VERTEX COVER POLYNOMIAL OF $K_n \times K_r$

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(Received On: 16-08-17; Revised & Accepted On: 11-09-17)

ABSTRACT

The vertex cover Polynomial of a graph $G$ of order $n$ has been already introduced in [3]. It is defined as the polynomial, 

$$C(G, x) = \sum_{i=0}^{\beta(G)} c(G, i) x^i,$$ 

where $c(G, i)$ is the number of vertex covering sets of $G$ of size $i$ and $\beta(G)$ is the covering number of $G$. In this paper, we derived a formula for finding the vertex cover polynomial of $K_n \times K_r$. Also we proved that $x^{r-n} [C(K_n \times K_r, x)]$ is log concave.

Key words: Vertex covering set, vertex covering number, vertex cover polynomial.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V/uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a vertex covering of $G$ if every edge $uv \in E$ is adjacent to at least one vertex in $S$. The vertex covering number $\beta(G)$ is the minimum cardinality of the vertex covering sets in $G$. A vertex covering set with cardinality $\beta(G)$ is called a $\beta$-set. Let $C(G, i)$ be the family of vertex covering sets of $G$ with cardinality $i$ and let $c(G, i) = |C(G, i)|$. The polynomial, $C(G, x) = \sum_{i=0}^{\beta(G)} c(G, i) x^i$, is defined as the vertex cover polynomial of $G$. In [3], many properties of the vertex cover polynomials have been studied.

Theorem 2.1: The vertex cover polynomial of $K_n \times K_r$ is 

$$C(K_n \times K_r, x) = \sum_{i=0}^{r} rC_{r-i} \frac{n!}{i!} x^{r-i} x^{n-r+i}.$$ 

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Proof:

Let the vertices of $G = K_n \times K_r$ are denoted by
\[ \{v_{11}, v_{12}, v_{13}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{r1}, v_{r2}, \ldots, v_{rn}\} \]

Now the vertices of $G$ can be partitioned into $r$ sets are denoted by $S_1, S_2, \ldots, S_r$ where
\[
S_1 = \{v_{11}, v_{12}, v_{13}, \ldots, v_{1n}\} \\
S_2 = \{v_{21}, v_{22}, v_{23}, \ldots, v_{2n}\} \\
S_3 = \{v_{31}, v_{32}, v_{33}, \ldots, v_{3n}\} \\
\vdots \\
S_r = \{v_{r1}, v_{r2}, v_{r3}, \ldots, v_{rn}\}
\]

Now each sub graph $H_i$ of $G$ consists the vertices of $S_i$, $i = 1, \ldots, r$ is complete sub graph with $n$-vertices. That is the graph $G$ contains $n$ complete sub graphs $Q_i$, $i = 1, \ldots, n$ whose vertices are
\[
Q_1 = \{v_{11}, v_{21}, v_{31}, \ldots, v_{r1}\} \\
Q_2 = \{v_{12}, v_{22}, v_{32}, \ldots, v_{r2}\} \\
Q_3 = \{v_{13}, v_{23}, v_{33}, \ldots, v_{r3}\} \\
\vdots \\
Q_n = \{v_{1n}, v_{2n}, v_{3n}, \ldots, v_{rn}\}
\]

Since each sub graph of $G$ containing the vertices of $S_i$ are complete, the maximum independent set of $G$ with cardinality of $r$ elements are as follows. Let us take the element $v_{11} \in S_1$, each element $v_{j1}, j = 2, 3, \ldots, n \in S_2$ are independent to element $v_{11} \in S_1$. For the fixed element $v_{11}$, $n - 1$ chances to select one element from $S_2$ which is independent to $v_{11}$. Suppose we select $v_{11}$ and $v_{22}$ be the first two elements of our maximum independent set from $S_1$ and $S_2$, the selected elements $v_{11} \in S_1$ and $v_{22} \in S_2$ are adjacent with $v_{31}$ and $v_{32}$ in $S_3$ respectively.

Since $H_2$ is complete, the third element in our independent set from $S_3$ which is independent to $v_{11}$, $v_{22}$ are other than the elements of $v_{31}, v_{32} \in S_3$. Therefore, $n - 2$ choices to select one element from $S_3$ which are independent to $v_{11}$ and $v_{22}$. Similarly, the number of choices to select independent sets to the fixed vertex $v_{11} \in S_1$ are
\[
(n - 1)(n - 2)(n - 3) \ldots (n - r - 1).
\]

Therefore, for all the elements of $S_1$, the number of maximum independent sets with cardinality $r$ are $n (n - 1)(n - 2) \ldots \, (n - r - 1)$.

It is equal to the number of minimum covering sets with cardinality $r n - r$.

Therefore,
\[
c(G, r n - r) = n(n - 1)(n - 2) \ldots (n - r - 1).
\]
To find the number of independent sets with cardinality \( r - 1 \), since each sub graph \( G_i, i = 1, \ldots, r \) is complete, we can choose independent set containing \( r - 1 \) elements from any \( r - 1 \) sub graph of \( G_i, i = 1, 2, \ldots, r \). From \( r - 1 \) sub graphs \( G_i \) of \( G \) can be chosen in \( \binom{r}{r-1} \) ways. Let \( S_1, S_2, \ldots, S_r \) be the \( r - 1 \) sub graphs of \( G \), then a fixed vertex \( v_{i_1} \in S_i \), \( i = 1, \ldots, n \) the vertices \( v_{j_2} \in S_2, j = 1, \ldots, n \); \( i \neq j \) are independent \( v_{i_1} \in S_i \).

Similarly for the fixed vertices \( v_{i_1} \in S_1 \) and \( v_{j_2} \in S_2 \), we can choose \( v_{k_3} \in S_3, i \neq j \neq k \); \( k = 1, 2, \ldots, n \) which are independent to both \( v_{i_1} \in S_1 \) and \( v_{j_2} \in S_2 \) proceeding this way \( n( n - 1 ) (n - 2) \ldots (n - 2) \) choices to select an independent set, of the fixed vertex \( v_{i_1} \in S_1 \). Therefore, for all vertices \( v_{i_1} \in S_1 \), the number of choices are

\[
\binom{n}{n-1} (n-2) \ldots (n-r) = n( n - 1 ) (n - 2) \ldots (n - r).
\]

Therefore, the total number of independent sets with cardinality \( r - 1 \) are

\[
\binom{r}{r-1} \cdot n(n-1)(n-2) \ldots (n-r).
\]

Therefore, the covering sets with cardinality \( r n - (r - 1) \) are

\[
c(G, r n - (r - 1)) = \binom{r}{r-1} \cdot n(n-1)(n-2) \ldots (n-r).
\]

The same procedure for the number of independent sets with cardinality \( r - 2 \) is, among \( r \) sets we can select \( r - 2 \) sets in \( \binom{r}{r-2} \) ways and for a fixed element in one set, independent sets with cardinality \( r - 2 \) are \( (n - 1) (n - 2) (n - 3) \ldots (n - 4) \) choices to select an independent set, of the fixed vertex \( v_{i_1} \in S_1 \). Therefore, for all vertices \( v_{i_1} \in S_1 \), the number of choices are

\[
(n - 1)(n - 2)(n - 3) \ldots (n - 4).
\]

Similarly, \( c(G, r n - (r - 2)) = \binom{r}{r-2} \cdot n(n-1)(n-2) \ldots (n-r) \). Therefore, the total number of independent sets with cardinality \( r - 2 \) of \( G \) is same as the covering sets with cardinality \( r n - (r - 2) \).

That is, \( c(G, r n - (r - 2)) = \binom{r}{r-2} \cdot n(n-1)(n-2) \ldots (n-r) \)

Similarly,

\[
\binom{r}{r-3} \cdot n(n-1)(n-2) \ldots (n-r).
\]

Therefore,

\[
c(G, r n - r) = \binom{r}{r-1} \cdot n(n-1)(n-2) \ldots (n-r-1)
\]

\[
c(G, r n - r + 1) = \binom{r}{r-1} \cdot n(n-1)(n-2) \ldots (n-r-2)
\]

\[
c(G, r n - r + 2) = \binom{r}{r-1} \cdot n(n-1)(n-2) \ldots (n-r-3)
\]

\[
c(G, r n - r + 3) = \binom{r}{r-1} \cdot n(n-1)(n-2) \ldots (n-r-4)
\]

\[
\vdots
\]

\[
c(G, r n - (r - 1)) = \binom{r}{r-1} \cdot n(n-1)(n-2) \ldots (n-r)
\]

\[
c(G, r n - (r)) = \binom{r}{r} \cdot n
\]

\[
c(G, r n) = \binom{r+1}{r}
\]

Therefore the vertex cover polynomial

\[
C(G,x) = n(n-1)(n-2) \ldots (n-r-1) x^{r n - r} + \binom{r}{r-1} n(n-1)(n-2) \ldots (n-r-2) x^{r n - r-1} + \binom{r}{r-2} n(n-1)(n-2) \ldots (n-r-3) x^{r n - r-3} + \ldots + \binom{r}{r} n x^{r n - r}.
\]
Corollary 2.2: The vertex cover polynomial of $K_n \times K_n$ is

$$C(K_n \times K_n, x) = \sum_{i=0}^{n} nC_{n-i} \frac{n!}{i!} x^{n^2 - n + i}$$

Proof: Put $r = n$ in equation (A) we get

$$C(K_n \times K_n, x) = \sum_{i=0}^{n} nC_{n-i} \frac{n!}{i!} x^{n^2 - n + i}$$

Theorem 2.3: The vertex cover polynomial of $K_n \times K_2$ satisfies the following identities

(i) $c(K_n \times K_2, 2n - 2) = n(n - 1)$

(ii) $c(K_n \times K_2, 2n - 1) = 2n$

(iii) $c(K_n+1 \times K_2, 2n) = c(K_n \times K_2, 2n - 2) + c(K_n \times K_2, 2n - 1)$

Proof:

(i) From equation (A)

$$C(G, x) = \sum_{i=0}^{r} rC_{r-i} \frac{n!}{(n-r+i)!} x^{r(n-r+i)}$$

Put $r = 2$ we get

$$R.H.S = 2C_2 \cdot \frac{n!}{(n-2)!} \cdot x^{2(n-2)} + 2C_1 \cdot \frac{n!}{(n-1)!} \cdot x^{2n-1} + 2C_0 \cdot \frac{n!}{n!} x^{2n}$$

($\ast$)

∴ $c(K_n \times K_2, 2n - 2) = \frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1)$

(ii) From equation (A)

$$c(K_n \times K_2, 2n - 1) = 2C_1 \cdot \frac{n!}{(n-1)!} = 2 \cdot \frac{n(n-1)!}{(n-1)!} = 2n$$

(iii) From equation (A)

$$c(K_n+1 \times K_2, x) = \sum_{i=0}^{r} \frac{(n+1)!}{(n+1-r+i)!} x^{(n+1)(n-r+i)}$$

$$= \sum_{i=0}^{r} \frac{(n+1)!}{(n+1-r+i)!} x^{n+1+i}$$

Put $r = 2$

$$c(K_n+1 \times K_2, x) = \frac{(n+1)!}{(n-1)!} x^{2n} + \frac{(n+1)!}{n!} x^{2n+1} + \frac{(n+1)!}{(n+1)!} x^{2n+2}$$

($\ast\ast$)
From (**)
\[ c(K_{n+1} \times K_n, 2n) = \frac{(n+1)!}{(n-1)!} \]
\[ = \frac{(n+1) (n) (n-1) !}{(n-1)!} \]
\[ = n (n+1) \]
\[ = n (n+1) + 2n \]
\[ = C(K_n \times K_2, 2n-2) + C(K_n \times K_2, 2n) . \]

**Theorem 2.4:** The vertex cover polynomial \( x^{r-n} [C(K_n \times K_r, x)] \) is log-concave.

**Proof:** By (A), \( C(G, x) = \sum_{i=0}^{r} rC_{r-i} \frac{n!}{(n-r-i)!} x^{n-r-i} \)

We prove this result on induction.

When \( r = 2 \) and \( i = 0, 1, 2 \)

We have to prove
\[ [c(K_n \times K_2, n-n+1)]^2 \geq c[K_n \times K_2, n-n] \times c[K_n \times K_2, n-n+2] \]

R.H.S = \( c[K_n \times K_2, n-n] \times c[K_n \times K_2, n-n+2] \)
\[ = nC_n \cdot \frac{n!}{(n-1)!} \]
\[ = n! \cdot nC_2 \cdot \frac{n!}{2} \]
\[ = n! \cdot nC_2 \cdot \frac{n!}{2} \]

\[ \text{L.H.S} = [c(K_n \times K_2, n-n+1)]^2 \]
\[ = \left[ rC_{r-1} \frac{n!}{(n-r+1)!} \right]^2 \]
\[ = \left[ r \cdot \frac{n!}{(n-1)!} \right]^2 \]
\[ = (r n^2) \]
\[ = r n^2 \]

\[ \text{(2)} / \text{(1)} \Rightarrow \frac{[c(K_n \times K_n, n-n+1)]^2}{c(K_n \times K_n, n-n) (c(K_n \times K_2), n-n+2)} = \frac{r^2 n^2}{n(n-1)} \text{ for every } n > 1 \]

Therefore, \( [c(K_n \times K_n, n-n+1)]^2 \geq [c(K_n \times K_n, n-n) \times c(K_n \times K_2, n-n+2) \]

Assume the result is true for all \( r < n \) and prove \( r = n. \)

**Case-(i):** \( r = n; \ i = 0, 1, 2 \)

We have to prove
\[ [c(K_n \times K_n, n^2-n+1)]^2 \geq c(K_n \times K_n, n^2-n) \times c(K_n \times K_n, n^2-n+2) \]

R.H.S = \( c(K_n \times K_n, n^2-n) \times c(K_n \times K_n, n^2-n+2) \)
\[ = nC_n \cdot \frac{n!}{0!} \cdot nC_{n+2} \frac{n!}{2!} \]
\[ = n! \cdot nC_2 \cdot \frac{n!}{2} \]
\[ = (r n^2) \]
\[ = r n^2 \]
L.H.S. = \([c(K_n \times K_n, n^2 - n + 1)]^2 = \left[ \binom{n!}{n-1} \right]^2 \]
= \(n^2 \cdot (n!)^2\)

(4) / (3) \(\Rightarrow\) \(\frac{n^2(n!)}{n! \cdot nC_2 \cdot n^2} \cdot 2! = \frac{2! \cdot n^2 \cdot 2!}{n(n-1)}\)
= \(\frac{4n}{n-1}\) > 1 for all \(n > 1\)

\(\Rightarrow\) \([c(K_n \times K_n, n^2 - n + 1)]^2 \geq c(K_n \times K_n, n^2 - n) \cdot c(K_n \times K_n, n^2 - n + 2)\)

Assume the result is true for \(i < k\) and prove for \(i = k\)

That is for prove,
\([c(K_n \times K_n, n^2 - n + k)]^2 \geq c(K_n \times K_n, n^2 - n + k-1) \cdot c(K_n \times K_n, n^2 - n + k + 1)\)

R.H.S = \(c(K_n \times K_n, n^2 - n + k-1) \cdot c(K_n \times K_n, n^2 - n + k+1)\)
= \(nC_{n-(k-1)} \cdot n! \cdot nC_{n-(k+1)} \cdot n!\) \(\frac{1}{(k-1)!} \cdot \frac{1}{(k+1)!}\) where \(k+1 \leq n\).
= \(nC_{n-(k-1)} \cdot n! \cdot nC_{n-(k+1)} \cdot n!\) \(\frac{1}{(k-1)!} \cdot \frac{1}{(k+1)!}\)
= \{nC_{k-1} \cdot n(n-1) \ldots (n-k+2)\} \{nC_{k+1} \cdot n(n-1) \ldots (n-k)\}

(5)

L.H.S. = \([c(K_n \times K_n, n^2 - n + k)]^2 = \left[ \binom{n!}{n-k} \right]^2 \]
= \([nC_{n-k} \cdot n(n-1) \cdot (n-k+1)]^2\)

(6) / (5) \(\Rightarrow\) \(\frac{[c(K_n \times K_n, n^2 - n + k)]^2}{c(K_n \times K_n, n^2 - n + k-1) \cdot c(K_n \times K_n, n^2 - n + k+1)}\)
= \(\frac{nC_k \cdot n(n-1) \ldots (n-k+1)]^2}{\{nC_{k-1} \cdot n(n-1) \ldots (n-k+2)\} \cdot \{nC_{k+1} \cdot n(n-1) \ldots (n-k)\}}\)
= \(\frac{nC_{k-1} \cdot n(n-1) \ldots (n-k+2) \cdot nC_{k+1} \cdot n(n-1) \ldots (n-k)}{nC_k \cdot n(n-1) \ldots (n-k+1) \cdot nC_{k-1} \cdot n(n-1) \ldots (n-k+2) \cdot nC_{k+1} \cdot n(n-1) \ldots (n-k)}\)
= \(\frac{(nC_k)^2}{nC_{k-1} \cdot nC_{k+1}} \cdot \left(\frac{n-k+1}{n-k}\right) \geq 1\) for every \(n > k\).

That is
\(\Rightarrow\) \([c(K_n \times K_n, n^2 - n + k)]^2 \geq c(K_n \times K_n, n^2 - n + k-1) \cdot c(K_n \times K_n, n^2 - n + k + 1)\)

The result is true for all \(i\).

Therefore, \(x_{n-i}^{-1} [C(K_n \times K_n, x)]\) is log-concave.

Hence the proof.
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Source of support: Nil, Conflict of interest: None Declared.

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