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MATHEMATICAL MODELING OF HERSCHEL-BULKLEY MODEL OF BLOOD FLOW THROUGH STENOSED ARTERIES

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ABSTRACT

Blood flow through a stenosed artery has been investigated in this paper. The present paper contains a study of blood flow in arteries from the heart keeping in view the nature of blood flow circulation in human body. Blood has been represented by a non-Newtonian fluid obeying Herschel-Bulkley equation. The usual blood flow in arteries is obstructed by abnormal tissue development on the walls of these vessels, called as stenosis in bio transport system. There will be a plug flow (the flow in the entry length portion consists of two parts. First part is called boundary layer flow and second part is called plug flow or core flow) on the axis of blood vessels which produces yield stress. An improved shape of the time-varient stenosis present in the tapered arterial lumen is given mathematically in order to update resemblance to the in vivo situation. Integral method has been used to solve the unsteady non linear Navier-Stokes equations in cylindrical coordinates system governing flow assuming axial symmetry under laminar flow condition. The effect of the stenosis geometry is assumed to overshadow any influence of wall distensibilty.

Key words: *Artery, stenosis, tapered, distensibility.*

Classification Number: *76Z05.*

INTRODUCATION

It is observed by medical scientists that non-Newtonian character of blood is typical in small arteries and veins where the pressure of the cells induces that specific behavior. It has been well established that many cardiovascular diseases are closely associated with the flow conditions in the blood vessels. One major type of arterial disease is atherosclerosis in which localized deposits and accumulation of cholesterol and lipid substances, as well as proliferation of connective tissues, cause a partial reduction in the arterial cross-sectional area (stenosis) and a considerable increase in the wall stiffness.

Hemo-dynamics characteristics of blood flow through arterial stenosis are numerically investigated by Moayeri and Zendehbudi [12]. Normal blood flow through the artery and there is considerable evidence that hydrodynamic factors can play a significant role in the development and progression of this disease. Mandal [11] considered unsteady analysis of non-Newtonian blood flow through tapered arteries by using finite difference scheme. Several flow characteristics, such as wall shearing stress. Reese and Thompson [7] investigated the model which applicable to any axisymmetric stenosis geometry in all laminar physiological flow regime.

In most of the studies, the flowing blood is assumed to be Newtonian. Hemorheological studies have three types of non-Newtonian blood properties: Thixotropy, shear thinning and viscoelasticity. Thixotropy, a transient property of blood, is exhibited at low shear rates and has a fairly long time scale. This suggests that thixotropy is of secondary importance in physiological blood flow. However, the viscoelasticity of blood flow diminishes very rapidly as shear rate rises and at hematocrit values. The shear thinning properties of blood, however are not transient and are exhibited in normal blood at all shear rates upto about $100S^{-1}$.

Shukla *et. al* [3] have investigated the flow of blood through stenosed tube and studied the effect of viscous terms and non-Newtonian effects. Morgan and Young [5] have used integral method to study the blood flow in arterial stenosis. Mishra and Kar (5) used momentum integral method for studying the strenosed vessel. It is therefore; appropriate to analyze the role of velocity distribution in blood flow through stenosed arteries with inertial effects.

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In this present paper the radius of geometry of stenosis have been taken. The model justifies as function is of exponential decay $\mathrm{R} = \mathrm{R}_o - \frac{\delta}{2}\exp.\left(\frac{-16}{D_0^{\,2}} z^2\right)$

2. GOVERNING EQUATIONS

The equations that govern flow under the assumed conditions are the continuity equation and the Navier-Stokes equations. Consider axisymmetric steady, laminar flow of blood flow through stenosed artery. Thus, the governing equations is dimensionless form are as follows [Chaturani and Samy. [1]]

The continuity equation

$$\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \tag{1}$$

Similarly, the Navier – Stokes Equations are

$$u\frac{\partial u}{\partial z} + V\,\frac{\partial u}{\partial r} = -\,\frac{\partial p}{\partial z} + \frac{2}{\mathrm{Re}}\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right] \tag{2}$$

In the axial direction

$$V\frac{\partial v}{\partial r} + u\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial r} + \frac{2}{\mathrm{Re}}\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right] \tag{3}$$

Where r and z are the radial and axial coordinate with z -axis and u and v are the axial and radial components of velocity, p is pressure and Re is Reynolds Number. To the pressure of the non linear terms representing convective acceleration an analytical solution of the above equations is not feasible. However an attempt is made to put forward on approximate solution to the problem by preserving the principal considerations regarding the stenosis geometry.

Boundary conditions of velocity - profile constraints can be established for axisymmetric tube are

(i) $u = U$ at $r = 0$

(ii) $u = 0$ at $r = R$

(iii) $\frac{\partial u}{\partial r} = 0$ at $r = 0$

(iv) $\frac{\partial^3 u}{\partial r^3} = 0$ at $r = 0$

(v) $\int_o^R r\,u\;dr = \frac{1}{2}$

The first of these is the centre line velocity, second is the no-slip condition at the wall. The third is derived from a consideration of the forces on a cylindrical element having its axis along the tube centre line. If the pressure and the inertial forces are to be finite as the radius of element approaches zero, the viscous force, which is proportional to $\frac{\partial u}{\partial r}$, must approach zero. The four constraints can be obtained by eliminating the pressure between equations (2) and (3). The fifth constraint arises from the condition that the net flow through any cross section must be the same for any incompressible fluid.

3. FORMULATION OF THE PROBLEM

To make the problem more manageable experimentally and mathematically, steady flow is assumed. Assuming the axial viscous component of the normal stress negligible and

$$\int_o^R r\,\frac{\partial p}{\partial z}\,dr \;\cong\; R^2\int_o^R ru\,\frac{\partial p}{\partial z}\,dr \tag{4}$$

Where $\quad R = R_o - \frac{\delta}{2}\exp.\left(\frac{-16}{D_0^{\,2}} z^2\right)$

And $D_0 = 2R_0$

Based on assumptions (1) and (3) integral momentum and energy equations can be combined to yield a single equation in term of the axial velocity:

$$\frac{1}{2} R^2 \frac{\partial}{\partial z} \int_o^R r\, u^3\, dr - \frac{\partial}{\partial z} \int_o^R r\, u^2\, dr = -\frac{2}{\mathrm{Re}} \left[R^2 \int_o^R r \left(\frac{\partial u}{\partial r} \right)^2 dr + R \left(\frac{\partial u}{\partial z} \right)_r \right] \quad (5)$$

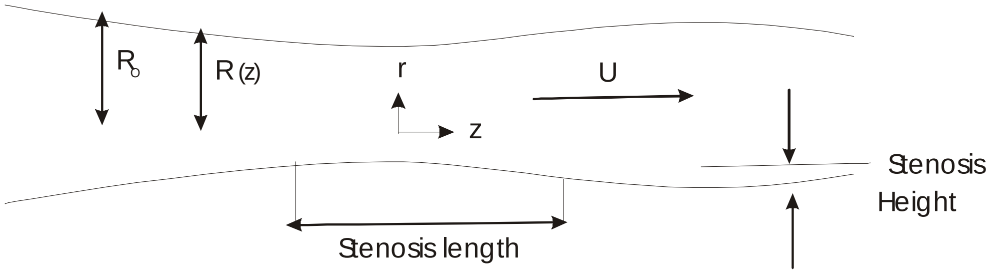

Figure-1: Geometry of arterial stenosis

Figure-1: Illustrates the stenosis geometry together with cylindrical polar- coordinate system used to describe the problem.

Where U is centre line velocity and the local radius of the axisymmetric tube is R(z) and R_0 is the radius of the unconstructed sections upstream and downstream of the stenosis.

The assumed dimensionless polynomial velocity profile, which satisfy the boundary conditions (iii) and (iv), we get

$$\frac{u}{U} = A + B \left(\frac{r}{R} \right)^m + C \left(\frac{r}{R} \right)^l \quad (6)$$

Where $R = R_0 - \frac{\delta}{2} \exp.\left(\frac{-16}{D_0^{\,2}} z^2 \right)$

And $D_0 = 2\ R_o,\ m = \frac{n+1}{n},\ l = \frac{n+3}{n}$

Now, using boundary conditions (i), (ii) and (v), we get

$$u = R^{-2} \left[S + \frac{3(3n+1)\,(n+1) - S\,(3(n+1)^2 + 4n)}{4n} \left(\frac{r}{R} \right)^m - \frac{3m}{4} \left((3n+1) - S\ (n+1) \right) \left(\frac{r}{R} \right)^l \right] \quad (7)$$

Where $S = R^2 U$

For $n = 1$ and $R^2 U$ = 2 in equation (7) a parabolic profile is obtained. Now the blunted (i.e profile is not sharp) can be expressed as,

$$u = \begin{cases} U \quad ; & 0 \le \frac{r}{R} \le \lambda \\ a + b \left(\frac{r}{R} \right)^m + c \left(\frac{r}{R} \right)^l ; & \lambda < \frac{r}{R} \le 1 \end{cases} \quad (8)$$

The coefficients a, b and c are evaluated from the no-slip condition along with two compatibility conditions,

$u = U$ and $\frac{\partial u}{\partial r} = 0$ at $\frac{r}{R} = \lambda$ then

Equitation (8) becomes

$$u = \begin{cases} U & ; \quad 0 \le \dfrac{r}{R} \le \lambda \\ \dfrac{U}{\varphi}\left(1 + k.\lambda^{2/n}\left(\dfrac{r}{R}\right)^m - 1\right) - \left(\dfrac{r}{R}\right)^l ; & \lambda < \dfrac{r}{R} \le 1 \end{cases} \tag{9}$$

Where $R = R_0 - \dfrac{\delta}{2} \exp.\left(\dfrac{-16}{D_0^{\ 2}} z^2\right)$

$D_0 = 2 R_0$

and $\phi = \left(1 - K.\ \lambda^{2/n} + \dfrac{2}{n+1} \lambda^K\right),\ K = \dfrac{n+3}{n+1}$

and $\lambda^{2/3} = \dfrac{(3n+1)\ (n+1)}{(3n+1)\ (n+3) - 3\ (n+1)^2}\left(\dfrac{3}{S.K} - 1\right)$

There remains now only the tedious, but straight-forward task of substituting the assumed profile into equation (5), we get

$$U^{'} = \left[-R^{-3}\ R^{'}\left[W.\ (S)^3 - 2L\ (S)^2 - D.S - 2\ (Y - G)\right] - \frac{2\eta}{\mathrm{Re}}\right]\left(\frac{3W}{2}\ (S)^2 - (2L - I).S + \frac{D}{2} - 1\right)^{-1} \tag{10}$$

Where $R^2 U \ge \dfrac{3}{k}$

$$\eta = R^{-2}\left[\frac{3(3n+1)\ (n+1)\ (n-3)\ (n-1)}{16n^3\ (n+2)} + \frac{9n^4 + 20n^3 + 30n^2 + 26n + 7}{16n^3\ (n+2)}\left((S)^2 - 2S\right)\right]$$

From the centre line velocity U, obtained from equation (10) the desired flow characteristics can be calculated. The velocity profiles are obtained directly from substituting U into equation (7) and (9). The average pressure gradient is found numerically,

$$\left(\frac{\partial p}{\partial z}\right)_{av} = \frac{\int_o^R r\left(\dfrac{\partial p}{dz}\right) dr}{\int_o^R r\ dr} = 2\ R^{-2} \int_o^R r\ \frac{\partial p}{\partial z}\ dr \tag{11}$$

Now, numerical integration of equation (11) gives the axial pressure distribution.

The wall shear stress can be expressed as,

$$\tau_w = \frac{R^{-3}}{8n^2}\left[m(3m-2).S - 3m(3n+1)/n\right]\left[1 + 1(R^{'})^2\right],\ S \supseteq \frac{3}{K} \tag{12}$$

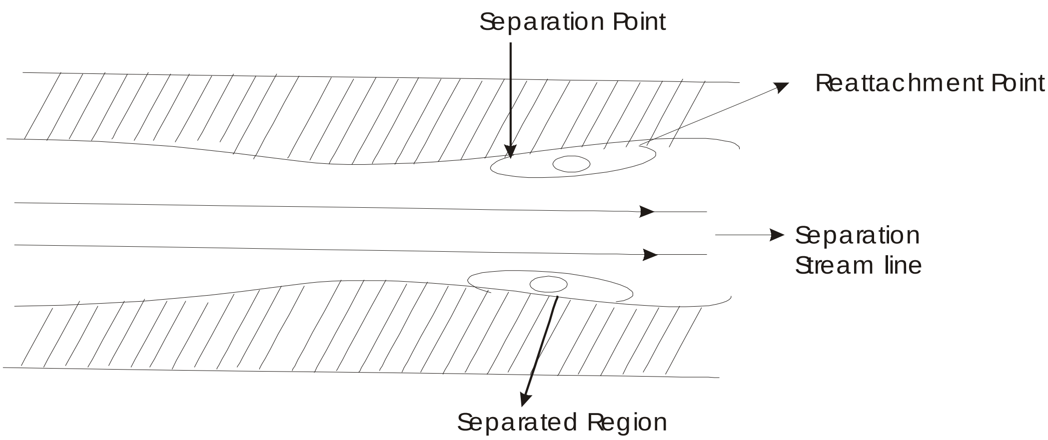

Figure-2: Separation in an axisymmetric stenosis

Along the wall shear τ_w = o flow separation and reattachment will occur as shown in Figure 2. with localized regions of recirculating flow. Wall shear stress is negative, indicating back flow and separated region is clearly delineated. Since there is no flow into the separated region, continuity requires that

$$\int_o^{r_s} ru \; dr = \frac{1}{2} \tag{13}$$

where r_s is radial coordinate of the separated stream line. From equation (7) and (13) to obtain implicit, expression for the separation stream line

$$R^2 U = \frac{1-\left(\frac{r_s}{R}\right)^2 \left[\frac{3(n+1)}{2}\left(\frac{r_s}{R}\right)^m \; \frac{3n+1}{2}\left(\frac{r_s}{R}\right)^l\right]}{\left(\frac{r_s}{R}\right)^2 \left[1-\frac{3\,(n+1)^2+4n}{2\,(3n+1)}\left(\frac{r_s}{R}\right)^m + \frac{n+1}{2}\left(\frac{r_s}{R}\right)^l\right]} \tag{14}$$

4. EXPERIMENTS

The particular stenosis geometry used was selected from Young (4) and given by the expressions

$$R\,(z) = 1 - \frac{\delta}{2}\left(1 + Cos\,\frac{\Pi z}{z_0}\right), \qquad |z| \le z_0 \tag{15}$$

where δ = stenosis height, z_0 = stenosis length.

Three model stenosis, with geometries defined by equation (15), but with different values of δ and z_0 were obtained for the experimental test (Table I)

Model Number	R_0 (in)	z_0	δ	Percent Stenosis
M -1	.37	3	1/3	63
M -2	.37	3	2/3	75
M -3	.37	3	2/3	90

Table-I: Geometric parameters of the Model Stenosis

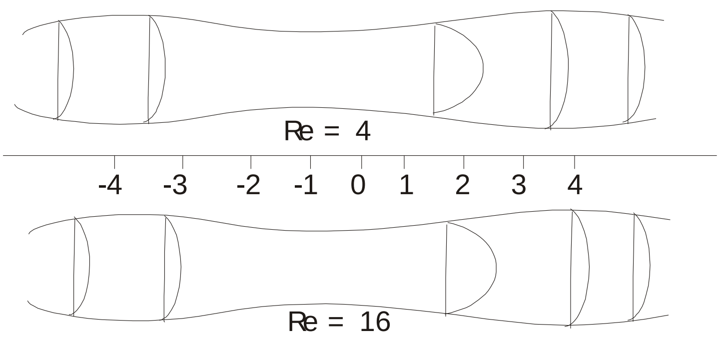

Figure-3: Velocity for Model M-2 at Re = 4 and 16

5. RESULTS AND DISCUSSIONS

A theoretical velocity profile is shown in Figure 3, for M-2 at Reynolds Number 4 and 16 the profile remain nearly parabolic throughout the constriction at the lower Reynolds Number. Since for M-2 it was determined that separation first occurs when Re = 16 the profiles downstream from the throat for this Reynolds Number show some departure from the parabolic form, but only a slight flattening can be detected upstream. The theory is fairly well for M-2and M-3, but there is a large discrepancy between the predicted critical Reynolds Numbers for initial separation and experimental value of M-1.From Figure 4.it is observed that as the axial distance increases the velocity (R^2U) also increases in a large scale for different values of parameters (n). From Figure 5, it may be noted that for the axial distance ($z = 0$), the wall shear stress is positive and there is no back flow. But as the axial distance increases then the wall shear stress increases and back flow develops for different values of parameter (n). So the nature of velocity distribution is very harmful to vessel walls.

6. CONCLUSION

Blood flow in stenosed arteries has been studied in present analysis. We have observed that the velocity field is highly dependent on the flow wave form, particularly downstream from the stenosis. It is worth noticing that the wall shear stress oscillates between positive and negative values, the arterial wall is therefore submitted to opposite force which could lead to lesions on the elastic characteristics of the wall, with the possibility of aneurism development in the post-stenotic zone. Results of this study agree well with published results regarding variations in shear stress with the amount of stenosis, length of stenosis sand Reynolds Number.

The model presented in this study provides reasonably accurate values of wall shear stress in stenosis using minimal time and resources and may be of use to clinical researchers for initial estimates of wall shear stress.

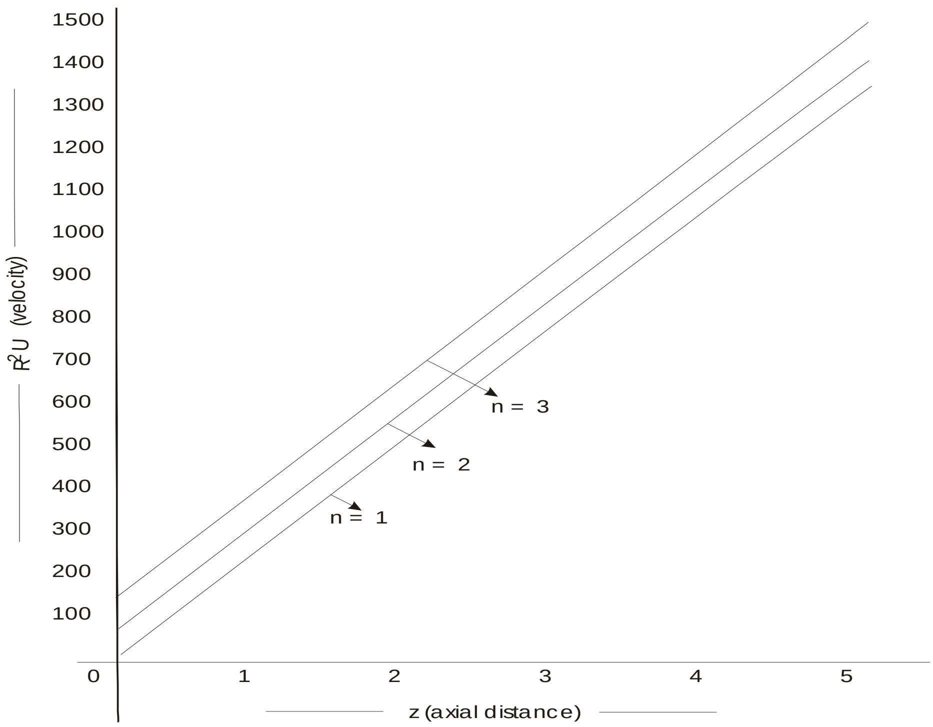

Figure.4 Variation between axial distance and velocity

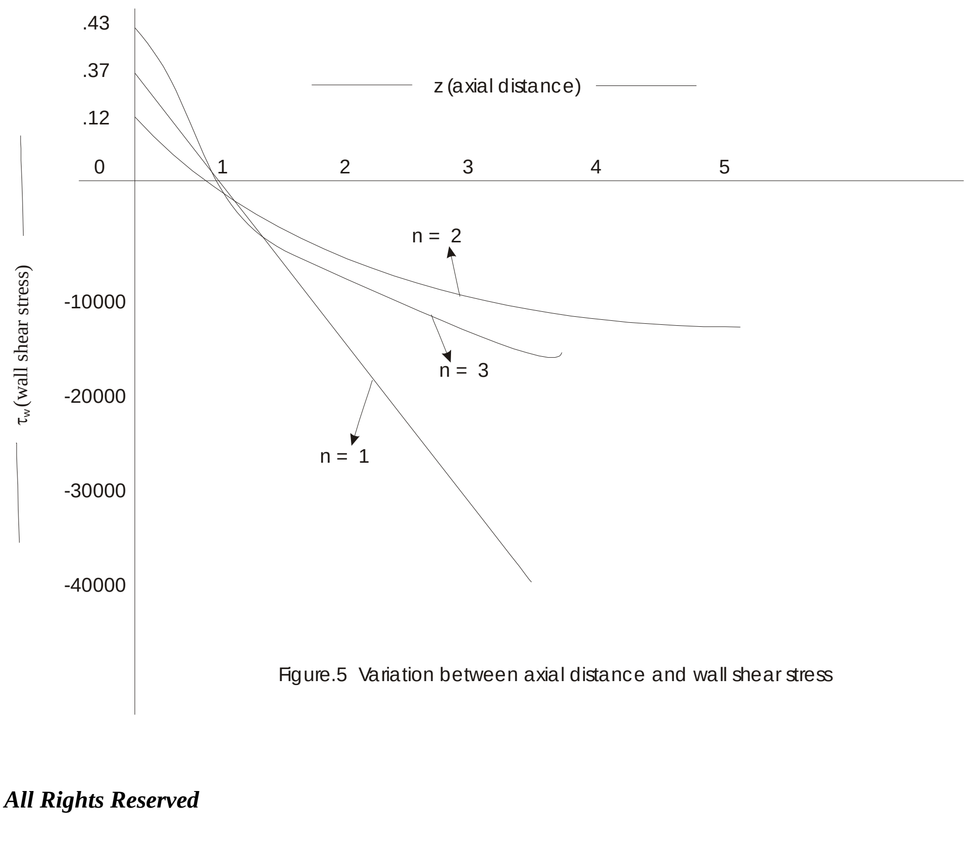

Figure.5 Variation between axial distance and wall shear stress

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