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VERTEX COVER POLYNOMIAL OF $K_n \times K_r$

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ABSTRACT

The vertex cover Polynomial of a graph G of order n has been already introduced in [3]. It is defined as the polynomial, $C(G, i) = \sum_{i=\beta(G)}^{|v(G)|} c(G, i)x^i$ *, where c (G, i) is the number of vertex covering sets of G of size i and β(G) is the covering number of G. In this paper, we derived a formula for finding the vertex cover polynomial of* $K_n \times K_r$*. Aslo we proved that* x^{r-rn} *[*$C(K_n \times K_r, x)$*] is log concave.*

Key words: *Vetex covering set, vertex covering number, vertex cover polynomial.*

1. INTRODUCTION

Let G = (V, E) be a simple graph. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V/uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subset V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subset V$ is a vertex covering of G if every edge $uv \in E$ is adjacent to at least one vertex in S. The vertex covering number β(G) is the minimum cardinality of the vertex covering sets in G. A vertex covering set with cardinality β(G) is called a β - set. Let C (G , i) be the family of vertex covering sets of G with cardinality i and let $c(G, i) = |C(G, i)|$. The polynomial, $C(G, x) = \sum_{i=\beta(G)}^{|v(G)|} c(G, i)x^i$, is defined as the vertex cover polynomial of G. In [3], many properties of the vertex cover polynomials have been studied.

Theorem 2.1: The vertex cover polynomial of $K_n \times K_r$ is

$$C(K_n \times K_r, x) = \sum_{i=0}^{r} rC_{r-i} \frac{n!}{i!} x^{rn-r+i}.$$

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Proof:

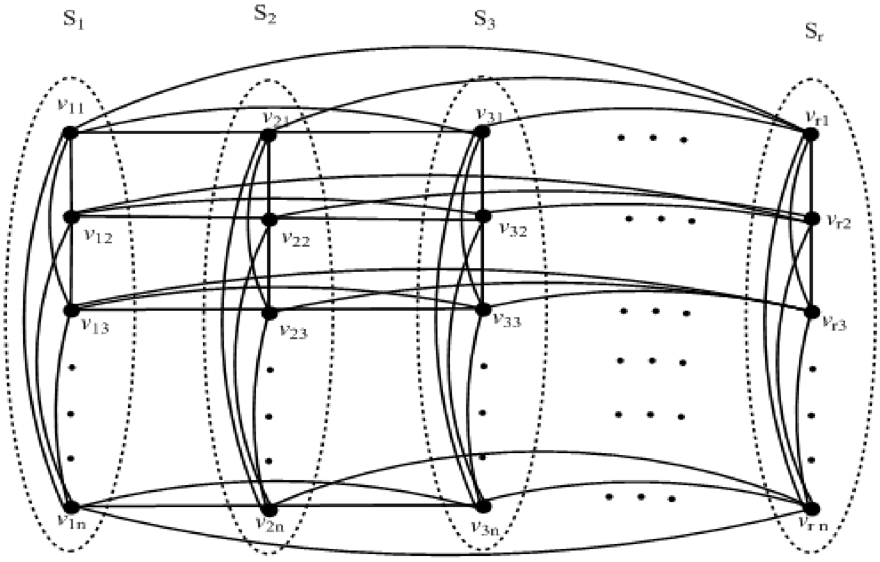

Figure: 1

Let the vertices of $G = K_n \times K_r$ are denoted by

$\{v_{11}, v_{12}, v_{13}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{r1}, v_{r2}, \ldots, v_{rn}\}$

Now the vertices of G can be partitioned into r sets are denoted by $S_1, S_2, \ldots, S_r$ where

$S_1 = \{v_{11}, v_{12}, v_{13}, \ldots, v_{1n}\}$
$S_2 = \{v_{21}, v_{22}, v_{23}, \ldots, v_{2n}\}$
$S_3 = \{v_{31}, v_{32}, v_{33}, \ldots, v_{3n}\}$
. . . .
. . . .
. . . .
$S_r = \{v_{r1}, v_{r2}, v_{r3}, \ldots, v_{rn}\}$

Now each sub graph H_i of G consists the vertices of S_i, i = 1, ...r is complete sub graph with n-vertices. That is the graph G contains n complete sub graphs Q_i, i =1,… , n whose vertices are

$Q_1 = \{v_{11}, v_{21}, v_{31}, \ldots, v_{r1}\}$
$Q_2 = \{v_{12}, v_{22}, v_{32}, \ldots, v_{r2}\}$
$Q_3 = \{v_{13}, v_{23}, v_{33}, \ldots, v_{r3}\}$
. . . .
. . . .
. . . .
$Q_n = \{v_{in}, v_{2n}, v_{3n}, \ldots, v_{rn}\}$

Since each sub graph of G containing the vertices of S_i are complete, the maximum independent set of G with cardinality of r elements are as follows. Let us take the element $v_{11} \in S_1$, each element $\{v_{2j}\}$, j = 2, 3, . . . , n $\in S_2$ are independent to element $v_{11} \in S_1$. For the fixed element v_{11}, n–1 chances to select one element from S_2 which is independent to v_{11}. Suppose we select v_{11} and v_{22} be the first two elements of our maximum independent set from S_1 and S_2, the selected elements $v_{11} \in S_1$ and $v_{22} \in S_2$ are adjacent with v_{31} and v_{32} in S_3 respectively.

Since H_3 is complete, the third element in our independent set from S_3 which is independent to v_{11}, v_{22} are other than the elements of $v_{31}, v_{32} \in S_3$. Therefore. n – 2 choices to select one element from S_3 which are independent to v_{11} and v_{22}. Similarly, the number of choices to select independent sets to the fixed vertex $v_{11} \in S_1$ are

$$(n-1)(n-2)(n-3)\ldots(n-\overline{r-1}).$$

Therefore, for all the elements of S_1, the number of maximum independent sets with cardinality r are $n(n-1)(n-2)\ldots(n-\overline{r-1})$.

It is equal to the number of minimum covering sets with cardinality r n – r.

Therefore,

$$c(G, rn-r) = n(n-1)(n-2)\ldots(n-\overline{r-1}).$$

To find the number of independent sets with cardinality r – 1, since each sub graph G_i, i = 1, . . . , r is complete, we can choose independent set containing r – 1 elements from any r – 1 sub graph of G_i, i = 1, 2, . . . , r. From r – 1 sub graphs G_i of G can be chosen in rC_{r-1} ways. Let $S_1, S_2, \ldots, S_{r-1}$ be the r – 1 sub graphs of G, then a fixed vertex $v_{1i} \in S_1$, i = 1, ..., n the vertices $v_{2j} \in S_2$, j = 1, . . . , n; $i \neq j$ are independent $v_{1i} \in S_1$.

Similarly for the fixed vertices $v_{1i} \in S_1$ and $v_{2j} \in S_2$, we can choose $v_{3k} \in S_3$, $i \neq j \neq k$; k =1, 2, . . . ,n which are independent to both $v_{1i} \in S_1$ and $v_{2j} \in S_2$ proceeding this way (n – 1) (n – 2) (n – 3) . . . (n - $\overline{r-2}$) choices to select an independent set, of the fixed vertex $v_{1i} \in S_1$. Therefore, for all vertices $v_{1i} \in S_1$, the number of choices are

$$n(n-1)(n-2)\ldots(n-\overline{r-2}).$$

Therefore, the total number of independent sets with cardinality

r –1 are $rC_{r-1} \cdot n(n-1)(n-2)\ldots(n-\overline{r-2})$.

Therefore, the covering sets with cardinality r n – (r – 1) are

$$c(G, rn-\overline{r-1}) = rC_{r-1} \cdot n \cdot (n-1)(n-2)\ldots(n-\overline{r-2})$$

The same procedure for the number of independent sets with cardinality r – 2 is, among r sets we can select r – 2 sets in rC_{r-2} ways and for a fixed element in one set, independent sets with cardinality r – 2 are (n – 1) (n – 2) (n – 3) . . . (n – $\overline{r-3}$).

Therefore, for all n elements to a fixed set S_i the number of independent set with cardinality r – 2 are

$$(n-1)(n-2)(n-3)\ldots(n-\overline{r-3})$$

Therefore, the total number of independent sets with cardinality (r – 2) of G is same as the covering sets with cardinality r n – $\overline{r-2}$.

That is, $c(G, rn-\overline{r-2}) = rC_{r-2} \cdot n(n-1)(n-2)\ldots(n-\overline{r-2})$

Similarly,

$$c(G, rn-\overline{r-3}) = rC_{r-3} \cdot n(n-1)(n-2)\ldots(n-\overline{r-4})$$

$$\vdots$$

$$c(G, rn-(r-\overline{r-2})) = rC_{r-\overline{r-2}} \cdot n(n-1)$$

$$c(G, rn-(r-\overline{r-1})) = rC_{r-\overline{r-1}} \cdot n \quad \text{and}$$

$$c(G, rn-r+r) = rC_{r-r}.$$

Therefore,

$$c(G, rn-r) = n(n-1)(n-2)\ldots(n-\overline{r-1})$$

$$c(G, rn-r+1) = rC_{r-1} \cdot n(n-1)(n-2)\ldots(n-\overline{r-2})$$

$$c(G, rn-r+2) = rC_{r-2} \cdot n(n-1)(n-2)\ldots(n-\overline{r-3})$$

$$c(G, rn-r+3) = rC_{r-3} \cdot n(n-1)(n-2)\ldots(n-\overline{r-4})$$

$$\vdots$$

$$c(G, rn-2) = rC_2 \; n(n-1)$$

$$c(G, rn-1) = rC_1 \; n$$

$$c(G, rn) = rC_0 \cdot nC_0$$

Therefore the vertex cover polynomial

$$\begin{aligned} C(G, x) = {} & n(n-1)(n-2)\ldots(n-\overline{r-1})\, x^{rn-r} \\ & + rC_{r-1}\, n(n-1)(n-2)\ldots(n-\overline{r-2})\, x^{rn-r+1} \\ & + rC_{r-2}\, n(n-1)(n-2)\ldots(n-\overline{r-3})\, x^{rn-r+2} \\ & + rC_{r-3}\, n(n-1)(n-2)\ldots(n-\overline{r-4})\, x^{rn-r+3} + \ldots + \end{aligned}$$

$$+ \; rC_{r - \overline{r-2}} \, .n\,(n-1)\, x^{rn - (r - \overline{r-2})} + rC_{r - \overline{r-1}} \, .n \; x^{rn - (r - \overline{r-1})}$$

$$+ rC_{r-r}\, nC_o . x^{rn}.$$

$$C(G, x) = \sum_{i=0}^{r} rC_{r-i} \frac{n!}{(n-r+i)!} x^{rn - r + i} \qquad \text{(A)}$$

Corollary 2.2: The vertex cover polynomial of $K_n \times K_n$ is

$$C(K_n \times K_n, x) = \sum_{i=0}^{n} nC_{n-i} \frac{n!}{i!} x^{n^2 - n + i}$$

Proof: Put $r = n$ in equation (A) we get

$$C(K_n \times K_n, x) = \sum_{i=0}^{n} nC_{n-i} \frac{n!}{i!} x^{n^2 - n + i}$$

Theorem 2.3: The vertex cover polynomial of $K_n \times K_2$ satisfies the following identities

(i) $c(K_n \times K_2,\ 2n-2) = n(n-1)$
(ii) $c(K_n \times K_2,\ 2n-1) = 2n$
(iii) $c(K_{n+1} \times K_2,\ 2n) = c(K_n \times K_2,\ 2n-2) + c(K_n \times K_2,\ 2n-1)$

Proof:

(i) From equation (A)

$$C(G, x) = \sum_{i=0}^{r} rC_{r-i} \frac{n!}{(n-r+i)!} x^{rn - r + i}$$

Put $r = 2$ we get

$$\text{R.H.S} = 2C_2 \cdot \frac{n!}{(n-2)!} \cdot x^{2n-2} + 2C_1 \cdot \frac{n!}{(n-1)!} \cdot x^{2n-1} + 2C_0 \cdot \frac{n!}{n!} x^{2n} \qquad (*)$$

$$\therefore c(K_n \times K_2,\ 2n-2) = \frac{n!}{(n-2)!}$$

$$= \frac{n(n-1)\,(n-2)!}{(n-2)!}$$

$$= n(n-1)$$

(ii) From equation (A)

$$c(K_n \times K_2,\ 2n-1) = 2C_1 . \frac{n!}{(n-1)!}$$

$$= 2 . \frac{n(n-1)\,!}{(n-1)!}$$

$$= 2n$$

(iii) From equation (A)

$$c(K_{n+1} \times K_2,\ x) = \sum_{i=0}^{r} \frac{(n+1)!}{(n+1-r+i)!} x^{r(n+1) - r + i}$$

$$= \sum_{i=0}^{r} \frac{(n+1)!}{(n+1-r+i)!} x^{rn + i}$$

Put $r = 2$

$$c(K_{n+1} \times K_2,\ x) = \frac{(n+1)!}{(n-1)!} x^{2n} + \frac{(n+1)!}{n!} x^{2n+1} + \frac{(n+1)!}{(n+1)!} x^{2n+2} \qquad (**)$$

From (**)

$$c(K_{n+1} \times K_n, 2n) = \frac{(n+1)!}{(n-1)!}$$
$$= \frac{(n+1)\,(n)\,(n-1)\,!}{(n-1)!}$$
$$= n\,(n+1)$$
$$= n\,(n+2-1)$$
$$= n\,(n-1) + 2n$$
$$= C(K_n \times K_2,\ 2n-2) + C(K_n \times K_2,\ 2n)\,.$$

Theorem 2.4: The vertex cover polynomial $x^{\,r-r\,n}\,[C(K_n \times K_r,\ x)]$ is log-concave.

Proof: By (A), $C(G, x) = \sum_{i=0}^{r} rC_{r-i}\,\frac{n!}{(n-r+i)!}\,x^{r\,n-r+i}$

We prove this result on induction.

When $r = 2$ and $i = 0, 1, 2$

We have to prove

$$[c(K_n \times K_2, r\,n-r+1)]^2 \geq c[K_n \times K_2, r\,n-r] \times c[K_n \times K_2, r\,n-r+2]$$

$$\text{R.H.S} = c[K_n \times K_2,\ r\,n-r] \times c[K_n \times K_2, r\,n-r+2]$$
$$= rC_r\,\frac{n!}{(n-r)!}\,rC_{r-2}\,\frac{n!}{(n-r+2)!}$$
$$= \frac{n\,!}{(n-2)!}\cdot\frac{n!}{n!} \quad [\because r = 2]$$
$$= n(n-1) \tag{1}$$

$$\text{L.H.S} = [c(K_n \times K_2, r\,n-r+1)]^2 = \left[rC_{r-1}\,\frac{n!}{(n-r+1)!}\right]^2$$
$$= \left[r\,.\,\frac{n!}{(n-1)!}\right]^2 \qquad [\because r = 2]$$
$$= (r\,n)^2$$
$$= r^2\,n^2 \tag{2}$$

$$(2)/(1) \Rightarrow \frac{[c(K_n \times K_n, r\,n-r+1)]^2}{c(K_n \times K_n, r\,n-r)\,(c(K_n \times K_2)\,, r\,n-r+2)} = \frac{r^2\,n^2}{n(n-1)} \quad \text{for every } n > 1$$

Therefore, $[c(K_n \times K_n, r\,n-r+1)]^2 \geq [c(K_n \times K_n, r\,n-r)\,.\,c(K_n \times K_2, r\,n-r+2)$

Assume the result is true for all $r < n$ and prove $r = n$.

Case-(i): $r = n$; $i = 0, 1, 2$

We have to prove

$$[c(K_n \times K_n, n^2-n+1)]^2 \geq c(K_n \times K_n, n^2-n)\,.\,c(K_n \times K_n, n^2-n+2)$$

$$\text{R.H.S} = c(K_n \times K_n, n^2-n)\,.\,c(K_n \times K_n, n^2-n+2)$$
$$= nC_n\,.\,\frac{n!}{0!}\,.\,nC_{n-2}\,\frac{n!}{2!}$$
$$= n!\,.\,nC_2\,.\,\frac{n!}{2!} \tag{3}$$

$$\text{L.H.S.} = [c(K_n \times K_n, n^2 - n + 1)]^2 = \left[nC_{n-1} \frac{n!}{1!} \right]^2$$

$$= n^2 . (n!)^2 \tag{4}$$

$$(4) / (3) \Rightarrow \quad \frac{n^2 (n!)^2}{n!\ nC_2\ n!} \ .\ 2! = \frac{2!\ n^2\ 2!}{n(n - 1)}$$

$$= \frac{4n}{n - 1} > 1 \text{ for all } \quad n > 1$$

$$\Rightarrow [c(K_n \times K_n, n^2 - n + 1)]^2 \geq c(K_n \times K_n, n^2 - n) . c(K_n \times K_n, n^2 - n + 2)$$

Assume the result is true for $i < k$ and prove for $i = k$

That is for prove,

$$[c(K_n \times K, n^2 - n + k)]^2 \geq c(K_n \times K_n, n^2 - n + k-1) . c(K_n \times K_n, n^2 - n + k + 1)$$

$$\text{R.H.S} = c(K_n \times K_n, n^2 - n + k-1) . c(K_n \times K_n, n^2 - n + k+1)$$

$$= nC_{n-(k-1)} \frac{n\ !}{(k-1)!} .\ nC_{n-(k+1)} \frac{n\ !}{(k+1)!} \quad \text{where } k + 1 \leq n.$$

$$= nC_{n-(k-1)} \frac{n\ !}{(k-1)!} .\ nC_{n-(k+1)} \frac{n\ !}{(k+1)!}$$

$$= \{nC_{k-1} . n(n-1) . . . (n-k+2)\} \{nC_{k+1} . n(n-1) . . . (n-k)\} \tag{5}$$

$$\text{L.H.S.} = [c(K_n \times K_n, n^2 - n + k)]^2 = \left[nC_{n-k} \frac{n!}{k!} \right]^2$$

$$= [nC_k . n(n-1) . (n-k+1)]^2 \tag{6}$$

$$(6) / (5) \Rightarrow \frac{[c(K_n \times K_n, n^2 - n + k)]^2}{c(K_n \times K_n, n^2 - n + k - 1)\ c(K_n \times K_n, n^2 - n + k + 1)}$$

$$= \frac{[nC_k\ .\ n(n-1)\ .\ .\ .\ (n-k+1)]^2}{\{nC_{k-1}\ .\ n(n-1)\ .\ .\ .\ (n\text{-}k+2)\}\ .\ \{nC_{k+1}\ .\ n(n-1)\ .\ .\ .\ (n-k)\}}$$

$$= \frac{nC_k . n(n-1)...(n-k+1).nC_k n(n-1)\ .\ .\ .\ (n-k+1)}{nC_{k-1}.n(n-1)...(n-k+2).nC_{k+2}.n(n-1)\ .\ .\ .\ (n-k)}$$

$$= \frac{nC_k . n(n-k+1)}{nC_{k-1}} \times \frac{nC_k}{nC_{k+1}(n-k)}$$

$$= \frac{(nC_k)^2}{nC_{k-1} .\ nC_{k+1}} \times \left(\frac{n-k+1}{n-k} \right) \geq 1 \text{ for every } n > k.$$

That is

$$\Rightarrow [c(K_n \times K_n, n^2 - n + k)]^2 \geq c(K_n \times K_n, n^2 - n + k - 1) . c(K_n \times K_n, n^2 - n + k + 1)$$

The result is true for all i.

Therefore, $x^{r - rn} [C(K_n \times K_n, x)]$ is log-concave.

Hence the proof.

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