed point theorem. First we prove the existence of solutions for the control system and then show that under certain assumptions, the approximate controllability of the stochastic control system (2.1)-(2.3) is implied by the approximate controllability of the associated linear part.

Definition 3.1: Let $x_b(\phi, u)$be the state value of (2.1)-(2.3) at the terminal time b corresponding to the control u and the initial value ϕ. Introduce the set

$$\mathcal{R}(b, \phi) = \{x_b(\phi; u)(0) : u(.) \in \mathcal{L}(J, U)\},$$

which is called the reachable set of (2.1)-(2.3) at the time b and its closure in H is denoted by $\overline{\mathcal{R}(b, \phi)}$. The system (2.1)-(2.3) is said to be approximately controllable on J if $\overline{\mathcal{R}(b, \phi)} = H$.

In order to establish the result, we need the following hypotheses:

(H_1) $T(t), t > 0$ is compact.

(H_2) The function $f, g : J \times \mathcal{B}_h \to H$ are continuous and there exists two positive constants M_1 and M_2 such that the function satisfies that

$$E\|f(t,x) - f(t,y)\|^2 \le M_1 \|x - y\|_{\mathcal{B}_h}^2$$
$$E\|f(t,x)\|^2 \le M_1 (1 + \|x\|_{\mathcal{B}_h}^2)$$

and

$$E\|AL^{-1}g(t,x) - AL^{-1}g(t,y)\|^2 \le M_2 \|x - y\|_{\mathcal{B}_h}^2$$
$$E\|g(t,x)\|^2 \le M_2 (1 + \|x\|_{\mathcal{B}_h}^2)$$

for every $x, y \in \mathcal{B}_h, t \in J$.

(H_3) The function σ is continuous and there exists two positive constants M_3 such that the function satisfies that

$$E\|\sigma(t, x_t) - \sigma(t, y_t)\|_{\mathcal{L}_2^0}^2 \le M_3 \|x - y\|_{\mathcal{B}_h}^2$$
$$E\|\sigma(t, x_t)\|_{\mathcal{L}_2^0}^2 \le M_3 (1 + \|x\|_{\mathcal{B}_h}^2)$$

(H_4) $I_k \in \mathcal{C}(H, H)$ and there exist and continuous nondecreasing functions $M_k : [0, +\infty) \to (0, +\infty)$ such that, for each $x \in H$,

$$E\|I_k(x)\|^2 \le M_k(E\|x\|^2) \quad \text{and}$$
$$\lim_{r\to\infty} inf \frac{M_k(r)}{r} = \beta_k < \infty, k = 1,2, \dots m.$$

Lemma 3.2: For any $\bar{x}_b \in \mathcal{L}^2(\mathcal{F}_b, H)$, there exists $\bar{\phi} \in \mathcal{L}_2^{\mathcal{F}}(\Omega, \mathcal{L}^2(J, \mathcal{L}(K, H)))$ such that $\bar{x}_b = E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s)$.

Now for any $\alpha > 0$, $\bar{x}_b \in \mathcal{L}^2(\mathcal{F}_b, H)$, we define the control function

$$\begin{aligned} u_\alpha(t,x) = {} & B^* L^{-1} T^*(b - s)\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)[L\phi(0) - g(0, \phi)] \\ & - L^{-1}g(s, x_s) - \int_0^t L^{-1}AL^{-1}T(b - s)g(s, x_s)ds - \int_0^t L^{-1}T(t - s)f\big(s, y_s + \hat{\phi}_s\big)ds \\ & + \int_0^t L^{-1}T(t - s)\, \sigma\big(s, y_s + \hat{\phi}_s\big)dw(s)ds + \sum_{0<t_k<t} L^{-1}T(t - t_k)I_k\big(x_{t_k}\big)\}(s)ds. \end{aligned}$$

Theorem 3.3: Suppose that the hypotheses $(H_1)-(H_4)$ are satisfied, then the system (2.1)-(2.3) has a mild solution on J provided that

$$4M_L^2 l^2[M_2 + M^2[b^2[M_1 + M_2 + M_3] + m\sum_{k=1}^{m}\beta_k]\left[7+49\left(\frac{M^2M_L^2M_B^2}{\alpha}\right)^2\right] < 1 \tag{3.3}$$

and where $\|B\| = M_B$.

Proof: For any $\alpha > 0$, we consider the operator $\Phi: \mathcal{B}_h{}' \to 2^{\mathcal{B}_h{}'}$defined by

$$\Phi(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ L^{-1}T(t)[L\phi(0) - g(0,\phi)] + L^{-1}g(t,x_t) + \int_0^t L^{-1}AL^{-1}T(b-s)g(s,x_s)ds & \\ \quad + \int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds \ + \int_0^t L^{-1}T(t-s)Bu_\alpha(s,y_s+\hat{\phi}_s)ds & \\ + \int_0^t L^{-1}T(t-s)\sigma(s,y_s+\hat{\phi}_s)dw(s)\ ds + \sum_{0<t_k<t} L^{-1}T(t-t_k)I_k(x_{t_k}), & t \in J \end{cases}$$

We shall show that the operator Φ has a fixed point, which is then a solution of (2.1) -(2.3). Clearly $x_1 = x(b) \in (\Phi x)(b)$, which means that $u_\alpha(t,x)$ steers system (2.1) - (2.3) from x_0 to x_b in finite time b. For $\phi \in \mathcal{B}_h$, we define $\hat{\phi}$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ L^{-1}T(t)L\phi(0) & t \in J \end{cases}$$

then $\hat{\phi} \in \mathcal{B}_h^{'}$. Let $x(t) = y(t) + \hat{\phi}(t)$, $-\infty < t \leq b$. It is easy to see that y satisfies $y_0 = 0$ and

$$\begin{aligned} y(t) = &-L^{-1}T(t)g(0,\phi) + L^{-1}g(t,y_t+\hat{\phi}_t) + \int_0^t L^{-1}AL^{-1}T(b-s)g(s,y_s+\hat{\phi}_s)ds \\ &+ \int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds + \int_0^t L^{-1}T(t-s)Bu_\alpha(s,y_s+\hat{\phi}_s)ds \\ &+ \int_0^t L^{-1}T(t-s)\sigma(s,y_s+\hat{\phi}_s)dw(s)\ ds + \sum_{0<t_k<t} L^{-1}T(t-t_k)I_k(x_{t_k}),\ t \in J \end{aligned}$$

if and only if x satisfies

$$\begin{aligned} x(t) = &L^{-1}T(t)[L\phi(0) - g(0,\phi)] + L^{-1}g(t,y_t+\hat{\phi}_t) + \int_0^t L^{-1}AL^{-1}T(b-s)g(s,y_s+\hat{\phi}_s)ds \\ &+ \int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds + \int_0^t L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)\ R(\alpha,\gamma_0^b) \\ &\times \{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)[L\phi(0) - g(0,\phi)] \\ &- \int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds \ + \int_0^t L^{-1}T(t-s)\sigma(s,y_s+\hat{\phi}_s)dw(s)\ ds \\ &+ \sum_{0<t_k<t} L^{-1}T(t-t_k)I_k(x_{t_k})\} + \int_0^t L^{-1}T(t-s)\sigma(s,y_s+\hat{\phi}_s)dw(s)ds \\ &+ \sum_{0<t_k<t} L^{-1}T(t-t_k)I_k(x_{t_k}),\ t \in J \end{aligned}$$

and $x(t) = \phi(t), t \in (-\infty, 0]$.

Let $\mathcal{B}_h^{''} = \{y \in \mathcal{B}_h^{'}: y_0 = 0 \in \mathcal{B}_h\}$. For any $y \in \mathcal{B}_h^{''}$, we have

$$\begin{aligned} \|y\|_b &= \|y_0\|_{\mathcal{B}_h} + sup_{s\in[0,b]}\{(E|y(s)|^2): 0 \leq s \leq b\} \\ &= sup_{s\in[0,b]}\{(E|y(s)|^2): 0 \leq s \leq b\}. \end{aligned}$$

thus $(\mathcal{B}_h^{''}, \|\cdot\|_b)$ is a Banach space. Set $\mathfrak{B}_r = \{y \in \mathcal{B}_h^{''}: \|y\|_b \leq r\}$ for some $r > 0$, then $\mathfrak{B}_r \subset \mathcal{B}_h^{''}$ is a uniformly bounded and for $y \in \mathfrak{B}_r$,from lemma 2.2 we have

$$\begin{aligned} \|y_t + \hat{\phi}_t\|_{\mathcal{B}_h}^2 &\leq 2\left(\|y_t\|_{\mathcal{B}_h}^2 + \|\hat{\phi}_t\|_{\mathcal{B}_h}^2\right) \\ &\leq 4\left(l^2\ sup_{s\in[0,t]}(E\|y(s)\|^2) + \|y_0\|_{\mathcal{B}_h}^2 + l^2\ sup_{s\in[0,t]}\left(E\|\hat{\phi}(s)\|^2\right) + \|\hat{\phi}_0\|_{\mathcal{B}_h}^2\right) \\ &\leq 4l^2(r + M^2E\|\phi(0)\|^2) + 4\|\phi\|_{\mathcal{B}_h}^2 \\ &\leq r' \end{aligned}$$

In view of Lemma 2.2 for each $t \in J$.

$$\|y(t) + \hat{\phi}(t)\| \leq l^{-1}\|y_t + \hat{\phi}_t\|_{\mathcal{B}_h} \leq l^{-1}r'$$

Therefore

$$\begin{aligned} \|I_k(y(t_k^-)) + \hat{\phi}(t_k^-)\| &\leq M_k(\|y(t_k^-) + \hat{\phi}(t_k^-)\|) \\ &\leq M_k(sup_{t\in J}|y(t) + \hat{\phi}(t)|) \\ &\leq M_k(l^{-1}r'), k = 1,2,\ldots,m. \end{aligned}$$

For the sake of convience, we subdivide the proof into several steps.

Step-1: We show that there exist some $r > 0$ such that $\Phi(\mathfrak{B}_r) \subset \mathfrak{B}_r$. If it is not true, then, for every positive number, there exists a function $y^r \in \mathfrak{B}_r$, but $\Phi \in \mathfrak{B}_r$, that is, $E\|(\Phi y^r))(t)\|^2 > r$ for some $t \in (-\infty, b]$, t may depending upon r . However, on the other hand, we have

$$
\begin{aligned}
r < E\|(\Phi y^r)(t)\|^2 &\\
\le 7\{&E\|L^{-1}T(t)g(0,\phi)\|^2 + E\left\|L^{-1}g(t, y_t^r+\hat{\phi}_t)\right\|^2 \\
&+E\|\int_0^b L^{-1}AL^{-1}T(b-s)g(s,y_t^r+\hat{\phi}_s)ds\|^2 + E\|\int_0^t L^{-1}T(t-s)f(s,y_t^r+\hat{\phi}_s)ds\|^2 \\
&+E\|\int_0^t L^{-1}T(t-s)Bu_\alpha(s,y_t^r+\hat{\phi}_s)ds\|^2 + E\|\int_0^t L^{-1}T(t-s)\sigma(s,y_t^r+\hat{\phi}_s)dw(s)ds\|^2 \\
&+E\|\sum_{0<t_k<t} L^{-1}T(t-t_k)I_k\left(y(t_k^-)+\hat{\phi}(t_k^-)\right)\|^2\}
\end{aligned}
$$

$$
\begin{aligned}
\le\ & 7M_L^2M^2E\|g(0,\phi)\|^2 + 7M_L^2M_2(1+\|\phi\|^2) + 7b^2M_L^2M_2(1+\|\phi\|^2) \\
&+7b^2M_L^2M_1(1+\|\phi\|^2) + 7b^2M_L^2M_3(1+\|\phi\|^2) + 7\,m\,M_L^2M^2\sum_{k=1}^m M_k(r') \\
&+49\left(\frac{M^2M_L^2M_B^2}{\alpha}\right)^2\{2E\|\bar{x}_b\|^2 + 2\int_0^b E\|\phi(s)\|dw(s) + M^2M_L^2[\phi(0)-g(0,\phi)] \\
&+M_L^2M_2(1+\|\phi\|^2) + b^2M_L^2M_2(1+\|\phi\|^2) + b^2M_L^2M_1(1+\|\phi\|^2) \\
&+b^2M_L^2M_3(1+\|\phi\|^2) + m\,M_L^2M^2\sum_{k=1}^m M_k(l^{-1}r')\}
\end{aligned}
$$

$$
\begin{aligned}
\le\ & 7M_L^2M^2E\|g(0,\phi)\|^2 + 7M_L^2M_2(1+r') + 7b^2M_L^2M_2(1+r') \\
&+7b^2M_L^2M_1(1+r') + 7b^2M_L^2M_3(1+r') + 7\,m\,M_L^2M^2\sum_{k=1}^m M_k(r') \\
&+49\left(\frac{M^2M_L^2M_B^2}{\alpha}\right)^2\{2E\|\bar{x}_b\|^2 + 2\int_0^b E\|\phi(s)\|dw(s) + M^2M_L^2[\phi(0)-g(0,\phi)] \\
&+M_L^2M_2(1+r') + b^2M_L^2M_2(1+r') + b^2M_L^2M_1(1+r') \\
&+b^2M_L^2M_3(1+r') + m\,M_L^2M^2\sum_{k=1}^m M_k(l^{-1}r')\}
\end{aligned}
$$

Dividing both sides of the above inequality by r and taking $r \to \infty$ we have

$$\lim_{r\to\infty} inf \sum_{k=1}^m \frac{M_k(l^{-1}r)}{r} = \lim_{r\to\infty} inf \sum_{k=1}^m \frac{M_k(l^{-1}r)}{l^{-1}r}.\frac{l^{-1}r}{r} = \sum_{k=1}^m \beta_k.$$

We obtain

$$4M_L^2l^2[M_2 + M^2[b^2[M_1+M_2+M_3]] + m\sum_{k=1}^m \beta_k]\left[7+49\left(\frac{M^2M_L^2M_B^2}{\alpha}\right)^2\right] \ge 1.$$

which is a contradiction to our assumption. Thus, for each $\alpha > 0$, there exists some positive number $r > 0$ such that $\Phi(\mathfrak{B}_r) \subset \mathfrak{B}_r$.

Next, we show that the operator Φ is condensing, for convenience, we decompose Φ as $\Phi = \Phi_1 + \Phi_2$, where

$$(\Phi_1 y)(t) = L^{-1}g(t, y_t+\hat{\phi}_t) + \int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds + \int_0^t L^{-1}T(t-s)\sigma(s,y_s+\hat{\phi}_s)dw(s)\,ds$$

$$
\begin{aligned}
(\Phi_2 y)(t) = &L^{-1}T(t)[L\phi(0) - g(0,\phi)] + \int_0^t L^{-1}AL^{-1}T(b-s)g(s,y_s+\hat{\phi}_s)ds \\
&+\int_0^t L^{-1}T(t-s)Bu_\alpha(s,y_s+\hat{\phi}_s)ds + \sum_{0<t_k<t} L^{-1}T(t-t_k)I_k(x_{t_k})
\end{aligned}
$$

Step-2: We prove that Φ_1 is a contraction $\mathfrak{B}_r$. Let $t \in J$ and $y_1, y_2 \in \mathfrak{B}_r$, we have

$$
\begin{aligned}
E\|\Phi_1 y_1(t) - \Phi_1 y_2(t)\| \le\ & 3E\left\|L^{-1}[g(t,y_{1,t}+\hat{\phi}_t) - g(t,y_{2,t}+\hat{\phi}_t)\right\|^2 \\
&+3E\left\|\int_0^t L^{-1}T(t-s)[f(s,y_{1,s}+\hat{\phi}_s) - f(s,y_{2,s}+\hat{\phi}_s)ds\right\|^2 \\
&+3E\left\|\int_0^t L^{-1}T(t-s)[\sigma(s,y_{1,s}+\hat{\phi}_s) - \sigma(s,y_{2,s}+\hat{\phi}_s)ds\right\|^2 \\
\le\ & 3M^2M_L^2\left\|y_{1,t}-y_{2,t}\right\|_{\mathcal{B}_h}^2 + 3M^2M_1M_L^2\int_0^t\left\|y_{1,s}-y_{2,s}\right\|_{\mathcal{B}_h}^2 ds \\
&+3M^2M_3M_L^2\int_0^t\left\|y_{1,s}-y_{2,s}\right\|_{\mathcal{B}_h}^2 dw(s) \\
\le\ & L\,sup_{t\in J}E\|y_1(s)-y_2(s)\|^2.
\end{aligned}
$$

where $L = 3l^2M_L^2[M_2 + M^2M_1 + M^2M_3 < 1$. Hence Φ_1 is a contraction.

Step-3: Φ_2 maps bounded sets into bounded sets in $\mathfrak{B}_r$.

$$
\begin{aligned}
E\|\Phi_2 y(t)\|^2 \le\ & 4E\|L^{-1}T(t)[-g(0,\phi)]\|^2 \\
&+4E\left\|\int_0^b L^{-1}AL^{-1}T(b-s)g(s,y_s+\hat{\phi}_s)ds\right\|^2 \\
&+4E\left\|\int_0^t L^{-1}T(t-s)Bu_\alpha(s,y_s+\hat{\phi}_s)ds\right\|^2 \\
&+4E\left\|\sum_{0<t_k<t} L^{-1}T(t-t_k)I_k(x_{t_k})\right\|^2
\end{aligned}
$$

$$\leq 4M^2M_L^2E\|[-g(0,\phi)]\|^2+4M^2M_L^2M_2(1+r')$$
$$+32\,\frac{M^2M_L^2}{\alpha^2}b\{2E\|\bar{x}_b\|^2+2\int_0^b E\|\phi(s)\|dw(s)+M^2M_L^2[\phi(0)-g(0,\phi)]$$
$$+M_L^2M_2(1+\|\phi\|^2)+b^2M_L^2M_2(1+\|\phi\|^2)+b^2M_L^2M_1(1+\|\phi\|^2)$$
$$+b^2M_L^2M_3(1+\|\phi\|^2)+m\,M_L^2M^2\sum_{k=1}^m M_k(l^{-1}r')\}$$
$$+4\,m\,M_L^2M^2\sum_{k=1}^m M_k(l^{-1}r')$$

$$=\Lambda.$$

Therefore, for each $y\in\mathfrak{B}_r$, we get $E\|\Phi_2\mathrm{y}(\mathrm{t})\|^2=\Lambda$.

Step-4: The map $\Phi(\mathfrak{B}_r)$ is equicontinuous. Indeed $\epsilon>0$ be small, $0<\tau_1<\tau_2\leq b$. for each $y\in\mathfrak{B}_r$ and let $0<\tau_1<\tau_2\leq b$ and $\tau_1,\tau_2\in J\{\tau_1,\tau_2,\ldots\ldots\tau_n\}$. Then we have

$$E\|\Phi_2\mathrm{y}(\tau_2)-\Phi_2\mathrm{y}(\tau_1)\|^2=9E\|L^{-1}[T(\tau_2)-T(\tau_1)][-g(0,\phi)]\|^2$$
$$+9E\left\|\int_{\tau_1}^{\tau_2}L^{-1}T(\tau_2-s)AL^{-1}g(s,y_s+\hat{\phi}_s)ds\right\|^2$$
$$+9E\left\|\int_{\tau_1-\epsilon}^{\tau_1}L^{-1}[T(\tau_2-s)-T(\tau_1-s)]AL^{-1}g(s,y_s+\hat{\phi}_s)ds\right\|^2$$
$$+9E\left\|\int_0^{\tau_1-\epsilon}L^{-1}[T(\tau_2-s)-T(\tau_1-s)]AL^{-1}g(s,y_s+\hat{\phi}_s)ds\right\|^2$$
$$+9E\left\|\int_0^{\tau_1-\epsilon}L^{-1}[T(\tau_2-s)-T(\tau_1-s)]Bu_\alpha(s,y_s+\hat{\phi}_s)ds\right\|^2$$
$$+9E\left\|\int_{\tau_1-\epsilon}^{\tau_1}L^{-1}[T(\tau_2-s)-T(\tau_1-s)]Bu_\alpha(s,y_s+\hat{\phi}_s)ds\right\|^2$$
$$+9E\left\|\int_{\tau_1}^{\tau_2}L^{-1}T(\tau_2-s)Bu_\alpha(s,y_s+\hat{\phi}_s)ds\right\|^2$$
$$+9E\left\|\sum_{0<\tau_k<t}L^{-1}[T(\tau_2-t_k)-T(\tau_1-t_k)]I_k\left(y(t_k^-)+\hat{\phi}(t_k^-)\right)\right\|^2$$
$$+9E\left\|\sum_{0<\tau_k<t}L^{-1}T(\tau_2-t_k)I_k\left(y(t_k^-)+\hat{\phi}(t_k^-)\right)\right\|^2$$
$$\leq 9M_L^2[T(\tau_2)-T(\tau_1)]E\|[-g(0,\phi)]\|^2$$
$$+9M_L^4\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|^2E\|Ag(s,y_s+\hat{\phi}_s)\|^2ds$$
$$+9M_L^4\int_{\tau_1-\epsilon}^{\tau_1}\|[T(\tau_2-s)-T(\tau_1-s)]\|^2E\|Ag(s,y_s+\hat{\phi}_s)\|^2ds$$
$$+9M_L^4\int_0^{\tau_1-\epsilon}\|[T(\tau_2-s)-T(\tau_1-s)]\|^2E\|Ag(s,y_s+\hat{\phi}_s)\|^2ds$$
$$+9M_L^2\int_0^{\tau_1-\epsilon}\|[T(\tau_2-s)-T(\tau_1-s)]\|^2E\|Bu_\alpha(s,y_s+\hat{\phi}_s)\|^2ds$$
$$+9M_L^2\int_{\tau_1-\epsilon}^{\tau_1}\|[T(\tau_2-s)-T(\tau_1-s)]\|^2E\|Bu_\alpha(s,y_s+\hat{\phi}_s)\|^2ds$$
$$+9M_L^2\int_{\tau_1}^{\tau_2}\|T(\tau_2-s)\|^2E\|Bu_\alpha(s,y_s+\hat{\phi}_s)\|^2ds$$
$$+9M_L^2\|[T(\tau_2-t_k)-T(\tau_1-t_k)]\|^2E\|\sum_{0<\tau_k<t}I_k\left(y(t_k^-)+\hat{\phi}(t_k^-)\right)\|^2$$
$$+9M_L^2\|T(\tau_2-t_k)\|^2E\|\sum_{0<\tau_k<t}I_k\left(y(t_k^-)+\hat{\phi}(t_k^-)\right)\|^2$$

Therefore for ϵ sufficiently small, we can verify that the right-hand side of the above inequality tends to zero as $\tau_2\to\tau_1$. On the otherhand, the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology. Thus Φ_2 maps $\mathfrak{B}_r$ into an equicontinuous family of functions.

Step-5: The set $V(t)=\{\Phi_2\mathrm{y}(\mathrm{t}),\Phi_2\mathrm{y}\in\mathfrak{B}_r\}$ is relatively compact in H. Let $t\in[0,b]$ be fixed and ϵ a real number satisfying $0<\epsilon<t$. For $x\in\mathfrak{B}_r$, we define

$$\Phi_2\mathrm{y}(\mathrm{t})=\int_0^{\mathrm{t}-\epsilon}L^{-1}T(t)L[-g(0,\phi)]+\int_0^{\mathrm{t}-\epsilon}L^{-1}AL^{-1}T(t-s)[g(s,y_s+\hat{\phi}_s)]ds$$
$$+\int_0^{t-\epsilon}L^{-1}T(t-s)Bu_\alpha(s,y_s+\hat{\phi}_s)ds+\sum_{0<t_k<t}L^{-1}T(t-t_k)I_k\left(y(t_k^-)+\hat{\phi}(t_k^-)\right)$$
$$=T(\epsilon)\int_0^{\mathrm{t}-\epsilon}L^{-1}T(t-\epsilon)L[-g(0,\phi)]$$
$$+T(\epsilon)\int_0^{\mathrm{t}-\epsilon}L^{-1}AL^{-1}T(t-s-\epsilon)[g(s,y_s+\hat{\phi}_s)]ds$$
$$+T(\epsilon)\int_0^{t-\epsilon}L^{-1}T(t-s-\epsilon)Bu_\alpha(s,y_s+\hat{\phi}_s)ds$$
$$+T(\epsilon)\sum_{0<t_k<t}L^{-1}T(t-t_k-\epsilon)I_k\left(y(t_k^-)+\hat{\phi}(t_k^-)\right)$$

Since $T(t)$ is a compact operator, the set $V_\epsilon(t)=\{\Phi_{2,\epsilon}\mathrm{y}(\mathrm{t}),\Phi_2\mathrm{y}\in\mathfrak{B}_r$ is relatively compact in H for each ϵ, $0<\epsilon<t$. Moreover, for , we have $\Phi_2\mathrm{y}\in\mathfrak{B}_r$, we can easily prove that $\Phi_{2,\epsilon}\mathrm{y}(\mathrm{t})$ is convergent to $\Phi_2\mathrm{y}(\mathrm{t})$ in $\mathfrak{B}_r$ as $\epsilon\to0^+$, hence the set $V(t)=\{\Phi_{2,\epsilon}\mathrm{y}(\mathrm{t}),\Phi_2\mathrm{y}\in\mathfrak{B}_r\}$ is also relatively compact in $\mathfrak{B}_r$. Thus, by Arzela-Ascoli theorem Φ_2 is completely continuous. Consequently, Φ has a fixed point, which is a mild solution of (2.1) - (2.3).

Theorem 3.4: Assume that $(H_1)-(H_4)$ are satisfied and the conditions of Theorem 3.3 holds. Further, if the functions f and g are uniformly bounded and $T(t)$ is compact, then the system (2.1) - (2.3) is approximately controllable on J.

Proof: Let $\hat{x}^\alpha(\cdot)$ be a solution of (2.1)-(2.3), we can easily get that

$$\begin{aligned}\hat{x}^\alpha(b) = \bar{x}_b - R(\alpha,\gamma_0^b)\times\Big\{&E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - L^{-1}T(t)[L\phi(0)-g(0,\phi)]\\ &+L^{-1}g\big(t,y_t+\hat{\phi}_t\big)+\int_0^t L^{-1}AL^{-1}T(b-s)g\big(s,y_s+\hat{\phi}_s\big)ds\\ &+\int_0^t L^{-1}T(t-s)f\big(s,y_s+\hat{\phi}_s\big)ds+\int_0^t L^{-1}T(t-s)\,\sigma\big(s,y_s+\hat{\phi}_s\big)dw(s)\,ds\\ &+\sum_{0<t_k<b}L^{-1}\,T(t-t_k)I_k\Big(y(t_k^-)+\hat{\phi}(t_k^-)\Big)\Big\}\end{aligned}$$

Moreover by assumption that f and σ are uniformly bounded on J, hence there is a subsequence still denoted by $f(s,x_s^\alpha)$ and $\sigma(s,x_s^\alpha)$ which converges to say $f(s)$ in H and $\sigma(s)$ in $\mathcal{L}(U,H)$.

$$\begin{aligned}E\|\hat{x}^\alpha(b)-\bar{x}_b\|^2 = 7E\|R(\alpha,\gamma_0^b)\times\{&E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s) - L^{-1}T(t)[L\phi(0)-g(0,\phi)]\\ &+L^{-1}g\big(t,y_t+\hat{\phi}_t\big)+\int_0^t L^{-1}AL^{-1}T(b-s)g\big(s,y_s+\hat{\phi}_s\big)ds\\ &+\int_0^t L^{-1}T(t-s)f\big(s,y_s+\hat{\phi}_s\big)ds+\int_0^t L^{-1}T(t-s)\,\sigma\big(s,y_s+\hat{\phi}_s\big)dw(s)\,ds\\ &+\sum_{0<t_k<t}L^{-1}\,T(t_2-t_k)I_k\Big(y(t_k^-)+\hat{\phi}(t_k^-)\Big)\}\|^2\end{aligned}$$

On the otherhand, by the assumption (H_0) the operator $\alpha(\alpha I+\gamma_0^b)^{-1}\to 0$ strongly as $\alpha\to 0^+$ and moreover $\|\alpha(\alpha I+\gamma_0^b)^{-1}\|\le 1$. It follows from Lebesgue dominated convergence theorem and the compactnesss of $T(t)$ that $E\|\hat{x}^\alpha(b)-\bar{x}_b\|^2\to 0$ as $\alpha\to 0^+$. This proves the approximate controllability of the differential equation (2.1)-(2.3).

Remark 3.5: There exists an extensive literature of differential equations with nonlocal conditions. Motivated by physical applications, Byszewski [8] studied a nonlocal Cauchy problem modeled in the form

$$\dot{x}(t) = Ax(t)+f\big(t,x(t)\big),\, t\in(\sigma,T]$$
$$x(0)=x_0+q(t_1,t_2,\ldots\ldots,t_n,u(\cdot))\in X$$

where A is the infinitesimal generator of a C_0 semigroup of linear operators on H, $f:[\sigma,T]\times X\to X$, $q:[\sigma,T]^n\times X\to X$ are appropriate functions and the symbol $q(t_1,t_2,\ldots\ldots,t_n,u(\cdot))$ is used in the sense that " $\cdot$ " can be substitute only for the points t_i, for instance

$$q\big(t_1,t_2,\ldots\ldots,t_n,u(\cdot)\big)=\sum_{k=1}^n \alpha_i u(t_i)$$

Byszewski & Akca [9], studied the existence, uniqueness and continuous dependence on initial data of solutions to the nonlocal Cauchy problem for functional differential equations with delay. Hernández [21] studied the existence results for nonlocal neutral functional differential equations with infinite delay modeled in the form

$$\frac{d}{dt}[x'(t)+F(t,x_t)]=Ax(t)+G(t,x_t),\qquad t\in[0,T]$$
$$x_\sigma=\psi+q(u_{t_1},u_{t_2},\ldots\ldots,u_{t_n},u(\cdot))\in\Omega$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, on a Banach space X; the histories $x_t(\theta)=x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, $\Omega\subset\mathcal{B}$ is open; $0\le\sigma<T$; $\sigma<t_0<t_1\ldots\ldots<t_n\le T$ and $q:\mathcal{B}^n\to\mathcal{B}$, $F,G:[\sigma,T]\times\Omega\to X$ are appropriate continuous functions. Since the appearance of these papers, several authors studied the issue of existence and uniqueness results for various types of nonlocal differential equations with control or without control, see [27, 34, 35, 52].

Inspired by the Remark 3.5, we establish a set of sufficient conditions for the approximate controllability of stochastic functional impulsive neutral differential equations of Sobolev-type with nonlocal conditions of the form

$$\frac{d}{dt}[Lx(t)-g(t,x_t)]=Ax(t)+Bu(t)+f(t,x_t)+\sigma(s,x_s)dw(s),\quad t\in J:=[0,b], t\neq t_k, k=1,2,..m. \quad (3.4)$$

$$x(t)=\phi(t)+q(x_{t_1},x_{t_2},\ldots\ldots,x_{t_n})\in\mathcal{B}_h,\ t\in(-\infty,0] \quad (3.5)$$

$$\Delta x|_{t=t_k}=I_k\big(x(t_k^-)\big), k=1,2,\ldots m. \quad (3.6)$$

where $0<t_1<t_2\ldots\ldots<t_n\le b$, $q:\mathcal{B}_h^n\to\mathcal{B}_h$ is a given function which satisfies the given condition

$(\boldsymbol{H_5})$ $q:\mathcal{B}_h^n\to\mathcal{B}_h$ is continuous and exist positive constants $L_i(q)$ such that

$$E\|q(\psi_1,\psi_2,\ldots.,\psi_n)-q(\Psi_1,\Psi_2,\ldots.\Psi_n)\|^2\le\sum_{k=1}^n L_i(q)\|\psi-\Psi\|_{\mathcal{B}_h}.$$

for every $\psi,\Psi\in\mathcal{B}_h$ and assume that $N_q=\sup\{\|q(\psi_{t_1},\psi_{t_2},\ldots.,\psi_{t_n})\|:\psi\in\mathcal{B}_h\}$.

Definition 3.6: A continuous H- valued process x is said to be a mild solution of (3.4)-(3.6) if

(i) $x(t)$ is $\mathcal{F}_t$- adapted and $\{x_t : t \in [0,b]\}$ is $\mathcal{B}_h$-valued.

(ii) for each $t \in J$, $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) = \; & L^{-1}T(t)[L\phi(0) + q(x_{t_1}, x_{t_2}, \dots\dots, x_{t_n}) - g(0,\phi)] + L^{-1}g(t,x_t) \\ & + \int_0^t L^{-1}AL^{-1}T(t-s)g(s,x_s)ds + \int_0^t L^{-1}T(t-s)f(s,x_s)ds \\ & + \int_0^t L^{-1}T(t-s)Bu_\alpha(s)ds + \int_0^t L^{-1}T(t-s)\sigma(s,x_s)dw(s)ds \\ & + \sum_{0<t_k<t} L^{-1}T(t-t_k)I_k(x_{t_k}),\; t \in J \end{aligned}$$

(iii) $x(t) = \phi(t) + q(x_{t_1}, x_{t_2}, \dots\dots, x_{t_n})$ on $(-\infty, 0]$ satisfying $\|\phi\|^2_{\mathcal{B}_h} < \infty$.

Theorem 3.7: Suppose that the hypotheses $(H_1)-(H_5)$ are satisfied, then the system (3.4)-(3.6) has a mild solution on J provided that

$$4M_L^2 l^2[M_2 + M^2[b^2[M_1 + M_2 + M_3] + m\sum_{k=1}^m \beta_k]\left[7+49\left(\frac{M^2M_L^2M_B^2}{\alpha}\right)^2\right] < 1.$$

Proof: The proof is similar to the proof of Theorem 3.3 and Theorem 3.4, we can omit the proof.

4. AN APPLICATION

Consider a control system of stochastic neutral impulsive differential equation with unbounded delay of the form

$$\frac{\partial}{\partial t}[z(t,x) - z_{xx}(t,x)] = z_{xx}(t,x) + F(t, z(t-r), x)ds + \mu(t,x) + G(t, y(t-r), x)dw(t),\; t \in [0,b], r > 0, x \in [0,\pi] \quad (4.1)$$

$$z(t,0) = z(t,\pi) = 0, 0 \le t \le b \quad (4.2)$$

$$z(t,x) = \phi(t,x), 0 \le x \le \pi, -\infty \le t \le 0 \quad (4.3)$$

$$z(t_k^+, x) - z(t_k^-, x) = I_k(z(t_k^-, x)), k = 1,2,..,m \quad (4.4)$$

where $w(t)$ denotes a standard cylindrical wiener process in H defined on a stochastic process $(\omega, \mathcal{F})$ and $H = K = \mathcal{L}^2([0,\pi])$. Define the operators $A: D(A) \subset H \to H$ and $L: D(L) \subset H \to H$ by $Ay = -y''$ and $Ly = y - y'$, where each domain $D(A)$ and $D(L)$ is given by $\{y \in H, y, y'$ are absolutely continuous $y'' \in H, y(0) = y(\pi) = 0\}$.

Further A and L can be witten as $y = \sum_{n=1}^\infty n^2 \langle y, z_n\rangle z_n, y \in D(A)$, $Ly = \sum_{n=1}^\infty (1+n^2)\langle y, z_n\rangle z_n, y \in D(L)$, where

$z_n(x) = \sqrt{\frac{2}{\pi}} \sin nx, n = 1,2,3, \dots$ is the orthogonal set of vectors of A. Also for $z \in H$, we have

$$L^{-1}z = \sum_{n=1}^\infty \frac{1}{1+n^2}\langle z, z_n\rangle z_n$$

and

$$AL^{-1}z = \sum_{n=1}^\infty \frac{n^2}{1+n^2}\langle z, z_n\rangle z_n$$

and

$$T(t)z = \sum_{n=1}^\infty exp\frac{n^2 t}{1+n^2}\langle z, z_n\rangle z_n$$

Further, we consider the phase space $\mathcal{B}_h$, with norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 g(s) \sup_{s\le\theta\le 0}(\mathrm{E}\|\phi(t)\|^2)^{1/2}\mathrm{ds}$$

where $g(s) = e^{2s}, s < 0$ and $\int_{-\infty}^0 g(s)ds = \frac{1}{2}$. Let $z(t)(x) = z(t,x)$. Define the function $f, g : J \times \mathcal{B}_h \to H$ and $\sigma : J \times \mathcal{B}_h \to \mathcal{L}_Q^0$ by $f(t,z)(\cdot) = f(t, z(\cdot))$, $g(t,z)(\cdot) = g(t, z(\cdot))$, $\sigma(\mathrm{t}, \mathrm{y}(\cdot)) = \sigma(\mathrm{t}, \mathrm{y}(\cdot))$ and the bounded linear operator $Bu(t)(x) = \mu(t,x)$ respectively. Moreover, it can be easily seen that AL^{-1} is compact and bounded with $\|L^{-1}\| \le 1$ and AL^{-1} generates a strongly continuous semigroup $T(t), t \ge 0$ with $\|T(t)\| \le e^{-t} \le 1$.

Thus with the above choices (4.1)-(4.4) can be written in the abstract from of (2.1)-(2.3). Further, we can impose some suitable conditions on the above defined functions to verify the assumptions on Theorem 3.4, we can conclude that (4.1)-(4.4) is approximately controllable on $[0,b]$.

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