International Journal of Mathematical Archive-8(8), 2017, 120-125
Available online through www.ijma.info ISSN 2229 – 5046

EFFICIENTLY DOMINATING STEINER NUMBER OF GRAPHS

K. RAMALAKSHMI*[1], K. PALANI[2]

[1]Assistant Professor, Department of Mathematics,
Sri Sarada College for Women, Tirunelveli 627 011, Tamil Nadu, India.

[2]Assistant Professor, Department of Mathematics,
A.P.C Mahalakshmi College, Tuticorin 628 002, Tamil Nadu, India.

(Received On: 18-07-17; Revised & Accepted On: 08-08-17)

ABSTRACT

In this paper, efficiently dominating steiner number of a graph is defined and this number is found for some standard graphs and their subdivision graphs.

Key words: *Domination, Steiner number, Steiner domination number and efficiently dominating Steiner number.*

1. INTRODUCTION

The concept of domination in graphs was introduced by Ore and Berge [4]. Let G = (V, E) be a finite undirected graph with neither loops nor multiple edges. A subset D of V(G) is a dominating set of G if every vertex in V- D is adjacent to atleast one vertex in D. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$.

The concept of Steiner number of a graph was introduced by G. Chatrand and P. Zhang [1]. For a nonempty set W of vertices in a connected graph G, the Steiner distance d(W) of W is the minimum size of a connected sub graph of G containing W. Necessarily each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W -tree. The set of all vertices of G that lie on some Steiner W -tree is denoted by S(W). If S(W) = V, then W is called a Steiner set for G and is denoted by s-set of G. A Steiner set of minimum cardinality is the Steiner number s(G) of G.

The concept of Steiner domination number of a graph was introduced by John, *et al.*[3]. For a connected graph G, a set of vertices W in G is called a Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of G is its Steiner domination number and is denoted by $\gamma_s(G)$. A steiner dominating set of cardinality $\gamma_s(G)$ is said to be a γ_s- set.

A subset S of V(G) is called an efficient dominating set of G if for every $v \in V(G)$, $|N[v] \cap S|=1$. A graph G is efficient if G has an efficient dominating set.

For a connected graph G, let W be a γ_s- set of G. Then, W is an efficiently dominating steiner set of G if for every $v \in V(G)$, $|N[v] \cap W|=1$. The cardinality of W is the efficiently dominating steiner number of G and is denoted by $E\gamma_s(G)$.

A subset S of V is called an independent set of G if no two vertices of S are adjacent in G. A subdivision of an edge e = uv of a graph G is the replacement of the edge e by a path {u, v, w}. If every edge of G is subdivided exactly once, then the resulting graph is called the subdivision graph and is denoted by S(G). Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and $V_2$and edge sets E_1 and E_2 respectively. Then their union $G=G_1 \cup G_2$ is a graph with vertex set $V=V_1 \cup V_2$ and edge set $E= E_1 \cup E_2$. Clearly $G_1 \cup G_2$ has p_1+p_2 vertices and q_1+q_2 edges. Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then the join of G_1 and G_2 is denoted by G_1+G_2 and is defined as $V(G_1+G_2) = V_1 \cup V_2$ and $E(G_1+G_2) = E_1 \cup E_2 \cup \{uv: u \in V_1, v \in V_2\}$.

Corresponding Author: K. Ramalakshmi[1],*
[1]Assistant Professor, Department of Mathematics,
Sri Sarada College for Women, Tirunelveli 627 011, Tamil Nadu, India.

Theorem 1.1[6]: For a Wheel graph $W_{1,p}=\{v,v_1,v_2,v_3,\ldots,v_p\}$, $p\geq 4$, the set $W=\{v,v_1,v_3,\ldots,v_{p-1}\}$ is the unique minimum steiner dominating set and $\gamma_s(W_{1,p}) = p-2$.

Theorem 1.2[3]: For the complete bipartite graph $G = K_{m,n}$, $\gamma_s(G)=\begin{cases} 2 & if\ m=n=1 \\ n & if\ n\geq 2, m=1 \\ \min\{m,n\}\ if\ m,n\geq 2 \end{cases}$

Theorem 1.3[5]: For the complete graph $K_p (p \geq 2)$, $\gamma_s(K_p) = p$.

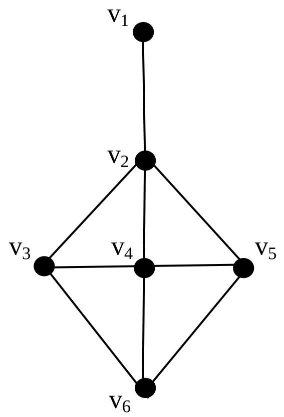
Theorem 1.4[5]: $\gamma_s(P_n)=\begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & if\ n\geq 5; \\ 2 & if\ n=2,3\ or\ 4. \end{cases}$

Theorem 1.5[5]: *For n > 5,* $\gamma_s(C_n)=\left\lceil \frac{n}{3} \right\rceil$.

2. EFFICIENTLY DOMINATING STEINER NUMBER OF A GRAPH

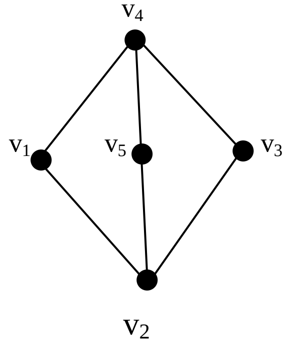
Definition 2.1: Let G be a connected graph and W be a γ_s-set of G. Then, W is an efficiently dominating steiner set of G if for every $v \in V(G)$, $|N[v]\cap W|=1$. The cardinality of W is the efficiently dominating steiner number of G and is denoted by $E\gamma_s(G)$. An efficiently dominating steiner set of cardinality $E\gamma_s(G)$ is called a $E\gamma_s$-set of G. A graph G is said to be an efficiently dominating steiner graph if it has an efficiently dominating steiner set.

Example 2.2: Consider the graph G in Figure 2.1. Here, $W=\{v_1, v_6\}$is the minimum steiner dominating set of G. $|N[v_i] \cap W| = 1$ for all $v_i \in V(G)$, $1 \leq i \leq 6$. Therefore, W is the minimum efficiently dominating steiner set of G and $E\gamma_s(G) = 2$. Therefore, the graph G is an efficiently dominating steiner graph.



G: Figure 2.1

Example 2.3: Consider the graph G in Figure 2.2. Here, $W= \{v_2, v_4\}$ is the minimum steiner dominating set of G. Here, $|N[v_1]\cap W|= |N[v_3]\cap W| = |N[v_5]\cap W| = 2$. Therefore, W is not an efficiently dominating steiner set of G. Here, G has no efficiently dominating steiner set and hence G is not an efficiently dominating steiner graph.



G: Figure 2.2

Theorem 2.4: The path P_n, $n\equiv1 \pmod 3$ has an efficiently dominating steiner set and $E\gamma_s(P_n) = \gamma_s(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2$.

Proof: Let $P_n = (v_1, v_2, v_3, ..., v_{3k+1})$, $k > 0$.
$W = \{v_1, v_4, v_7, ..., v_{3k+1}\}$ is the unique γ_s–set of P_n. Clearly we can see that $|N[v] \cap W| = 1$ for all $v \in P_n$. Hence W is the efficiently dominating steiner set of P_n. Therefore, $E\gamma_s(P_n) = \gamma_s(P_n)$. By Theorem 1.4, $E\gamma_s(P_n)=\left\lceil \frac{n-4}{3} \right\rceil+2$.

Theorem 2.5: The paths P_n, $n\equiv0 \pmod 3$ and $n\equiv2 \pmod 3$ have no efficiently dominating steiner set.

Proof: Let $P_n = (v_1, v_2, v_3, ..., v_n)$.

Case-(i) $n\equiv0 \pmod 3$ Then, $n = 3k$, $k > 0$

In this case, P_n has more than one steiner dominating set and in each set any one of the vertices $v_2, v_5, v_8, ..., v_{n-1}$ does not satisfy the condition $|N[v] \cap W| = 1$.

Therefore, P_n, $n \equiv 0 \pmod 3$ has no efficiently dominating steiner set.

Case-(ii) $n\equiv2 \pmod 3$

Then, $n = 3k + 2$, $k > 0$

In this case also, P_n has more than one steiner dominating set and in each set any one of the vertices $v_1, v_2, v_4, v_7, ..., v_{n-1}, v_n$ does not satisfy the condition $|N[v] \cap W| = 1$.

Therefore, P_n, $n \equiv 2 \pmod 3$ has no efficiently dominating steiner set.

Theorem 2.6: The graph $G = C_{3n}$, $n > 1$ has an efficiently dominating steiner set and $E\gamma_s(C_{3n}) = n$.

Proof: Let $V(C_{3n}) = \{v_1, v_2, ..., v_{3n}\}$. The γ_s-sets of C_{3n} are $W_1=\{v_1,v_4,...,v_{3(n-1)+1}\}$, $W_2=\{v_2,v_5,...,v_{3(n-1)+2}\}$ and $W_3=\{v_3,v_6,...,v_{3n}\}$. Obviously, $|N[v]\cap W_i| = 1$ for all $v \in C_{3n}$ and $i = 1, 2, 3$. Hence C_{3n} is an efficiently dominating steiner graph and W_i, $i=1,2,3$ are efficiently dominating steiner sets of C_{3n}. Therefore, $E\gamma_s(C_{3n}) = \gamma_s(C_{3n})$. By Theorem 1.5, $E\gamma_s(C_{3n}) =\left\lceil \frac{3n}{3} \right\rceil = n.$

Theorem 2.7: The cycles C_n, $n\equiv1 \pmod 3$ and $n\equiv2 \pmod 3$ have no efficiently dominating steiner set.

Proof: Let $C_n = (v_1, v_2, ..., v_n,v_1)$

Case-1: $n \equiv 1 \pmod 3$

The graph C_n has more than one γ_s-set. In each set, there exists elements u and v such that, $d(u,v) = 1$ or $d(u,v) = 2$.

Sub Case-1a: $d(u,v) = 1$

Let $u,v \in W_i$ where W_i is one of the γ_s-sets of C_n.

Therefore, $|N[u]\cap W_i| \neq 1$ and $|N[v]\cap W_i| \neq 1$. Therefore, C_n has no efficiently dominating steiner set.

Sub Case-1b: $d(u,v) = 2$

Let $u, v\in W_i$ where W_i is one of the γ_s-sets of C_n. Let $w \in C_n$ be the vertex which lies between u and v. Then, $|N[w] \cap W_i| \neq 1$.

Therefore, C_n has no efficiently dominating steiner set.

Case-2: $n \equiv 2$ (mod3)

Sub Case 2a: $n = 5$

Here the cycle is C_5 and every steiner dominating set of C_5 contains a pair of adjacent vertices.

Therefore, C_5 has no efficiently dominating steiner set.

Sub Case 2b: $n > 5$

Here, C_n has more than one γ_s-set. In each set, there exists elements u and v such that $d(u,v) = 2$. Let $u,v \in W_i$ where W_i is one of the γ_s-sets of C_n. Let $w \in C_n$ be the vertex which lies between u and v. Then, $|N[w] \cap W_i| \neq 1$. Therefore, C_n has no efficiently dominating steiner set.

Theorem 2.8: The complete graph K_n is not an efficiently dominating steiner graph.

Proof: By Theorem 1.3, the steiner domination number of K_n is $\gamma_s(K_n) = n$. Since, all the vertices belongs to the minimum steiner dominating set of K_n, it has no efficiently dominating steiner set. Hence, complete graphs are not efficiently dominating steiner graphs.

Theorem 2.9: The Wheel graph $W_{1,p}$ has no efficiently dominating steiner set.

Proof: Let $W_{1,p}=\{v,v_1,v_2,v_3,\ldots,v_p\}$. Let W be the minimal steiner dominating set of $W_{1,p}$. Then by theorem 1.1, $W=\{v,v_1,v_3,\ldots,v_{p-1}\}$ and $\gamma_s(W_{1,p}) = p-2$.

Let v be the central vertex of the wheel graph. Then, $|N[v] \cap W| \geq 2$. Therefore, Wheel graphs have no efficiently dominating steiner set.

Theorem 2.10: The star graph $K_{1,n}$, $n \geq 2$ is not an efficiently dominating steiner graph.

Proof: By Theorem 1.2, $\gamma_s(K_{1,n}) = n$. Let W be the unique minimum steiner dominating set and let v be the central vertex of the star graph. Then, $|N[v] \cap W| \geq 2 = n \geq 2$. Therefore, W is not an efficiently dominating steiner set. Hence, star graphs are not efficiently dominating steiner graphs.

Theorem 2.11: Complete bipartite graphs $K_{m,n}$ are not efficiently dominating steiner graphs.

Proof:

Case (i): $m = n = 1$

Now, the graph is isomorphic to K_2. Then by Theorem 2.8, it is not an efficiently dominating steiner graph.

Case (ii): $n \geq 2$, $m = 1$

We get a star graph. By Theorem 2.10, it is not an efficiently dominating steiner graph.

Case (iii): $m, n \geq 2$

Let $V(K_{m,n}) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m\}$. If $n<m$, by Theorem 1.2, $W = \{v_1,v_2, \ldots, v_n\}$ is the unique minimum steiner dominating set of $K_{m,n}$. Also, each of $u_1, u_2, \ldots, u_m$ is adjacent with $v_1,v_2,\ldots,v_n$. Therefore, $|N[u_i] \cap W| = n \geq 2$ for all u_i, $1 \leq i \leq m$ and $|N[v_j] \cap W| = m \geq 2$ for all v_j, $1 \leq j \leq n$. Therefore, W is not an efficiently dominating steiner set. Hence, complete bipartite graphs are not efficiently dominating steiner graphs.

Observation 2.12: A graph G has no efficiently dominating steiner set if it has a vertex with more than one pendant edge.

Theorem 2.13: Every efficiently dominating steiner set of a graph G is independent.

Proof: Let W be an efficiently dominating steiner set of G. Let u and v be any two distinct vertices of W. If u and v are adjacent then, $|N[u] \cap W| \geq 2$ and $|N[v] \cap W| \geq 2$, which is a contradiction.

Hence, every efficiently dominating steiner set is independent.

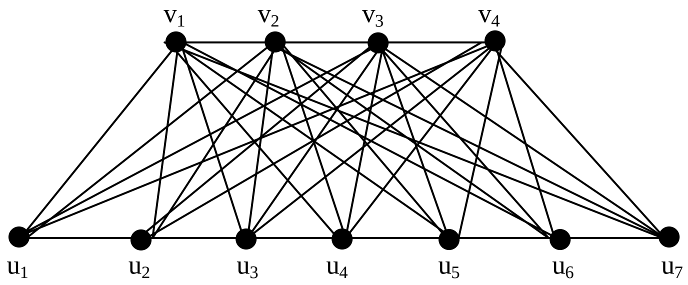
Observation 2.14:

(i) If G_1 and G_2 are efficiently dominating steiner graphs, then $G_1 \cup G_2$ is also an efficiently dominating steiner graph.

(ii) If G_1 and G_2 are efficiently dominating steiner graphs with disjoint vertex sets then, $E\gamma_s(G_1 \cup G_2) = E\gamma_s(G_1) + E\gamma_s(G_2)$.

Remark 2.15: If G_1 and G_2 are efficiently dominating steiner graphs then, G_1+G_2 need not be an efficiently dominating steiner graph.

For Example consider, the paths P_4 and P_7. By theorem 2.4, both P_4 and P_7 are efficiently dominating steiner graphs. Consider the graph P_4+P_7in Figure 2.3.



P_4+P_7: Figure 2.3

All the vertices of the graph P_4+P_7 belong to the minimum steiner dominating set. Hence, P_4+P_7 is not an efficiently dominating steiner graph.

3. EFFICIENTLY DOMINATING STEINER NUMBER OF SUBDIVISION GRAPHS

Theorem 3.1: $S(P_n)$, where $n \equiv 1 (\text{mod } 3)$ is an efficiently dominating steiner graph.

Proof: $S(P_n)$, where $n \equiv 1$ (mod 3) is also a path P_n with $n \equiv 1$ (mod 3). Therefore, by Theorem 2.4, $S(P_n)$ where $n \equiv 1$ (mod 3) is an efficiently dominating steiner graph.

Theorem 3.2: $S(P_n)$ is not an efficiently dominating steiner graphs where $n \equiv 0$ (mod 3) and $n \equiv 2$ (mod 3).

Proof:

Case-(i): $n \equiv 0$ (mod 3)

The graph $S(P_n)$, $n \equiv 0$ (mod 3) is a path P_n with $n \equiv 2$ (mod 3). Therefore, by Theorem 2.5 case (ii), $S(P_n)$ where $n \equiv 0$ (mod 3) is not an efficiently dominating steiner graph.

Case-(ii): $n \equiv 2$ (mod 3)

The graph $S(P_n)$, $n \equiv 2$ (mod 3) is a path P_n with $n \equiv 0$ (mod 3). Therefore, by Theorem 2.5 case (i), $S(P_n)$ where $n \equiv 2$ (mod 3) is not an efficiently dominating steiner graph.

Theorem 3.3: $S(C_{3n})$ where $n \geq 1$ is an efficiently dominating steiner graph and $E\gamma_s(S(C_{3n})) = 2n$.

Proof: The graph $S(C_{3n})$ where $n \geq 1$, is also a cycle isomorphic to C_{3m}, $m = 2, 4, \ldots, 2n$. Therefore by Theorem 2.6, $S(C_{3n})$ is an efficiently dominating steiner graph and also $E\gamma_s(S(C_{3n})) = E\gamma_s(C_{3m}) = 2n$.

Theorem 3.4: $S(C_n)$, where $n \equiv 1 (\text{mod } 3)$ or $n \equiv 2 (\text{mod } 3)$ have no efficiently dominating steiner set.

Proof: Let $C_n = (v_1, v_2, ..., v_n, v_1)$

Case-1: $n \equiv 1$(mod 3)

Then, $S(C_n)$ is a cycle C_n with $n \equiv 2$(mod 3). Therefore, by case 2 of Theorem 2.7, $S(C_n)$, $n \equiv 1$(mod 3) has no efficiently dominating steiner set.

Case-2: $n \equiv 2 (\text{mod } 3)$

Then, $S(C_n)$ is a cycle C_n with $n \equiv 1$ (mod 3). Therefore, by case1of Theorem 2.7, $S(C_n)$, $n \equiv 2$(mod 3) has no efficiently dominating steiner set.

Theorem 3.5: The subdivision graph $S(W_{1,p})$, $p \geq 3$ is not an efficiently dominating steiner graph.

Proof: Let v be the central vertex of the graph $W_{1,p}$. Let $\{u_1, u_2, ..., u_p\}$ be the vertices which subdivide the edges of the outer cycle of the graph $W_{1,p}$. Then, $W = \{v, u_1, u_2, ..., u_p\}$ is the unique minimum steiner dominating set of the graph $S(W_{1,p})$. If $\{v_1, v_2, ..., v_p\}$ are the rim vertices of $W_{1,p}$ then $N([v_i] \cap W) = 2$ for all $i=1,2,...,p$. Therefore, $S(W_{1,p})$ has no efficiently dominating steiner set. Hence, $S(W_{1,p})$ is not an efficiently dominating steiner graph.

Observation 3.6:

(i) $S(K_n)$ where $n \geq 4$ is not an efficiently dominating steiner graph.

(ii) $S(K_{1,n})$ is not an efficiently dominating steiner graph.

REFERENCES

1. G. Chatrand and P.Zhang, "The Steiner number of a graph", Discrete Mathematics, Vol.242,41-54(2002).
2. T.W.Haynes, S.T.Hedetniemi and P.J.Slater, "Fundamentals of Domination in graphs", Marcel Decker, Inc., New York 1998.
3. J. John, G. Edwin and P. Paul Sudhahar, "The Steiner domination number of a graph", International Journal of Mathematics and Computer Applications Research, Vol.3, Issue 3, Aug 2013, 37-42.
4. Ore and Berge, "Theory of Graphs", American Mathematical Society, Colloquium Publications Volume XXXVIII, 1962.
5. K. Ramalakshmi, K. Palani and T. Tamil Chelvam, "Steiner Domination Num-ber of Graphs", Proceedings of International Conference on Recent Trends in Mathematical Modelling, ISBN 13-978-93-82592-00-06, pp.128-134.
6. S.K.Vaidya and R.N.Mehta, Steiner domination number of some wheel related graphs, International journal of Mathematics and soft computing, Vol.5, No.2 (2015), 15-19.
7. A.Mahalakshmi, K.Palani, S.Somasundaram, "Efficiently dominating (γ, eD) number of graphs", International journal of Mathematics and its Applications, volume 5, Issue – I 2017, 59 – 64, ISSN: 2357-1557.

Source of support: Nil, Conflict of interest: None Declared.