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FIXED POINTS FOR CYCLIC WEAK ϕ-CONTRACTION IN COMPLETE GENERALIZED METRIC SPACE

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ABSTRACT

In this paper, when the number of cyclic sets is even, we obtain a fixed point theorem for a mapping satisfying cyclic weak ϕ-contraction in complete generalized metric spaces, which gives a positive answer to the question raised by Fei He and Alatancang Chen (Fixed point theory Appl. 2016:67).

Keywords: *Fixed point, comparison function, generalized metric space, Cyclic φ-contraction, cyclic weak ϕ-contraction.*

Mathematics subject classification: *Primary 47H10, Secondary 54H25.*

1. INTRODUCTION AND PRELIMINARIES

The main purpose of this paper is to answer an open question raised by He and Chen in [12]. We show these results are valid in complete generalized metric spaces when the number of cyclic sets is even.

Before going to the main results. Let us recall the basic definitions and theorems.

Definition 1.1: ([4]) A function $\varphi : [0, \infty) \to [0, \infty)$ is called a comparison function if it satisfies :

(i) φ is increasing,

(ii) $(\varphi^n(t))_{n\in N}$ converges to 0 as $n \to \infty$, for all $t \in (0, \infty)$.

If the condition (ii) is replaced by (iii) $\sum_{k=0}^{\infty} \varphi^k(t) < \infty$, for all $t \in (0, \infty)$ then φ is called a strong comparison function.

Definition 1.2: ([12]) Let (X, d) be a metric space, $p \in \mathbb{N}$, $A_1, \ldots, A_p$ non empty subsets of X, and $Y := \bigcup_{i=1}^{p} A_i$. An operator $f : Y \to Y$ is called a cyclic φ-contraction if :

(i) $\bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to f,

(ii) there exists a comparison function $\varphi : [0, \infty) \to [0, \infty)$ such that

$$d(f x, f y) \le \varphi(d(x, y)) \tag{1}$$

for any $x \in A_i$, $y \in A_{i+1}$, where $A_{p+1} = A_1$.

Theorem 1.1: ([12]) Let (X, d) be a complete metric space, $p \in \mathbb{N}$, $A_1, \ldots, A_p$ non empty closed subsets of X, and $Y := \bigcup_{i=1}^{p} A_i$. Assume that $f : Y \to Y$ is a cyclic φ-contraction. Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{p} A_i$ and a picard iteration $\{x_n\}_{n\ge1}$ given by $x_n = fx_{n-1}$ converging to x^* for any starting point $x_0 \in \bigcup_{i=1}^{p} A_i$.

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Theorem 1.2: ([12]) Let (X, d) be a complete metric space, $p \in \mathbb{N}$, $A_1, \ldots, A_p$ closed non empty subsets of X, $Y := \bigcup_{i=1}^{p} A_i$. and $f : Y \to Y$ an operator. Assume that:

(i) $\bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to f,

(ii) there exists a function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) < t$ and $t - \phi(t)$ is non decreasing for $t \in (0, \infty)$ and $\phi(0) = 0$ such that

$$d(f x, f y) \le \phi(d(x, y)) \tag{2}$$

for any $x \in A_i$, $y \in A_{i+1}$, where $A_{p+1} = A_1$. Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{p} A_i$.

Definition 1.3: ([12]) A function $\phi : [0, \infty) \to [0, \infty)$ is called a (w)-comparison function if it satisfies:

(i) $\phi(0) = 0$;
(ii) $\phi(t) < t$, for all $t \in (0, \infty)$;
(iii) the function $\psi(t) := t - \phi(t)$ is increasing, that is., $t_1 \le t_2$ implies $\psi(t_1) \le \psi(t_2)$, for $t_1, t_2 \in [0, \infty)$.

Lemma 1.1: ([12]) If $\phi : [0, \infty) \to [0, \infty)$ is a (w)-comparison function, then the following hold :

(i) $\phi(t) \le t$, for all $t \in [0, \infty)$;
(ii) for $k \ge 1$, $\phi^k(t) \le t$, for any $t \in (0, \infty)$;
(iii) $(\varphi^n(t))_{n\in N}$ converges to 0 as $n \to \infty$, for all $t \in (0, \infty)$.

Definition 1.4: ([12]) Let (X, d) be a generalized metric space, $p \in \mathbb{N}$, $A_1, \ldots, A_p$ non empty subsets of X, and $Y := \bigcup_{i=1}^{p} A_i$. An operator $f : Y \to Y$ is called a cyclic weak ϕ-contraction if :

(i) $\bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y with respect to f,
(ii) there exists a (w)-comparison function $\phi : [0, \infty) \to [0, \infty)$ such that

$$d(f x, f y) \le \phi(d(x, y)) \tag{3}$$

for any $x \in A_i$, $y \in A_{i+1}$, where $A_{p+1} = A_1$.

2. MAIN RESULTS

This section deals with fixed point theorem which provides a positive answer to the question raised by Fei He and Alatancang Chen [12].

Question: If the number of cyclic sets is even, then prove the following theorem is valid or not.

Theorem 2.1: Let (X, d) be a complete generalized metric space, p an odd number, $A_1, \ldots, A_p$ non empty closed subsets of X, and $Y := \bigcup_{i=1}^{p} A_i$. Assume that $f : Y \to Y$ is a cyclic weak ϕ-contraction. Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{p} A_i$ and a Picard iteration $\{x_n\}_{n\ge1}$ given by $x_n = fx_{n-1}$ converging to x^* for any starting point $x_0 \in \bigcup_{i=1}^{p} A_i$.

Theorem 2.2: Let (X, d) be a complete generalized metric space, p an even number, $A_1, \ldots, A_p$ non empty closed subsets of X, and $Y := \bigcup_{i=1}^{p} A_i$. Assume that $f : Y \to Y$ is a cyclic weak ϕ-contraction. Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{p} A_i$ and a Picard iteration $\{x_n\}_{n\ge1}$ given by $x_n = fx_{n-1}$ converging to x^* for any starting point $x_0 \in \bigcup_{i=1}^{p} A_i$.

Proof: Let $x_0 \in Y$ and $x_n = f x_{n-1}$; n = 1, 2, ... If there exists n_0 such that $x_{n_0+1} = x_{n_0}$ then $fx_{n_0} = x_{n_0+1} = x_{n_0}$ and the existence of the fixed point is proved. Consequently, we will assume that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$.

Step-1: We will prove that $x_n \ne x_m$ for all $n \ne m$

Suppose that $x_n = x_m$ for some $n \ne m$. Without loss of generality, we may assume that $n > m + 1$.

Due to the property of ϕ, we see that

$$\begin{aligned} d(x_m, x_{m+1}) &= d(x_m, f\,x_m) \\ &= d(x_n, f\,x_n) \\ &= d(f\,x_{n-1}, f\,x_n) \\ &\le \phi(d(x_{n-1}, x_n)) \\ &\le \ldots \\ &\le \phi^{n-m}(d(x_m, x_{m+1})) \end{aligned} \tag{4}$$

By Lemma 1.1(2), we get $\phi^{n-m}(d(x_m, x_{m+1})) < (d(x_m, x_{m+1}))$, which is a contradiction.

Step-2: We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0,\ \lim_{n\to\infty} d(x_n, x_{n+2}) = 0 \ldots \lim_{n\to\infty} d(x_n, x_{n+p}) = 0. \tag{5}$$

Using (3), we get

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \le \phi(d(x_{n-1}, x_n)) \text{ for all } n \in \mathbb{N}.$$

Using the definition of ϕ, we see that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \tag{6}$$

This implies the sequence $\{d(x_n, x_{n+1})\}$ is is decreasing and bounded below.

Consequently, $d(x_n, x_{n+1}) \to r$ for some $r \ge 0$.

Suppose that $r > 0$. Then $\phi(r) < r$.

Using the definition of ϕ and $d(x_n, x_{n+1}) \ge r$, we get

$$r - \phi(r) \le d(x_n, x_{n+1}) - \phi(d(x_n, x_{n+1})), \text{ for all } n \in \mathbb{N}.$$

From $d(x_{n+1}, x_{n+2}) \le \phi(d(x_n, x_{n+1}))$, we see that

$$r - \phi(r) \le d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2}), \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above inequality, we get $r - \phi(r) \le 0$, which is a contradiction with $\phi(r) < r$. Thus we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{7}$$

Using the triangle inequality, we get

$$d(x_n, x_{n+2}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$$

From (7), we see that $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$

Also since $d(x_n, x_{n+4}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})$

From (7), we see that $d(x_n, x_{n+4}) \to 0$ as $n \to \infty$

By induction, we deduce that

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0 \tag{8}$$

for all $k \in \{2, 4, \ldots, p\}$

Now to Prove

$$\lim_{n\to\infty} d(x_n, x_{n+p-1}) = 0 \tag{9}$$

Since x_n and x_{n+p-1} lie in different adjacently labeled sets A_i and A_{i+1} for certain $i \in \{1, 2, \ldots, p\}$, from(3) we get

$$d(x_n, x_{n+p-1}) = d(fx_{n-1}, x_{n+p-2}) \le \phi(d(x_{n-1}, x_{n+p-2}))$$

Similar to the proof of the conclusion (7), we can deduce that $\{d(x_n, x_{n+p-1})\}$ is decreasing and converges to 0. This means that (9) holds

For k = 2, 4, . . . , p – 4, using the rectangular inequality, we have

$$d(x_n, x_{n+k}) \le d(x_n, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p}) + d(x_{n+p}, x_{n+k}) \tag{10}$$

Since p – k is even, from (8) we get

$$\lim_{n\to\infty} d(x_{n+p}, x_{n+k}) = \lim_{n\to\infty} d(x_{n+p-k}, x_n) = 0 \tag{11}$$

Therefore, from (8), (9), (10) and (11) we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \text{for all } k \in \{2, 4, \ldots, p\text{-}2\} \tag{12}$$

Combining $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} d(x_n, x_{n+k}) = 0$

We see

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0,\ \lim_{n\to\infty} d(x_n, x_{n+2}) = 0,\ \ldots,\ \lim_{n\to\infty} d(x_n, x_{n+p}) = 0.$$

is proved.

Step-3: We will prove the following claim.

Claim: For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > m > N$ with $n - m \equiv 1 \bmod p$ then $d(x_n, x_m) < \varepsilon$.

In fact, if this is not true, then there exists $\varepsilon_0 > 0$ such that for any $N \in \mathbb{N}$ we can find $n > m > N$ with $n - m \equiv 1 \bmod p$ satisfying $d(x_n, x_m) \ge \varepsilon_0$.

By the definition of ϕ, we get

$$\varepsilon_0 - \phi(\varepsilon_0) \le d(x_n, x_m) - \phi(d(x_n, x_m)). \tag{13}$$

Using (3), we get

$$d(x_{n+1}, x_{m+1}) \le \phi(d(x_n, x_m)). \tag{14}$$

By (13), (14) and the rectangular inequality, we obtain

$$\begin{aligned}\varepsilon_0 - \phi(\varepsilon_0) &\le d(x_n, x_m) - d(x_{n+1}, x_{m+1})\\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) - d(x_{n+1}, x_{m+1})\\ &= d(x_n, x_{n+1}) + d(x_{m+1}, x_m).\end{aligned}$$

From (6), it follows that

$$\varepsilon_0 - \phi(\varepsilon_0) \le 2d(x_{m+1}, x_m)$$

and $$d(x_{m+1}, x_m) \ge \frac{\varepsilon_0 - \phi(\varepsilon_0)}{2} > 0$$

Therefore, $\{d(x_{m+1}, x_m)\}$does not converge to 0 as $m \to \infty$, which contradicts (7).

Step-4: We will prove $\{x_n\}$ is a Cauchy sequence in X. Let $\varepsilon > 0$ be given.

Using the claim, we find that $N_1 \in \mathbb{N}$ such that if $n > m > N_1$ with $n - m \equiv 1 \bmod p$ then $d(x_n, x_m) < \frac{\varepsilon}{3}$.

On the other hand, using (5), we also find $N_2 \in \mathbb{N}$ such that, for any $n > N_2$,

$$d(x_n, x_{n+1}) < \frac{\varepsilon}{3},\ d(x_n, x_{n+2}) < \frac{\varepsilon}{3}\ \ldots,\ d(x_n, x_{n+p}) < \frac{\varepsilon}{3}$$

Let $n, m > N = \max\{N_1, N_2\} + 1$ with $n > m$. Then we can find $s \in \{0, 1, 2, \ldots, p - 1\}$ such that $n - (m + s) \equiv 1 \bmod p$.

In the case where s = 0, we have

$$d(x_n, x_m) < \frac{\varepsilon}{3} < \varepsilon$$

In the other case where $s \geq 1$, using the rectangular inequality we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m+s}) + d(x_{m+s}, x_n) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon . \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence.

Step-5: We will prove that f has a unique fixed point $x^* \in \bigcap_{i=1}^{p} A_i$ and the Picard iteration $\{x_n\}$ converges to x^*.

Since X is a complete generalized metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Using the cyclic character of f, there exists a subsequence $\{x_n\}$ for which belongs to A_i for $i \in \{1, 2, \ldots, p\}$.

Hence, from A_i is closed for $i \in \{1, 2, \ldots, p\}$, we see that $x^* \in \bigcap_{i=1}^{p} A_i$ Now, we will prove $d(x_n, f x^*) \to 0$ as $n \to \infty$.

This means $\varphi\big(d(y,z)\big) = d(y,z)$. .

In fact using (3), we have

$$d(x_n, f x^*) = d(fx_{n-1}, fx^*) \leq \phi(d(x_{n-1}, x^*)) \leq d(x_{n-1}, x^*) \to 0 \text{ as } n \to \infty$$

which implies $d(x_n, f x^*) \to 0$ as $n \to \infty$.

Using Proposition 3 of [11], we deduce that $f x^* = x^*$, that is, x^* is a fixed point of f.

In order to prove that the uniqueness of the fixed point, we take $y, z \in Y$ such that y and z are fixed points of f.

The cyclic character of f implies that $y, z \in \bigcap_{i=1}^{p} A_i$

Using (3), $d(y, z) = d(f y, f z) \leq \phi(d(y, z)) \leq d(y, z)$.

Since $\phi(t) > 0$ for $t > 0$, we get $d(y, z) = 0$ and $y = z$.

This finishes the proof.

Remark 2.1: From theorem 2.2, we see that the open question raised by Fei He and Alatancang Chen has been answered.

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