International Journal of Mathematical Archive-8(8), 2017, 142-149
Available online through www.ijma.info ISSN 2229 – 5046

COMMON FIXED POINTS OF (ψ, φ)-WEAK CONTRACTIONS IN REGULAR CONE METRIC SPACES

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(Received On: 16-07-17; Revised & Accepted On: 21-08-17)

ABSTRACT

In this paper we have established the C-Class function of coincidence and common fixed point theorems of self maps for altering distance function and ultra altering distance function of weak contraction in regular cone metric spaces.

Key words *Cone metric space, weak contraction, altering distance function,ultra altering distance function, coincide point, common fixed point.*

2010 Mathematics Subject Classification: *Primary 47H10, Secondary 54H25.*

1. INTRODUCTION AND PRELIMINARIES

In 2007, Huang and Zhang [6] introduced the concept of cone metric spaces and fixed point theorems of contraction mappings; Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$ has a unique fixed point.Arslan Hojat Ansari,Sumit Chandok, Nawab Hussin and Ljiljana Paunovic are discuss the concept of fixed points of (ψ, ϕ)- weak contractions in regular cone metric spaces via new function[11,12].

In this paper, we discuss about common and coincidence fixed point theorems of self maps for altering distance functions and ultra altering distance functions of weak contractions in regular cone metric spaces via C-Class function. Also our result is supported by an example.

Definition 1.1: [6] Let E be a Banach space. A subset P of E is called a cone if and only if:
1. P is closed, nonempty and $P \neq 0$
2. $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b
3. $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0$ as $n \to 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose E is a Banach space, P is a cone in E with $intP \neq 0$ and $\leq$ is partial ordering with respect to P.

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Example 1.2: [15] Let $K > 1$ be given. Consider the real vector space with

$$E = \{ax + b : a, b \in R; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E. The cone P is regular and so normal.

Definition 1.3: [6] Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y\ \forall x, y \in X$,
2. $d(x, y) = d(y, x)$, $\forall x, y \in X$,
3. $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$,

Then (X, d) is called a cone metric space simply Cone metric space

Lemma 1.4: [9] Every regular cone is normal.

Example 1.5: Let $E = R^2$

$$P = \{(x, y) : x, y \geq 0\}$$

$X = R$ and $d: X \times X \to E$ such that

$$d(x, y) = (|\ x - y\ |, \alpha\ |\ x - y\ |)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a Cone metric space.

Definition 1.6: be a sequence in X. Then $\{x_n\}_{n=0}^{\infty}$ converges to x in X whenever for every $c \in E$ with $0 \ll \varepsilon$, there is a natural number $N \in N$ such that $d(x_n, x) \ll \varepsilon$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 1.7: Let (X, d) be a Cone metric space and $\{x_n\}_{n=0}^{\infty}$ be a sequence in X. $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll \varepsilon$, there is a natural number $N \in N$, such that $d(x_n, x_m) \ll \varepsilon$ for all $n, m \geq N$.

Lemma 1.8: Let (X, d) be a Cone metric space with regular cone P such that $d(x, y) \in intP$, Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$ is unique.

Definition 1.9: Let (X, d) be a Cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete Cone metric space.

Lemma 1.10: Let (X, d) be a Cone metric space with regular cone P such that $d(x, y) \in intP$, Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0 (n, m \to \infty)$.

Definition 1.11: Let T and S be two self maps defined on a set X maps T and S are said to be commuting of $TSx = STx$ for all $x \in X$

Definition 1.12: Let T and S be two self maps defined on a set X maps T and S are said to be weakly compatible if they commute at coincidence points. that is if $Tx = Sx$ forall $x \in X$ then $TSx = STx$

Definition 1.13: Let T and S be two self maps on set X. If $Tx = Sx$, for some $x \in X$ then x is called coincidence point of T and S

Lemma 1.14: Let T and S be weakly compatible self mapping of a set X. If T and S have a unique point of coincidence, that is $w = Tx = Sx$ then w is the unique common fixed point of T and S.

Definition 1.15: An altering distance function is a function $\varphi: P \to P$ which satisfies:

1. φ is continuous.
2. $\varphi(t) = 0$ if and only if $0 \ll t$

Definition 1.16: [11] An ultra-altering distance function is a function $\psi: P \to P$ which satisfies

1. ψ is continuous and non-decreasing
2. $\psi(0) > 0$.
3. $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$

Definition 1.17: [12] A mapping $F: P^2 \to P$ is called cone C-*class* function if it is continuous and satisfies following axioms:

1. $F(s, t) \leq s$;
2. $F(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in P$.

We denote cone C-class functions as $\mathcal{C}$.

Example 1.18: [11] The following functions $F: P^2 \to P$ are elements of $\mathcal{C}$, for all $s,t \in [0,\infty)$:

1. $F(s,t) = s - t$,
2. $F(s,t) = ks$, where $0 < k < 1$,
3. $F(s,t) = s\beta(s)$, where $\beta: [0,\infty) \to [0,1)$,
4. $F(s,t) = \Psi(s)$, where $: P \to P$,$\Psi(0) = 0$,$\Psi(s) > 0$ for all $s \in P$ with $s \neq 0$ and $\Psi(S) \leq s$ for all $s \in P$.,
5. $F(s,t) = s - \varphi(s)$, where $\varphi: [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
6. $F(s,t) = s - h(s,t)$, where $h: [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(s,t) = 0 \Leftrightarrow t = 0$ for all $t,s > 0$.
7. $F(s,t) = \varphi(s), F(s,t) = s \Rightarrow s = 0$, here $\varphi: [0,\infty) \to [0,\infty)$ is a upper semi continuous function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

Lemma 1.19:[12] Let ψ and φ are altering distance and ultra altering distance functions respectively , $F \in \mathcal{C}$ and $\{s_n\}$ a decreasing sequence in P such that

$$\psi(s_{n+1}) \leq F(\psi(s_n), \varphi(s_n))$$

for all $n \geq 1$. Then $\lim_{n\to\infty} s_n = 0$.

2. MAIN RESULTS

Theorem 2.1: Let (X,d) be a complete Cone metric space with regular cone P such that $d(x,y) \in intP$, for $x,y \in X$. Let $T: X \to X$ be a mapping satisfying the inequality

$$\psi\big((Tx,Ty)\big) \leq F\Big(\psi\big(d(x,y)\big), \varphi\big(d(x,y)\big)\Big) \text{ for x, y \textbackslash in X} \tag{2.1}$$

where ψ and φ are altering distance and ultra altering distance functions respectively,$F \in \mathcal{C}$ such that

(i) $\psi(t+s) \leq \psi(t) + \psi(s)$.

(ii) either $\psi(t), \varphi(t) \leq d(x,y)$ or $d(x,y) \leq \psi(t), \varphi(t)$, for $t \in intP \cup \{0\}$ and $x,y \in X$. Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n = T^n x_0$.

Take $x = x_n, y = x_{n-1}$ case 1:

$$\psi(d(x_n,x_{n+1})) = \psi\big(d(Tx_{n-1},Tx_n)\big) \leq F(\psi(d(x_n,x_{n+1})), \varphi(d(x_n,x_{n+1})))$$

Since ψ is non-decreasing and monotone, we have

$$d(x_n,x_{n+1}) \quad < d(x_n,x_{n-1}) + d(x_{n+1},x_n)$$

Suppose the sequence $d(x_n,x_{n+1})$ is decreasing and monotone. Since regular cone P such that $0 \leq d(x_n,x_{n+1}) \in intP$, for all $n \in N$, there exists $\lambda \in P$ such that

$$d(x_n,x_{n+1}) \to \lambda \text{ as n } \to \infty$$

Since φ, ψ are continuous and

$$\begin{aligned}\psi(d(x_n\,,x_{n-1}) + d(x_{n+1},x_n)) &\leq \psi(d(x_n,x_{n-1})) + \psi\big(d(x_{n+1},x_n)\big)\\ &\leq F(\psi(d(x_{n-1},x_n\,)), \varphi(d(x_{n-1},x_n\,))) + \psi(d(x\_(n+1),x\_n)))\end{aligned}$$

We have by taking $n \to \infty$

$$\psi(\lambda) \leq F(\psi(\lambda), \varphi(\lambda)) + \psi(\lambda)$$

which is a contradiction unless $\lambda = 0$. Hence $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$.

Let $\varepsilon \in E$ with $0 \ll \varepsilon$ be arbitrary. Since $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$, there exists $m \in N$ such that

$$\psi(d(x_m,x_{m+1})) \ll \varphi(\varphi(\varepsilon/2)).$$

Let $\mathcal{B}(x_m,\varepsilon) = \{x \in X: \psi(d(x_m,x)) \ll \varepsilon\}$. Clearly, $x_m \in \mathcal{B}(x_m,\varepsilon)$. Therefore, $\mathcal{B}(x_m,\varepsilon)$ is nonempty. Now we will show that $Tx \in \mathcal{B}(x_m,\varepsilon)$, for $x \in \mathcal{B}(x_m,\varepsilon)$. Let $x \in \mathcal{B}(x_m,\varepsilon)$. Case (ii) Consider ψ, we have the following two possible sub cases.

Case-(i): $\varphi(d(x,x_m)) \leq \varphi(\varepsilon/2), \psi(d(x,x_m)) \leq \varphi(\varepsilon/2)$ or

Case-(ii): $\varphi(\varepsilon/2) \leq \varphi(d(x, x_m)), \varphi(\varepsilon/2) \leq \psi(d(x, x_m))$. Here we have, Case (i):

$$\begin{aligned}\psi(d(Tx, x_m)) &\leq \psi(d(Tx, Tx_m) + d(x_m, Tx_m)) \\ &\leq F\left(\psi\big(d(x, x_m)\big), \varphi\big(d(x, x_m)\big)\right) + \psi\big(d(x_m, Tx_m)\big) \\ &\leq F\left(\psi\big(d(x, x_m)\big), \varphi\big(d(x, x_m)\big)\right) + \psi\big(d(x_m, x_{m+1})\big) \\ &\leq F\left(\varphi\left(\frac{\varepsilon}{2}\right), \varphi\left(\frac{\varepsilon}{2}\right)\right) + \varphi\left(\varphi\left(\frac{\varepsilon}{2}\right)\right) \\ &\leq \varphi(\frac{\varepsilon}{2}) - \varphi(\frac{\varepsilon}{2}) + \varphi(\frac{\varepsilon}{2}) \leq \frac{\varepsilon}{2} \leq \varepsilon.\end{aligned}$$

Case-(ii): $\psi d(Tx, x_m)) \leq \psi(d(Tx, Tx_m) + \ d(x_m, Tx_m))$

$$\begin{aligned}&\leq \psi\big(d(Tx, Tx_m)\big) + \psi\ \big(d(x_m, Tx_m)\big) \\ &\leq\ F\left(\psi\big(d(x, x_m)\big), \varphi\big(d(x, x_m)\big)\right) + \psi\big(d(x_m, Tx_m)\big) \\ &\leq \psi(d(x, x_m) - \varphi\left(\tfrac{\epsilon}{2}\right) + \varphi\left(\varphi\left(\tfrac{\epsilon}{2}\right)\right) \\ &\leq \psi(d(x, x_m)\left(because\ \varphi\big(d(x, x_m)\big) \geq \varphi\left(\tfrac{\epsilon}{2}\right), \psi\big(d(x, Tx_m)\big) \leq \varphi\left(\varphi\left(\tfrac{\epsilon}{2}\right)\right)\right) < \epsilon.\end{aligned}$$

In any case $Tx \in \mathcal{B}(x_m, \varepsilon)$ for $x \in \mathcal{B}(x_m, \varepsilon)$. Therefore, T is a self mapping of $\mathcal{B}(x_m, \varepsilon)$. Since $x_m \in \mathcal{B}(x_m, \varepsilon)$ and $Tx_{n-1} = x_n, n \geq 1$, it follows that $x_n \in \mathcal{B}(x_m, \varepsilon)$, for all $n \geq m$. Again, c is arbitrary. This establish that $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Now,

$$(\psi(d(x_n, Tx)) = \psi(d(Tx_{n-1}, Tx)) \leq F(\psi(d(x_{n-1}, x)), \varphi(d(x_{n-1}, x)))).$$

Taking $n \to \infty$, we have,

$$\psi(d(x, Tx)) \leq 0.$$

But $\psi(d(x, Tx)) \geq 0$. This implies that $d(x, Tx) = 0$ and $x = Tx$. That is x is a fixed point of T.

If $y \in X$, with $y \neq x$, is a fixed point of T. Then $\varphi(d(x, y)) \in intP$ and so

$$\psi(d(x, y) = \psi(Tx, Ty) \leq F(\psi(d(x, y)), \varphi(d(x, y))) < k\psi(d(x, y)), \text{where } 0 < k < 1$$

which is a contradiction and hence $d(x, y) = 0$, i.e. $x = y$.

Definition 2.2: Let (X, d) be a Cone metric space and let $T, S: X \to X$ be a pair of mappings is said to be a weakly (ψ, φ)-pair, if

$$d(Tx, Ty) \leq F(\psi(M_{T,S}(x, y)), \varphi(M_{T,S}(x, y))), \tag{2.2}$$

for all $x, y \in X$, for some $z \in M_{T,S}$. where

$$M_{T,S} = \max\{d(Sx, Sy), d(Tx, Sx), d(Ty, Sy)\}.$$

Lemma 2.3: [2] Let (X, d) be a Cone metric space with regular cone P such that $d(x, y) \in intP$, for $x, y \in X$ with $x \neq y$. Let $\varphi: intP \cup \{0\} \to intP \cup \{0\}$ be a function with the following properties:

(i) $\varphi(t) = 0$ if and only if $t = 0$,
(ii) $(\varphi(t) \ll t$, for $t \in intP$ and
(iii) either $\varphi(t) \leq d(x, y)$ or $d(x, y) \leq \varphi(t)$, for $t \in intP \cup \{0\}$ and $x, y \in X$. Let $\{x_n\}$ be a sequence in X for which $\{d(x_n, x_{n+1})\}$ is monotonic decreasing. Then $\{d(x_n, x_{n+1})\}$ is convergent to either $\lambda = 0$ or $\lambda \in intP$.

Theorem 2.4: Let (X, d) be a Cone metric space with regular cone P such that $d(x, y) \in intP$, for all $x, y \in X$. Let $T, S: X \to X$ be a mapping and weakly (ψ, φ)-pair. If $Tx \subset Sx$ and Sx is a complete subspace of X, then T and S have a unique point of coincidence in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Proof: Suppose x_0 is an arbitrary initial point of X and construct $(T(x_n))$ be a T-S-sequence with initial point x_0. If $T(x_n) = T(x_{n-1})$ for some $n \in N$, then $T(x_m) = T(x_n)$ for all $m \in N$ with $m > n$ and so $(T(x_n))$ is a Cauchy sequence. Therefore we consider that $T(x_n) \neq T(x_{n-1})$ for all $n \in N$.

We have for all $n \geq 0$,

$$\psi(d(Tx_{n+1}, Tx_{n+2})) \leq F(\psi(z), \varphi(z))$$

where $z \in M_{T,S}(x_{n+1}, x_{n+2}) = \{d(Tx_n, Tx_{n+1}), d(Tx_{n+1}, Tx_n), d(Tx_{n+2}, Tx_{n+1})\}$.

If $z = d(Tx_{n+1}, Tx_{n+2})$, then we have

$$\psi\big(d(Tx_{n+1}, Tx_{n+2})\big) \leq F(\psi(d(Tx_{n+1}, Tx_{n+2})), \varphi(d(Tx_{n+1}, Tx_{n+2})). \tag{2.3}$$

Using a property of ψ and φ, the inequality (2.3) hold if and only if $d(Tx_{n+1}, f_{n+2}) = 0$ and $Tx_{n+1} = Tx_{n+2}$, a contradiction. Now if $z = d(Tx_n, Tx_{n+1})$, then,

$$\psi\big(d(Tx_{n+1}, Tx_{n+2})\big) \leq F(\psi(d(Tx_n, Tx_{n+1)}), \varphi(d(Tx_n, Tx_{n+1}))). \tag{2.4}$$

Using a property of φ, we have for all $n \geq 0$,

$$\psi(d(Tx_{n+1}, Tx_{n+2})) \leq \psi(d(Tx_n, Tx_{n+1})),$$

which implies that

$$d(Tx_{n+1}, Tx_{n+2}) \leq d(Tx_n, Tx_{n+1}),$$

Since ψ is strongly monotone increasing. Therefore, $\{d(Tx_n, Tx_{n+1})\}$ is a monotone decreasing sequence. Hence by Lemma (2.3), there exists an $r \in P$ with either $\lambda = 0$ or $r \in intP$, such that

$$d(Tx_n, Tx_{n+1}) \to \lambda \text{ as } n \to \infty. \tag{2.5}$$

Letting limit as $n \to \infty$ in (2.4), using (2.5) and the continuities of ψ and φ,

$$\psi(\lambda) \leq F(\psi(\lambda), \varphi(\lambda))$$

which is a contradiction unless $\lambda = 0$. So we must have,

$$d(Tx_n, Tx_{n+1}) \to \ 0 \text{ as } n \to \infty. \tag{2.6}$$

Now we claim that $\{Tx_n\}$ is a Cauchy sequence. If $\{Tx_n\}$ is not a Cauchy sequence, then there exists a $c \in E$ with $0 \ll \varepsilon$, such that for$n_0 \in N$,there exist$n, m \in N$ with $n > m \geq n_0$ such that $d(Tx_m, Tx_n) < \varphi(\epsilon)$. Hence by a property of φ, $\varphi(\varepsilon) \leq d(Tx_m, Tx_n)$. Therefore, there exist sequences $\{m_i\}$ and $\{n_i\}$ in N such that for all positive integers k,

$$n_i > m_i > k \text{ and } d(Tx_{m_i}, Tx_{n_i}) \geq \varphi(\varepsilon).$$

Assuming that n_i is the smallest such positive integer, we get

$$d(Tx_{m_i}, Tx_{n_i}) \geq \varphi(\varepsilon)$$

And$d\big(Tx_{m_i}, Tx_{n_i-1}\big) \leq \varphi(\varepsilon)$.

Now,

$$\varphi(\varepsilon) \leq d(Tx_{m_i}, Tx_{n_i}) \leq d(Tx_{m_i}, Tx_{n_i-1}) + d(Tx_{n_i-1}, Tx_{n_i})$$

that is,

$$\varphi(\varepsilon) \leq d(Tx_{m_i}, Tx_{n_i}) \leq \varphi(\varepsilon) + d(Tx_{n_i-1}, Tx_{n_i}).$$

Letting $i \to \infty$ in the above inequality, using inequality (2.6), we have

$$lim_{i\to\infty}\ d(Tx_{m_i}, Tx_{n_i}) = \varphi(\epsilon). \tag{2.7}$$

Again,

$$d(Tx_{m_i}, Tx_{n_i}) \leq d(Tx_{m_i}, Tx_{m_{i+1}}) + d(Tx_{m_{i+1}}, Tx_{n_i+1}) + d(Tx_{n_i+1}, Tx_{n_i})$$

and

$$d(Tx_{m_i+1}, Tx_{n_i+1}) \leq d(Tx_{m_i+1}, Tx_{m_i}) + d(Tx_{m_i}, Tx_{n_i}) + d(Tx_{n_i}, Tx_{n_i+1})$$

Letting $i \to \infty$ in the above inequalities, using (2.6) and (2.7), we have

$$lim_{i\to\infty} d(Tx_{m_{i+1}}, Tx_{n_{i+1}}) = \varphi(\epsilon). \tag{2.8}$$

Putting $x = x_{m_{i+1}}$ and $y = x_{n_i+1}$ in (2.2), we have$d(Tx_{m_{i+1}}, Tx_{n_i+1}) \leq F(\psi(z), \varphi(z))$
where

$$\begin{aligned} z \in M_{T,S}(x,y) &= \{d\big(Sx_{m_{i+1}}, Sx_{n_i+1}\big), d\big(Tx_{m_{i+1}}, Sx_{m_{i+1}}\big), d\big(Tx_{n_i+1}, Sx_{n_i+1}\big)\} \\ &= \{d(Tx_{m_i}, Tx_{n_i}), d\big(Tx_{m_{i+1}}, Tx_{m_i}\big), d\big(Tx_{n_i+1}, Tx_{n_i}\big)\}. \end{aligned}$$

Case-1: If $z = d(Tx_{m_i}, Tx_{n_i})$ and letting $i \to \infty$ the above inequality, using (2.7), (2.8) and the continuities of and φ, we have

$$\psi(\varphi(\varepsilon)) \leq F(\psi(\varphi(\varepsilon)), \varphi(\varphi(\varepsilon))).$$

Ii only true for $\varphi(\varepsilon) = 0$. This implies $c = 0$, a contradiction to $0 \ll \varepsilon$.

Case-2: If $z = d(Tx_{m_{i+1}}, Tx_{m_i})$ and letting $i \to \infty$ the above inequality, using (2.7), (2.8) and the continuities of φ and ψ, we have

$$\psi(\varphi(\varepsilon)) \leq F(\psi(0), \varphi(\varphi(0))) = 0.$$

This implies that $\psi(\varphi(\varepsilon)) = 0 \Rightarrow \varphi(\varepsilon) = 0 \Rightarrow c = 0$. It is again a contradiction.

Case-3: Similarly in case.2 we get a contradiction.
Therefore $\{Tx_n\}$ be a Cauchy sequence in Sx. Since Sx is complete, there exists a $q \in Sx$ such that $\{Tx_n\} \to q$ as $n \to \infty$. Since $q \in Sx$, we can find $p \in X$ such that $Sp = q$. Now, putting $x = x_{n+1}$ and $y = p$ in (2.2), we have

$$\psi(d(Tx_{n+1}, Tp)) \leq F(\psi(z), \varphi(z)),$$

where $z \in M_{T,S}(x_{n+1}, p) = \{d(Tx_n, Sp), d(Tx_{n+1}, Tx_n), d(Tp, Sp)\}$. Now

Case-1: If $z = d(Tx_n, Sp)$, then

$$\psi(d(Tx_{n+1}, Tp)) \leq F(\psi(d(Tx_n, Sp)), \varphi(d(Tx_n, Sp))),$$

Letting limit $n \to \infty$, we have

$$\psi(d(q, Tp)) \leq F(\psi(d(q, q)), \varphi(d(q, q))),$$

i.e. $\psi(d(q, Tp)) \leq 0$. By definition of ψ, $\psi(d(q, Tp)) \geq 0$, so we have $\psi(d(q, Tp)) = 0$ implies $Tp = q = Sp$.

Case-2: If $z = d(Tx_{n+1}, Tx_n)$, then

$$\psi(d(Tx_{n+1}, Tp)) \leq F(\psi(d(Tx_{n+1}, Tx_n)), \varphi(d(Tx_{n+1}, Tx_n))),$$

Letting limit $n \to \infty$, we have

$$\psi(d(q, Tp)) \leq F(\psi(d(q, q)), \varphi(d(q, q))),$$

i.e. $\psi(d(q, Tp)) \leq 0$. By definition of ψ, $\psi(d(q, Tp)) \geq 0$, so we have $\psi(d(q, Tp)) = 0$ implies $Tp = q$.

Case-3: If $z = d(Tp, Sp)$, then

$$\psi(d(Tx_{n+1}, Tp)) \leq F(\psi(d(Tp, Sp)), \varphi(d(Tp, Sp))),$$

Letting limit $n \to \infty$, we have

$$\psi(d(q, Tp)) \leq F(\psi(Tp, q), \varphi(Tp, q)).$$

This is contradiction if $(d(Tp, q)) \neq 0$. Hence $d(Tp, q) = 0$ and $Tp = q$. Therefore, we have

$$q = Tp = Sp.$$

Hence p is a coincidence point and q is a point of coincidence of T and S.

We next show that the point of coincidence is unique. For this, assume that there exists a point q in X such that $z_1 = Tq = Sq$. Then, from (2.2)

$$\psi\big(d(Tp, Tq)\big) \leq F(\psi(z), \varphi(z)) \tag{2.9}$$

where $z \in \{M_{T,S}(p, q) = \{d(Sp, Sq), d(Tp, Sp), d(Tq, Sq)\}$.

Case-1: If $z = d(Sp, Sq)$, then from (2.9)

$$\psi(d(q, z_1)) \leq F(\psi(d(q, z_1)), \varphi(d(q, z_1))),$$

it is only true for $d(q, z_1) = 0$. Hence $q = z_1$.

Case-2: If $z = d(Tp, Sp)$, then from (2.9)

$$\psi(d(q, z_1)) \leq F(\psi(d(q, q)), \varphi(d(q, q))) = 0,$$

i.e. $d(q, z_1) \leq 0$. But $d(q, z_1) \geq 0$. Hence $q = z_1$.

Case-3: If $z = d(Tq, Sq)$, then from (2.9)

$$\psi(d(q, z_1)) \leq F(\psi(d(z_1, z_1)), \varphi(d(z_1, z_1))) = 0,$$

i.e. $d(q, z_1) \leq 0$, but $d(q, z_1) \geq 0$. Hence $d = z_1$.

Therefore, q is the unique point of coincidence of T and S. Now, if T and S are weakly compatible, then by Lemma (1.14), z is the unique common fixed point of T and S. Hence the roof is completed.

Example 2.5: Let $X = [0,1] \cup \{2, 3, \cdots\}$, $E = \mathbb{R}^2$ with usual norm, be a real Banach space, $P = \{(x, y) \in E : x, y \geq 0\}$ be a regular cone and the partial ordering $\leq$ with respect to the cone P, be the usual partial ordering in E. We define $d: X \times X \to E$ as

$$f(x) = \begin{cases} (|x-y|, |x-y|), & if\ x, y \in [0,1], x \neq y \\ (x+y, x+y), & if\ at\ least\ one\ of\ x\ or\ y \notin [0,1] and\ x \neq y, \\ (0,0), & if\ x = y. \end{cases}$$

for $x, y \in X$. Then (X, d) is a complete Cone metric space with $d(x, y) \in intP$, for $x, y \in X$. Define $\psi, \varphi: P \cup \{0\} \to P \cup \{0\}$ as

$$\psi(t_1, t_2) = (t_1, t_2), if\ t_1, t_2 \in [0,1],$$
$$= (t_1^2, t_1^2),\ for\ otherwise.$$

$$\varphi(t_1,t_2)=\left(\frac{1}{2}t_1^2,\frac{1}{2}t_2^2\right), if\ \ t_1,t_2\in[0,1],$$
$$=\left(\frac{1}{2},\frac{1}{2}\right), for\ otherwise.$$

Let $T: X\to X$ be defined as

$$Tx=\ x-\frac{1}{2}x^2,\qquad if\ x\in\ [0,1],$$
$$=x-1,\ if\ x\in\{2,3\cdots\}\ .$$

Without loss of generality, we assume that $x\geq y$ and discuss the following cases.

Case-1: For $x,y\in[0,1]$. Then

$$\begin{aligned}(\psi\big(d(Tx,Ty)\big)&=F((x-\frac{1}{2}x^2)-\left(y-\frac{1}{2}y^2\right),\left(x-\frac{1}{2}x^2\right)-\left(y-\frac{1}{2}y^2\right))\\&=F\left((x-y)-\frac{1}{2}(x-y)(x+y),(x-y)-\frac{1}{2}(x-y)(x+y)\right)\\&\leq F\left(\left((x-y)-\frac{1}{2}(x-y)^2,(x-y)-\frac{1}{2}(x-y)^2\right)\right)\\&=\psi(d(x,y))-\varphi(d(x,y)))\end{aligned}$$

Case-2: For $x\in\{3,4,\cdots\}$. Then, If $y\in[0,1]$

$$d(Tx,Ty)=d(x-1,y-\frac{1}{2}y^2)=(x-1+y-\frac{1}{2}y^2,x-1+y-\frac{1}{2}y^2)\leq(x+y-1,x+y-1).$$

If $y\in\{2,3\cdots\}$

$$d(Tx,Ty)=d(x-1,y-1)=(x+y-2,x+y-2)<(x+y-1,x+y-1).$$

Therefore

$$\begin{aligned}(\psi(d(Tx,Ty))&\leq F((x+y-1)^2,(x+y-1)^2)\\&<F((x+y-1)(x+y-1),(x+y-1)(x+y-1))\\&<F((x+y)^2-1,(x+y)^2-1)\\&<F\left((x+y)^2-\frac{1}{2},(x+y)^2-\frac{1}{2}\right)\\&=F((x+y)^2,(x+y)^2)-\left(\frac{1}{2},\frac{1}{2}\right)=\psi(d(x,y))-\varphi(d(x,y)).)\end{aligned}$$

Case 3. For $x=2$ and $y\in[0,1]$. Then, $Tx=1$, and

$$d(Tx,Ty)=F(1-(y-\frac{1}{2}y^2),1-(y-\frac{1}{2}y^2))\leq(1,1).$$

So, we have$\psi(d(Tx,Ty))\leq\psi(1,1)=(1,1)$.

Again $d(x,y)=(2+y,2+y)$. So,

$$\begin{aligned}(\psi(d(Tx,Ty))&\leq F(\psi(d(x,y)),\varphi(d(x,y)))\\&=F((2+y)^2,(2+y)^2)-\varphi\big(d(x,y)\big)\\&=F((2+y)^2,(2+y)^2)-\left(\frac{1}{2},\frac{1}{2}\right)\\&=F\left(\frac{7}{2}+4y+y^2,\frac{7}{2}+4y+y^2\right)\\&>(1,1)=\psi(d(Tx,Ty)).)\end{aligned}$$

By Theorem 2.1 and 0 is the unique fixed point of T.

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Source of support: Nil, Conflict of interest: None Declared.