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SOFT CONE METRIC SPACES AND COMMON FIXED POINT THEOREMS

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ABSTRACT

Let V be an initial universe and A be a set of parameters, $(\tilde{V}, \|.\|, A)$ be a soft real Banach space and $(P, A) \in S(\tilde{V})$ be a soft subset of $\tilde{V}$, in this paper some sufficient conditions for the existence of common fixed point of multivalued mappings satisfying contractive type conditions in cone metric spaces are generalized and obtained.

Keywords: *Common fixed point, soft cone metric spaces, contractive multivalued mappings.*

1. INTRODUCTION

Molodtsov [15] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties and has shown several applications of this theory in solving many practical problems in various disciplines like as economics, engineering, etc. Maji *et al.* [13, 14] studied soft set theory in detail and presented an application of soft sets in decision making problems. Chen *et al.* [2] worked on a new definition of reduction and addition of parameters of soft sets, Shabir and Naz [18] studied about soft topological spaces and explained the concept of soft point by various techniques. Das and Samanta introduced a notion of soft real set and number [4], soft complex set and number [5], soft metric space [6, 7], soft normed linear space [8, 9]. Chiney and Samanta [3] introduced the concept of vector soft topology, Das, *et al.* [10] studied on soft linear space and soft normed space.

In 2007, Huang and Zhang [11] introduced cone metric spaces with normal cone as a generalization of metric space. Rezapour and Hamlbarani [16] presented the results of [11] for the case of cone metric space without normality in cone.

In this paper, concept of soft cone metric space which is based on soft elements is discussed. In Section 2, some preliminary definitions and results are given. In Section 3, concept of soft cone metric according to soft element is defined. In Section 4, some fixed point theorems for contractive multivalued mappings on soft cone metric spaces are proved.

2. DEFINITION

Definition 2.1: [15] Let V be an initial universe and A be a set of parameters. Let $P(V)$ denote the power set of V. A pair (F, A) is called a soft set over V, where F is a mapping given by $F: A \to P(V)$.

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Definition 2.2: [13] Let (F, A) and (G, A) be two soft sets over a common initial universe V.

a) (F, A) is said to be null soft set (denoted by ϕ), if $\forall\, \lambda \in A, F(\lambda) = \phi$. And (F, A) is said to be an absolute soft set (denoted by $\tilde{V}$), if $\forall\, \lambda \in A, F(\lambda) = V$.

b) (F, A) is said to be a soft subset of (G, A) if $\forall\, \lambda \in A, F(\lambda) \subseteq G(\lambda)$ and it is denoted by $(F, A) \tilde{\subseteq} (G, A)$. (F, A) is said to be a soft upperset of (G, A) if (G, A) is a soft subset of (F, A). We denote it by $(F, A) \tilde{\supseteq} (G, A)$. (F, A) and (G, A) is said to be equal if (F, A) is a soft subset of (G, A) and (G, A) is a soft subset of (F, A).

c) The union of (F, A) and (G, A) over V is (H, A) defined as $H(\lambda) = F(\lambda) \cup G(\lambda), \forall\, \lambda \in$ A. We write $(F, A) \tilde{\cup} (G, A) = (H, A)$.

d) The intersection of (F, A) and (G, A) over V is (H, A) defined as $H(\lambda) = F(\lambda) \cap G(\lambda), \forall\, \lambda \in$ A. We write $(F, A) \tilde{\cap} (G, A) = (H, A)$.

e) The Cartesian product (H, A) of (F, A) and (G, A) over V denoted by $(H, A) = (F, A) \tilde{\times} (G, A)$, is defined as $H(\lambda) = F(\lambda) \times G(\lambda), \forall\, \lambda \in$ A.

f) The complement of (F, A) is defined as $(F, A)^{C} = (F^{C}, A)$ where $F^{C}: A \to P(V)$ is a mapping given by $F^{C}(\lambda) = D \setminus F(\lambda), \forall\, \lambda \in$ A for all $\lambda \in$ A. Clearly, we have $\widetilde{V^{C}} = \varphi$ and $\widetilde{F^{C}} = \tilde{V}$.

g) The difference (H, A) of (F, A) and (G, A) denoted by $(F, A) \tilde{\setminus} (G, A) = (H, A)$ is defined as $H(\lambda) = F(\lambda) \setminus G(\lambda), \forall\, \lambda \in$ A.

Definition 2.3: [4, 6] Let A be a non-empty parameter set and Z be a non-empty set. Then a function $h: A \to Z$ is said to be a soft element of Z. A soft element h of Z is said to belongs to a soft set (F, A) of Z which is denoted by $h \tilde{\in} (F, A)$ if $h(\lambda) \in F(\lambda), \forall \lambda \in$ A. Thus for a soft set (F, A) of Z with respect to the index set A, we have $F(\lambda) = \{\mathrm{h}(\lambda)\colon \mathrm{h} \tilde{\in} (F, A), \lambda \in$ A. In that case, h is also said to be a soft element of the soft set (F, A). Thus every singleton soft set (a soft set (F, A) of E for which F(λ) is a singleton set, ∀ λ ∈ A) can be identified with a soft element by simply identifying the singleton set with the element that it contains ∀ λ ∈ A.

Definition 2.4: [4, 6] Let R be the set of real numbers and A be a set of parameters and $B(R)$ be the collection of non-empty bounded subsets of R. Then a mapping $F: A \to B(R)$ is called a soft real set, denoted by (F, A). If specifically (F, A) is a singleton soft set, then after identifying (F, A) with the corresponding soft element, it will be called a soft real number. The set of all soft real numbers is denoted by $R(A)$ and the set of non-negative soft real numbers by $R(A)^{*}$.

Let $\tilde{r}$ and $\tilde{s}$ be two soft real numbers. Then the following statements hold:

- $\tilde{r} \tilde{\leq} \tilde{s}$, if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda), \forall\, \lambda \in A$,
- $\tilde{r} \tilde{<} \tilde{s}$, if $\tilde{r}(\lambda) < \tilde{s}(\lambda), \forall\, \lambda \in A$,
- $\tilde{r} \tilde{\geq} \tilde{s}$, if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda), \forall\, \lambda \in A$,
- $\tilde{r} \tilde{>} \tilde{s}$, if $\tilde{r}(\lambda) > \tilde{s}(\lambda), \forall\, \lambda \in A$.

Proposition 2.5: [6]

(a) For any soft sets $(F, A), (G, A) \in S(\tilde{V})$, we have $(F, A) \tilde{\subset} (G, A)$ if and only if every soft element of (F, A) is also a soft elements of (G, A).

(b) Any collection of soft elements of a soft set can generate a soft subset of that soft set. The soft set constructed from a collection B of soft elements is denoted by $SS(B)$.

(c) For any soft set $(F, A) \in S(\tilde{E}), SS(SE(F, A)) = (F, A)$; whereas for a collection B of soft elements, $SE(SS(B)) \supset B$, but, in general, $SE(SS(B)) \neq B$.

Definition 2.6: [9, 10]

(a) A sequence $\{\widetilde{z_n}\}$ of soft elements in a soft normed linear space $(\tilde{Z}, \|.\|, A)$ is said to be convergent and converges to a soft element $\tilde{z}$ if $\| \widetilde{z_n} - \tilde{z} \| \to \bar{0}$ as $\mathrm{n} \to \infty$. This means for every $\tilde{\epsilon} \tilde{>} \bar{0}$, chosen arbitrarily, $\exists$ a natural number $N = N(\tilde{\epsilon})$ such that $\bar{0} \tilde{\leq} \| \widetilde{z_n} - \tilde{z} \| \tilde{<} \tilde{\epsilon}$ whenever $n > N$ i.e. $n > N \Rightarrow \widetilde{z_n} \in B(\tilde{z}, \tilde{\epsilon})$, (where $B(\tilde{z}, \tilde{\epsilon})$ is an open ball with centre $\tilde{z}$ and radius $\tilde{\epsilon}$).

(b) A sequence $\{\widetilde{z_n}\}$ of soft elements in a soft normed linear space $(\tilde{Z}, \|.\|, A)$ is said to be a Cauchy sequence in $\tilde{Z}$ if corresponding to every $\tilde{\epsilon} \tilde{>} \bar{0}\, \exists$ a natural number $N = N(\tilde{\epsilon})$ such that $\| \widetilde{z_n} - \widetilde{z_n} \| \tilde{\leq} \tilde{\epsilon}, \forall\, \mathrm{m}, \mathrm{n} > N$ i.e. $\| \widetilde{z_n} - \widetilde{z_n} \| \to \bar{0}$ as $\mathrm{n}, \mathrm{m} \to \infty$.

(c) Let$(\tilde{Z}, \|.\|, A)$ be a soft normed linear space. Then $\tilde{Z}$ is said to be complete if every Cauchy sequence of soft elements in $\tilde{Z}$ converges to a soft element of $\tilde{Z}$. Every complete soft normed linear space is called a soft Banach space.

Definition 2.7: [6] Let Z be a non-empty set and A be non-empty a parameter set. A mapping $d: SV(\tilde{Z}) \times SV(\tilde{Z}) \to R(A)^{*}$ is said to be a soft metric on the soft set $\tilde{Z}$ if d satisfies the following conditions:

- $d(\widetilde{z_1}, \widetilde{z_2}) \tilde{\geq} \bar{0}, \forall\, \widetilde{z_1}, \widetilde{z_2} \in \tilde{Z}$.
- $d(\widetilde{z_1}, \widetilde{z_2}) = \bar{0}$, if and only if $\widetilde{z_1} = \widetilde{z_2}$.

- $d(\widetilde{z_1}, \widetilde{z_2}) = d(\widetilde{z_2}, \widetilde{z_1}), \forall\, \widetilde{z_1}, \widetilde{z_2} \in \tilde{Z}$.
- $d(\widetilde{z_1}, \widetilde{z_2}) \leqq d(\widetilde{z_1}, \widetilde{z_3}) + d(\widetilde{z_3}, \widetilde{z_2}), \forall\, \widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3} \in \tilde{Z}$.

The soft $\tilde{Z}$ with a soft metric d on $\tilde{Z}$ is said to be a soft metric space and denoted by $(\tilde{Z}, d, A)$ or $(\tilde{Z}, d)$.

Proposition 2.8: [10] Let $(\tilde{Z}, \|.\|, A)$ be soft normed linear space. Let us define
$d: \tilde{Z} \times \tilde{Z} \to R(A)^*$ by $d(\widetilde{z_1}, \widetilde{z_2}) = k\widetilde{z_1} - \widetilde{z_2}k, \forall\, \widetilde{z_1}, \widetilde{z_2} \tilde{\in} \tilde{Z}$. Then d is a soft metric on $\tilde{Z}$.

3. SOFT CONE METRIC SPACES

Definition 3.1: Let $(\tilde{V}, \|.\|, A)$ be a soft real Banach space and $(P, A) \in S(\tilde{V})$ be a soft subset of $\tilde{V}$. Then (P, A) is called a soft cone if and only if

(1) (P, A) is closed, $(P, A) \neq \varphi$ and $(P, A) \neq SS\{(\theta)$,
(2) $\tilde{a}, \tilde{b} \in R(A)^*, \widetilde{z_1}, \widetilde{z_2} \in (P, A) \Longrightarrow \tilde{a}\widetilde{z_1} + \tilde{b}\widetilde{z_2} \in (P, A)$,
(3) $\widetilde{z_1} \tilde{\in} (P, A)$ and $-\widetilde{z_1} \tilde{\in} (P, A)$ implies $\widetilde{z_1} = \theta$.

Given a soft cone $(P, A) \in S(\tilde{V})$, we define a soft partial ordering $\tilde{\lesssim}$ with respect to (P, A) by $\widetilde{z_1} \tilde{\lesssim} \widetilde{z_2}$ if and only if $\widetilde{z_2} - \widetilde{z_1} \tilde{\in} (P, A)$. We write $\widetilde{z_1} \tilde{<} \widetilde{z_2}$ whenever, $\widetilde{z_1} \tilde{\lesssim} \widetilde{z_2}$ and $\widetilde{z_1} \neq \widetilde{z_2}$, while $\widetilde{z_1} \tilde{\ll} \widetilde{z_2}$ will stand for $\widetilde{z_2} - \widetilde{z_1} \tilde{\in} Int(P, A)$ where $Int(P, A)$ denotes the interior of (P, A). The cone (P, A) is called normal if there is a number $k > 0$, such that $\forall\, \widetilde{z_1}, \widetilde{z_2} \in \tilde{V}$,

We have $\theta \tilde{\lesssim} \widetilde{z_1} \tilde{\lesssim} \widetilde{z_2} \Longrightarrow \| \widetilde{z_1} \| \leqslant k \| \widetilde{z_2} \|$

The least positive number satisfying this inequality is called the soft normal constant of (P, A). The soft cone (P, A) is called regular if every increasing sequence which is bounded from above is convergent. Equivalently the cone (P, A) is called regular if every decreasing sequence which is bounded from below is convergent. Regular soft cones are soft normal and there exist soft normal cones which are not regular. Throughout the Banach space $\tilde{V}$ and the cone (P, A) will be omitted.

Definition 3.2: Let Z be a non-empty set and $\tilde{Z}$ be absolute soft set. A mapping $d: SV(\tilde{Z}) \times SV(\tilde{Z}) \to SV(\tilde{Z})$ is said to be a soft cone metric on $\tilde{Z}$ if d satisfies the following axioms:

(d1) $d(\widetilde{z_1}, \widetilde{z_2}) \tilde{\in} \tilde{Z}, i.e. \theta \tilde{<} d(\widetilde{z_1}, \widetilde{z_2}), \forall\, \widetilde{z_1}, \widetilde{z_2} \tilde{\in} \tilde{Z}$ and $d(\widetilde{z_1}, \widetilde{z_2}) = \theta$ if and only if $\widetilde{z_1} = \widetilde{z_2}$.
(d2) $d(\widetilde{z_1}, \widetilde{z_2}) = d(\widetilde{z_2}, \widetilde{z_1}), \forall\, \widetilde{z_1}, \widetilde{z_2} \tilde{\in} \tilde{Z}$.
(d3) $d(\widetilde{z_1}, \widetilde{z_2}) \tilde{\lesssim} d(\widetilde{z_1}, \widetilde{z_3}) + d(\widetilde{z_3}, \widetilde{z_2}), \forall\, \widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3} \tilde{\in} \tilde{Z}$.

Then, the soft set $\tilde{Z}$ with a soft cone metric d on $\tilde{Z}$ is called a soft cone metric space and is denoted by $(\tilde{Z}, d, A)$.

Hence, it is obvious that soft cone metric spaces generalize soft metric spaces.

4. MAIN RESULTS

Theorem 4.1: Let $(\tilde{Z}, d, A)$ be a complete soft cone metric space and $T, S: (\tilde{Z}, d, A) \to (\tilde{Z}, d, A)$ be any two contractive multivalued mappings, satisfying for each $\tilde{y}, \tilde{z} \tilde{\in} \tilde{Z}$

$$F[H(T\tilde{z}, S\tilde{y}) \leqq \alpha[F\{d(\tilde{z}, \tilde{y}) + d(T\tilde{z}, S\tilde{y})\}] + \beta[F\{d(\tilde{z}, T\tilde{z}) + d(\tilde{y}, S\tilde{y})] + \gamma[F\{d(\tilde{z}, S\tilde{y}) + d(\tilde{y}, T\tilde{z})\}] + \delta \frac{F\{d(\tilde{z}, S\tilde{y}) + d(T\tilde{z}, S\tilde{y})\}}{1 - F\{d(\tilde{z}, S\tilde{y}) d(T\tilde{z}, S\tilde{y})\}}$$

$$\forall \tilde{y}, \tilde{z} \tilde{\in} \tilde{Z}, \alpha + \beta + \gamma + \tfrac{1}{2}\delta < \tfrac{1}{2}; \alpha, \beta, \gamma, \delta \in \left[0, \tfrac{1}{2}\right] \ and\ 1 - F\{d(\widetilde{z_n}, T\widetilde{z^*}) d(T\widetilde{z_n}, T\widetilde{z^*})\} > 0.$$

Then $T\ and\ S$ has a common fixed soft element in $\tilde{Z}$. For each $\tilde{y}, \tilde{z} \tilde{\in} \tilde{Z}$, the iterative sequence $\{T^n \tilde{z}\}$ converges to the fixed soft element.

Proof: for each $\widetilde{z_0} \in \tilde{Z}$ and $n \geq 1$, $\widetilde{z_1} \in T(\widetilde{z_0})$,………, $\widetilde{z_{n+1}} \in T(\widetilde{z_n})$ by iteration method, we have a sequence $\{\widetilde{z_n}\}$ of soft elements in $\tilde{Z}$ by letting $\widetilde{z_1} = T\widetilde{z_0}$, $\widetilde{z_2} = T\widetilde{z_1} = T^2\widetilde{z_0}, \ldots\ldots\ldots\ldots, \widetilde{z_{n+1}} = T\widetilde{z_n} = T^{n+1}\widetilde{z_0}, \ldots\ldots\ldots\ldots\ldots$
Then,

$$\begin{aligned}
F[d(\widetilde{z_{n+1}}, \widetilde{z_n})] &\leq F[H(T\widetilde{z_n}, S\widetilde{z_{n-1}})] \\
&\leq \alpha[F\{d(\widetilde{z_n}, \widetilde{z_{n-1}}) + d(T\widetilde{z_n}, S\widetilde{z_{n-1}}) + \beta[F\{d(\widetilde{z_n}, T\widetilde{z_n}) + d(\widetilde{z_{n-1}}, S\widetilde{z_{n-1}})] \\
&\quad + \gamma[F\{d(\widetilde{z_n}, S\widetilde{z_{n-1}}) + d(\widetilde{z_{n-1}}, T\widetilde{z_n})\}] + \delta \frac{[F\{d(\widetilde{z_n}, S\widetilde{z_{n-1}}) + d(T\widetilde{z_n}, S\widetilde{z_{n-1}})\}]}{1 - d(\widetilde{z_n}, S\widetilde{z_{n-1}}) + d(T\widetilde{z_n}, S\widetilde{z_{n-1}})} \\
&\leq \alpha[F\{d(\widetilde{z_n}, \widetilde{z_{n-1}}) + d(\widetilde{z_{n+1}}, \widetilde{z_n})\}] + \beta[F\{d(\widetilde{z_n}, \widetilde{z_{n+1}}) + d(\widetilde{z_{n-1}}, \widetilde{z_n})] \\
&\quad + \gamma[F\{d(\widetilde{z_n}, \widetilde{z_n}) + d(\widetilde{z_{n-1}}, \widetilde{z_{n+1}})\}] + \delta \frac{F\{d(\widetilde{z_n}, \widetilde{z_n}) + d(\widetilde{z_{n+1}}, \widetilde{z_n})\}}{1 - F\{d(\widetilde{z_n}, \widetilde{z_n}) d(\widetilde{z_{n+1}}, \widetilde{z_n})\}}
\end{aligned}$$

$$\leq \alpha[F\{d(\widetilde{z_n},\widetilde{z_{n-1}})+d(\widetilde{z_{n+1}},\widetilde{z_n})\}]+\beta[F\{d(\widetilde{z_n},\widetilde{z_{n+1}})+d(\widetilde{z_{n-1}},\widetilde{z_n})]+\gamma[F\{d(\widetilde{z_{n-1}},\widetilde{z_{n+1}})\}] + \delta[F\{d(\widetilde{z_{n+1}},\widetilde{z_n})\}]$$
$$\leq \alpha[F\{d(\widetilde{z_n},\widetilde{z_{n-1}})+d(\widetilde{z_{n+1}},\widetilde{z_n})\}]+\beta[F\{d(\widetilde{z_n},\widetilde{z_{n+1}})+d(\widetilde{z_{n-1}},\widetilde{z_n})] +\gamma[F\{d(\widetilde{z_n},\widetilde{z_{n+1}})+d(\widetilde{z_{n-1}},\widetilde{z_n})\}]+\delta[F\{d(\widetilde{z_{n+1}},\widetilde{z_n})\}]$$
$$\leq (\alpha+\beta+\gamma)F[d(\widetilde{z_n},\widetilde{z_{n+1}})+d(\widetilde{z_n},\widetilde{z_{n-1}})]+\delta[F\{d(\widetilde{z_{n+1}},\widetilde{z_n})\}]$$

$$\Longrightarrow F[d(\widetilde{z_{n+1}},\widetilde{z_n})] \leq \left(\frac{\alpha+\beta+\gamma}{1-(\alpha+\beta+\gamma+\delta)}\right)F[d(\widetilde{z_n},\widetilde{z_{n-1}})]$$

Where $\frac{\alpha+\beta+\gamma}{1-(\alpha+\beta+\gamma+\delta)}=L$

Hence $F[d(\widetilde{z_{n+1}},\widetilde{z_n})]= L^n F[d(\widetilde{z_1},\widetilde{z_0})]$

For $n>m$ we have

$$F[d(\widetilde{z_n},\widetilde{z_m})] \leq F[d(\widetilde{z_n},\widetilde{z_{n-1}})+d(\widetilde{z_{n-1}},\widetilde{z_{n-2}})+ \dots\dots\dots\dots\dots + d(\widetilde{z_{m+1}},\widetilde{z_m})$$
$$\leq [L^{n-1}+L^{n-2}+L^{n-3}+L^{n-4}+ \dots\dots\dots\dots + L^m]F\{d(\widetilde{z_1},\widetilde{z_0})\}$$
$$\leq \frac{L^m}{1-L}F\{d(\widetilde{z_1},\widetilde{z_0})\}$$

For a natural number N_1 let $\tilde{c}<0$ such that $\frac{L^m}{1-L}F\{d(\widetilde{z_1},\widetilde{z_0})\} \mathrel{\tilde{<}} \tilde{c}, \forall\, m \geq N_1$.

Thus $d(\widetilde{z_n},\widetilde{z_m}) \mathrel{\tilde{<}} \tilde{c}$ for $n>m$. Therefore $\{\widetilde{z_n}\}$ is a Cauchy sequence in $\tilde{Z}$. Since $(\tilde{Z},d,A)$ is a complete metric space, $\exists\, \widetilde{z^*}\in\tilde{Z}$ such that $\widetilde{z_n}\to\widetilde{z^*}$ as $n\to\infty$. Choose a natural number N_2 such that $F\{d(\widetilde{z_{n+1}},\widetilde{z_n})\}<(1-\tilde{t})\frac{\tilde{c}}{3}$ and $F\{d(\widetilde{z_{n+1}},\widetilde{z^*})\}<(1-\tilde{t})\frac{\tilde{c}}{3}, \forall\, n\geq N_2$. We have

$$F[d(T\widetilde{z^*},\widetilde{z^*})] \leq F[H(T\widetilde{z_n},T\widetilde{z^*})+d(T\widetilde{z_n},\widetilde{z^*})$$
$$\leq \alpha[F\{d(\widetilde{z_n},\widetilde{z^*})+d(T\widetilde{z_n},T\widetilde{z^*})\}]+\beta[F\{d(\widetilde{z_n},T\widetilde{z_n})+d(\widetilde{z^*},T\widetilde{z^*})\}]+\gamma[F\{d(\widetilde{z_n},T\widetilde{z^*})+\ d(T\widetilde{z_n},\widetilde{z^*})\}] +\delta\frac{F\{d(\widetilde{z_n},T\widetilde{z^*})+d(T\widetilde{z_n},T\widetilde{z^*})\}}{1-F\{d(\widetilde{z_n},T\widetilde{z^*})d(T\widetilde{z_n},T\widetilde{z^*})\}}+F[d(T\widetilde{z_n},\widetilde{z^*})]$$
$$\leq \alpha[F\{d(\widetilde{z_n},\widetilde{z^*})+d(\widetilde{z_{n+1}},T\widetilde{z^*})\}]+\beta[F\{d(\widetilde{z_n},\widetilde{z_{n+1}})+d(\widetilde{z^*},T\widetilde{z^*})\}]+\gamma[F\{d(\widetilde{z_n},T\widetilde{z^*})+\ d(\widetilde{z_{n+1}},\widetilde{z^*})\}] +\delta\frac{F\{d(\widetilde{z_n},T\widetilde{z^*})+d(\widetilde{z_{n+1}},T\widetilde{z^*})\}}{1-F\{d(\widetilde{z_n},T\widetilde{z^*})d(\widetilde{z_{n+1}},T\widetilde{z^*})\}}+F[d(\widetilde{z_{n+1}},\widetilde{z^*})]$$
$$\leq \alpha[F\{d(\widetilde{z_n},\widetilde{z^*})+d(\widetilde{z_{n+1}},T\widetilde{z^*})\}]+\beta[F\{d(\widetilde{z_n},\widetilde{z_{n+1}})+d(\widetilde{z^*},T\widetilde{z^*})\}]+\gamma[F\{d(\widetilde{z_n},T\widetilde{z^*})+\ d(\widetilde{z_{n+1}},\widetilde{z^*})\}] +F[d(\widetilde{z_{n+1}},\widetilde{z^*})]$$
$$\leq \alpha[F\{d(\widetilde{z_n},\widetilde{z^*})+d(\widetilde{z_{n+1}},\widetilde{z^*})+d(\widetilde{z^*},T\widetilde{z^*})\}]+\beta[F\{d(\widetilde{z_n},\widetilde{z^*})+d(\widetilde{z^*},\widetilde{z_{n+1}})+d(\widetilde{z^*},T\widetilde{z^*})\}] +\gamma[F\{d(\widetilde{z_n},\widetilde{z^*})+d(\widetilde{z^*},T\widetilde{z^*})+\ d(\widetilde{z_{n+1}},\widetilde{z^*})\}]+F[d(\widetilde{z_{n+1}},\widetilde{z^*})]$$

$$\Rightarrow (1-\tilde{t})F[d(T\widetilde{z^*},\widetilde{z^*})] \leq \tilde{t}[F\{d(\widetilde{z_n},\widetilde{z^*})\}]+\tilde{t}[F\{d(\widetilde{z_{n+1}},\widetilde{z^*})\}]+F[d(\widetilde{z_{n+1}},\widetilde{z^*})]$$
$$\leq [F\{d(\widetilde{z_n},\widetilde{z^*})\}]+[F\{d(\widetilde{z_{n+1}},\widetilde{z^*})\}]+F[d(\widetilde{z_{n+1}},\widetilde{z^*})]$$

Where, $\tilde{t}=\alpha+\beta+\gamma$

$$\Rightarrow F[d(T\widetilde{z^*},\widetilde{z^*})] \leq \frac{\mathrm{F}\{\mathrm{d}(\widetilde{z_n},\widetilde{z^*})\}\ +\ \mathrm{F}\{\mathrm{d}(\widetilde{z_{n+1}},\widetilde{z^*})\}\ +\ \mathrm{F}\{\mathrm{d}(\widetilde{z_{n+1}},\widetilde{z^*})\}}{1-\tilde{t}}$$
$$\leq \frac{\tilde{c}}{3}+\frac{\tilde{c}}{3}+\frac{\tilde{c}}{3}=\tilde{c}, \forall\, n\geq N_1$$

Thus $F[d(T\widetilde{z^*},\widetilde{z^*})]\leq\frac{\tilde{c}}{m}, \forall\, m\geq 1$, So, $\frac{\tilde{c}}{m}-F[d(T\widetilde{z^*},\widetilde{z^*})]\in(P,A), \forall m\geq 1$. Since $\frac{\tilde{c}}{m}\to 0$ as $m\to\infty$ and (P,A) is closed, $-F\{d(T\widetilde{z^*},\widetilde{z^*})\}\in(P,A)$. But, $d(T\widetilde{z^*},\widetilde{z^*})\in(P,A)$. Therefore, $d(T\widetilde{z^*},\widetilde{z^*})\in(P,A)=0$ and so, $T\widetilde{z^*}=\widetilde{z^*}$. Now if $\widetilde{z^{**}}$ is another soft fixed point of T. Then $F[d\{\widetilde{z^*},\widetilde{z^{**}})]=F[d(T\widetilde{z^*},T\widetilde{z^{**}})]\leq t[d(T\widetilde{z^*},\widetilde{z^*})+d(T\widetilde{z^{**}},\widetilde{z^{**}})]=0$.

Hence $\widetilde{z^*}\in T\widetilde{z^*}=\widetilde{z^*}=\widetilde{z^*}\in T\widetilde{z^{**}}$. Therefore, $\widetilde{z^*}$ is a soft fixed point of T.

Similarly, it can be established that $\widetilde{z^*}\in\ T\widetilde{z^*}=\widetilde{z^*}=\widetilde{z^*}\in S\widetilde{z^*}$. Thus $\widetilde{z^*}$ is the common soft fixed point of T and S.

Theorem 4.2: Let $(\tilde{Z},d,A)$ be a soft cone metric space and let $T,S:(\tilde{Z},d,A)\to CB(\tilde{Z})$ be any two multivalued mappings satisfying $\tilde{y},\tilde{z}\in\tilde{Z}$,

$$F[H(T\tilde{z},S\tilde{y})]\leq q\ max\ [F\{d(\tilde{z},\tilde{y}),d(\tilde{z},T\tilde{z}),d(\tilde{y},S\tilde{y}),d(T\tilde{z},S\tilde{y})\}]$$

$\forall\tilde{y},\tilde{z}\in\tilde{Z}, and\ q\in[0,1]$.

Then $T\ and\ S$ has a common fixed soft element in $\tilde{Z}$. For each $\tilde{y},\tilde{z}\in\tilde{Z}$, the iterative sequence $\{T^n\,\tilde{z}\}$ converges to the fixed soft element.

Proof: for each $\tilde{z_0} \in \tilde{Z}$ and $n \geq 1$, $\tilde{z_0} \in T(\tilde{z_0})$,………, $\widetilde{z_{n+1}} \in T(\widetilde{z_n})$ by iteration method, we have a sequence $\{\widetilde{z_n}\}$ of soft elements in $\tilde{Z}$ by letting $\tilde{z_1} = T\tilde{z_0}, \tilde{z_2} = T\tilde{z_1} = T^2\tilde{z_0}, \dots\dots\dots, \widetilde{z_{n+1}} = T\widetilde{z_n} = T^{n+1}\tilde{z_0}, \dots\dots$
Then,

$$\begin{aligned} F[d(\widetilde{z_{n+1}}, \widetilde{z_n})] &\leq F[H(T\widetilde{z_n}, S\widetilde{z_{n-1}})] \\ &\leq q \max [F\{d(\widetilde{z_n}, \widetilde{z_{n-1}}), d(\widetilde{z_n}, T\widetilde{z_n}), d(\widetilde{z_{n-1}}, S\widetilde{z_{n-1}}), d(T\widetilde{z_n}, S\widetilde{z_{n-1}})\}] \\ &\leq q\, max[F\{d(\widetilde{z_n}, \widetilde{z_{n-1}}), d(\widetilde{z_n}, \widetilde{z_{n+1}}), d(\widetilde{z_{n-1}}, \widetilde{z_n}), d(\widetilde{z_{n+1}}, \widetilde{z_n})\}] \\ &\leq q\, max[F\{d(\widetilde{z_{n-1}}, \widetilde{z_n}), d(\widetilde{z_{n+1}}, \widetilde{z_n})\}] \\ &\leq q[F\{d(\widetilde{z_n}, \widetilde{z_{n-1}})\}] \end{aligned}$$

$$\Rightarrow F[d(\widetilde{z_{n+1}}, \widetilde{z_n})] \leq q^n F[d(\tilde{z_1}, \tilde{z_0})]$$

For $n > m$ we have

$$\begin{aligned} F[d(\widetilde{z_n}, \widetilde{z_m})] &\leq F[d(\widetilde{z_n}, \widetilde{z_{n-1}}) + d(\widetilde{z_{n-1}}, \widetilde{z_{n-2}}) + \dots\dots\dots\dots + d(\widetilde{z_{m+1}}, \widetilde{z_m}) \\ &\leq F[q^{n-1} + q^{n-2} + q^{n-3} + q^{n-4} + \dots\dots\dots + q^m]F[d(\tilde{z_1}, \tilde{z_0})] \\ &\leq \frac{q^m}{1-q} F[d(\tilde{z_1}, \tilde{z_0})] \end{aligned}$$

For a natural number N_1 let $\tilde{c} < 0$ such that $\frac{q^m}{1-q} d(\tilde{z_1}, \tilde{z_0}) < \tilde{c}, \forall\, m \geq N_1$.

Thus $d(\widetilde{z_n}, \widetilde{z_m}) < \tilde{c}$ for $n > m$. Therefore $\{\widetilde{z_n}\}$ is a Cauchy sequence in $\tilde{Z}$. Since $(\tilde{Z}, d, A)$ is a complete soft metric space, $\exists\, \widetilde{z^*} \in \tilde{Z}$ such that $\widetilde{z_n} \to \widetilde{z^*}$ as $n \to \infty$. Choose a natural number N_2 such that $d(\widetilde{z_{n+1}}, \widetilde{z_n}) < (1 - \tilde{t})\frac{\tilde{c}}{3}$ and $d(\widetilde{z_{n+1}}, \widetilde{z^*}) < (1 - \tilde{t})\frac{\tilde{c}}{3}, \forall\, n \geq N_2$. We have

$$\begin{aligned} F[d(T\widetilde{z^*}, \widetilde{z^*})] &\leq F[H(T\widetilde{z_n}, T\widetilde{z^*}) + d(T\widetilde{z_n}, \widetilde{z^*}) \\ &\leq q\, max[F\{d(\widetilde{z_n}, \widetilde{z^*}), d(\widetilde{z_n}, T\widetilde{z_n}), d(\widetilde{z^*}, T\widetilde{z^*}), d(T\widetilde{z_n}, T\widetilde{z^*})\}] + F[d(T\widetilde{z_n}, \widetilde{z^*})] \\ &\leq q\, max\, [F\{d(\widetilde{z_n}, \widetilde{z^*}), d(\widetilde{z_n}, \widetilde{z_{n+1}}), d(\widetilde{z^*}, T\widetilde{z^*}), d(\widetilde{z_{n+1}}, T\widetilde{z^*})\}] + F[d(\widetilde{z_{n+1}}, \widetilde{z^*})] \\ &\leq q\, max\, [F\{d(\widetilde{z_n}, \widetilde{z^*}), d(\widetilde{z_n}, \widetilde{z^*})\} + F\{d(\widetilde{z^*}, \widetilde{z_n}) d(\widetilde{z^*}, T\widetilde{z^*})\}] \end{aligned}$$

$\Rightarrow F[d(T\widetilde{z^*}, \widetilde{z^*})] \leq \tilde{c}, \forall\, n \geq N_1]$. Thus $F[d(T\widetilde{z^*}, \widetilde{z^*})] \leq \frac{\tilde{c}}{m}, \forall\, m \geq 1$, So, $\frac{\tilde{c}}{m} - F[d(T\widetilde{z^*}, \widetilde{z^*})] \in (P, A), \forall m \geq 1$. Since $\frac{\tilde{c}}{m} \to 0$ as $m \to \infty$ and (P, A) is closed, $-d(T\widetilde{z^*}, \widetilde{z^*}) \in (P, A)$. But, $d(T\widetilde{z^*}, \widetilde{z^*}) \in (P, A)$. Therefore, $d(T\widetilde{z^*}, \widetilde{z^*}) \in (P, A) = 0$ and so, $T\widetilde{z^*} = \widetilde{z^*}$. Now if $\widetilde{z^{**}}$ is another soft fixed point of T. Then $F[d(\widetilde{z^*}, \widetilde{z^{**}})] = F[d(T\widetilde{z^*}, T\widetilde{z^{**}})] \leq t[d(T\widetilde{z^*}, \widetilde{z^*}) + d(T\widetilde{z^{**}}, \widetilde{z^*})] = 0$.

Hence $\widetilde{z^*} \in T\widetilde{z^*} = \widetilde{z^*} = \widetilde{z^*} \in T\widetilde{z^{**}}$. Therefore, $\widetilde{z^*}$ is a soft fixed point of T.

Similarly, it can be established that $\widetilde{z^*} \in T\widetilde{z^*} = \widetilde{z^*} = \widetilde{z^*} \in S\widetilde{z^*}$. Thus $\widetilde{z^*}$ is the common soft fixed point of T and S.

CONCLUSION

In this paper the concept of soft cone metric spaces via soft element is introduced and the work has been done on the soft convergence of soft sequences and soft Cauchy sequences in such spaces. Some fixed point theorems of contractive multivalued mappings on soft cone metric spaces has been discussed. There is ample scope for further research on soft cone metric spaces. This paper is a basis to works on the above mentioned ideas.

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