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DIFFERENT TYPES OF MEASURES ON MEASURABLE SPACE AND THEIR INTER-RELATIONSHIP

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ABSTRACT

This paper is a review on historical development of the role of measure functions on measurable spaces, different types of measures on different measurable spaces and their inter-relationship.

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1. INTRODUCTION

The mathematical notion of a measure is intended to represent concepts such as length, area, volume, mass, weight, electric charge, etc., in measuring the sizes of various structures / objects of the physical word. The objects to be measured are represented by subsets and a measure is an additive function of subsets. These are non-negative values attached to certain elementary figures such as intervals, rectangles, spheres or balls. When the objects are more abstract, then we assign the corresponding numerical values by approximating them, whenever possible with the above types of figures. More abstract structures appear in real problems. They are measured and assigned suitable numerical values. This leads to analysis of objects which are composed of elementary figures in the sense of using sums (unions), differences (intersections) and other operations. These ideas motivate an abstraction and use of set theoretical operations on them, to analyse new and abstract figures. A class of subsets can be measured to study their volumes, areas and their extensions for figures defined by various functions. Both measurable classes of subsets and functions depend on the measuring instrument for measuring length, area, volume, mass, density, etc., prescribed by the problem being investigated. Measure functions or measures will provide measures for the simplest figures with a certain additivity property. These measures are studied on measurable spaces.

Measures are the generalizations of volume. Fundamental example is the Lebesgue measure on R^n. Measures are important because of their intrinsic geometrical significance and also allow us to define integrals [46][50][53][60].

2. HISTORICAL DEVELOPMENT

In the end of the 19th century, E. Borel introduced a notion of measure on the real line and in 1898 he extended the notion of length of intervals to a measure called Borel measure on a wide class of subsets called Borel subsets on the real line. Lebesgue extended the definition of Borel measure and developed a theory of integration and differentiation. Since an open subset is the union of a disjoint sequence of intervals he defined the measure of an open subset as the sum of the lengths of these intervals. Since a closed subset is the complement of an open subset, he defined the measure of a closed subset as one minus the measure of its complement. Then he defined the outer measure of any subset as the infimum of the measures of open subsets containing it and the inner measure of the subset as the supremum of the measures of closed subsets contained in it. If the inner and outer measures of a set coincide, then the exact measure of subset is obtained. He extended Borel measure to the whole real line and in Euclidean spaces of higher dimensions. Then he defined the integral of a positive function on the real spaces.

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In 1904, he turned his attention to differentiation. A statement holds almost everywhere if the set of points at which it does not hold has measure zero. He first proved that a monotone function, and hence the sum or difference of two monotone functions, is differentiable almost everywhere. Soon the field had explosive growth and wider applications of these ideas to other branches of mathematics. The need to construct measures is to solve problems arising in all fields within and outside mathematics. Construction of measure for some specific purpose frequently motivated by problems in another field. Therefore, methods for constructing measures are of great importance. Hausdorff developed the theory of measures called as Hausdorff measures. They play a significant role in differential geometry where line and surface integrals are frequently used.

In 1910, Lebesgue extended the fundamental Theorem of calculus from the real line to Euclidean spaces of higher dimensions. In 1913 it was extended by J. Radon to other measures on Euclidean spaces and in 1915 M. Frechet replaced Euclidean space by any abstract space.

In 1933, S. Bochner initiated the study of measures which assigns to each set a vector in Banach space instead of a number. Such measures are studied in physics and differential geometry where the Banach space is finite dimensional which is Euclidean space.

In 1940, A. Haar has extended the theory and constructed a translation invariant measure called Haar measure on a metric group and then to any locally compact group.

In 1940's and 1950's, the work of H. Witney and others have given rise to a field known as geometric measure theory. Its methods are more related to ideas in differential geometry like shape, holes, direction, orientation, etc [53][60].

In 1958, V. S. Varadarajan [61] had discussed weak convergence of measures on separable metric spaces. Every purely non-atomic measure (Radon measure) defined on a σ-compact space is extended to a measure on a countably compact spaces. In 1964, Kenneth A. Ross and Karl Stromberg [54] have studied Baire sets and Baire measures on σ-compact, locally compact space in particular to all locally compact topological groups.

In 1967, W. Moran [49] have examined how certain properties of completely regular spaces are preserved under mappings and products. Also studied the transformation of measures from one space to another by means of Baire measurable mappings on measure compact space and strongly measure-compact spaces. In 1969 and 1972, Chandra Gowrisankaran [23][24] have studied Quasi-Invariant Radon measures on Groups and a real valued Radon measures on Hausdorff topological group which is locally compact. In 1971, Peter Ganssler [22] have studied the compactness and sequentially compactness in spaces of measures. In 1975, Casper Goffman and George Pedrick [25] have given a proof that non-atomic Lebesgue-Stieltjes measure on n-dimensional space, for which open sets have positive measure, that is, homeomorphic with Lebesgue measure.

Some authors have studied different types of measures on manifold. Based on Neumann-Oxtoby-Ulam Theorem [51] which states that given two good measures μ and ϑ on M^n, there exists a homeomorphism of a compact connected manifold M^n such that $h_*\mu = \vartheta$. In 1980, A. Fathi [15] have made a comprehensive study on topological and algebraic properties of groups of measure-preserving homeomorphisms of compact n-manifolds. In 2003, R. Berlanga [5] has extended this work to the non compact manifolds. In 2006, 2007 and 2010, Tatsuhiko Yagasaki [64][65][66][67] using R. Berlanga's results obtained results on topological properties of groups of Radon measure-preserving homeomorphisms of non compact 2-manifolds and Groups of measure-preserving homeomorphisms and volume-preserving Diffeomorphisms of non compact manifolds and mass flow toward ends. In 2011, H. N. Mhaskar studied Marcinkiewicz-Zygmund measures on manifolds and Florian Stampfer studied on pull back of measures on Riemannian manifolds.

In 2010, Stan Gudder [26] had discussed a generalization called finite quantum measure spaces and also discussed more general spaces called super-quantum measure spaces and he had presented some basic properties of a quantum measure space [27][28]. Compatibility of sets with respect to a quantum measure is studied and the centre of a quantum measure space is characterised in terms of signed product measures. Also, super-quantum measure space is introduced and shown that quantum measure is useful for computing and predicting elementary particle masses. In 2011, Sumati Surya [58] had discussed the extension of vector valued measures on finite time events to a measure over infinite time, or equivalently, covariant events. Fay Dower, Steven Johnston and Sumati Surya [57][58], had worked on the extension of the quantum measures to applications in physics and proposed an appropriate generalisation, the quantum cover, which in addition to being a cover in Quantum measure theory. In 2012, S. Gudder [27][28] had shown that quantum measures and integrals appear naturally in any L_2 −Hilbert space. In 2013, Yongjian Xie-Aili Yang and Fang Ren [62] had studied on the properties of the super quantum measures. In 2014 and 2015, Yongjian Xie-Aili Yang [63] had studied Quantum measures on finite effect algebra with the Riez Decomposition properties and in 2015, Gregg Jaeger had studied measurement and fundamental process in Quantum.

In the next section, we discuss some of the concepts and results on different types of measure structures on measurable space.

3. SOME RESULTS ON DIFFERENT TYPES OF MEASURES ON MEASURABLE SPACE

We use the following basic concepts and results to construct different measure structures on measure manifold:

Definition 3.1: σ-algebra [46][53][59][60]
A σ-algebra Σ on a topological space (R^n, τ) is a collection of subsets of $(R^n, \ )$ such that,

(i) $\emptyset, R^n \in \Sigma$,
(ii) $A \in \Sigma$, then $A^C \in \Sigma$,
(iii) If $A_i \in \Sigma$, for $i \in N$, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$, $\bigcap_{i=1}^{\infty} A_i \in \Sigma$.
The triplet (R^n, τ, Σ) is called a measurable space.

Definition 3.2: G_δ-set [46][53][59][60]
A subset $A \subseteq R^n$ is called G_δ-set if it is the countable intersection of open sets. That is, $A = \{\bigcap_{i=1}^{\infty} A_i : A_i \in \tau\}$.

Definition 3.3: F_σ-set [46][53][59][60]
A subset $E \subseteq R^n$ is called F_σ-set if it is the countable union of closed sets. That is, $= \{\bigcup_{i=1}^{\infty} E_i : E_i \in \tau^c\}$.

Definition 3.4: Borel σ -algebra [46][53][59][60]
The Borel σ -algebra $\mathcal{B}(R^n)$ on (R^n, τ) is the smallest $\boldsymbol{\sigma}$ -algebra generated by the open sets belonging to τ such that $\mathcal{B}(R^n) = \Sigma(\tau\ (R^n))$. A set that belongs to the $\boldsymbol{\sigma}$ -algebra is called a Borel set.

Definition 3.5: Measurable Topological Space [19][20]
The space (R^n, τ, Σ) is called a measurable topological space if the space (R^n, τ) is a topological space equipped with σ -algebra where the members of τ which belongs to σ-algebra Σ are the Borel subsets in (R^n, τ, Σ).

Definition 3.6: Measurable Hausdorff Space [19][20]
The space (R^n, τ, Σ) is called a measurable Hausdorff space provided that if x and y are distinct members of (R^n, τ, Σ) then, $\exists$ disjoint Borel open subsets A and B such that $x \in A$ and $y \in B$.

Definition 3.7: Measure Space [46][53][59][60]
A measure μ on a measurable space (R^n, τ, Σ) is a function $\mu : \Sigma \longrightarrow [0, \infty]$ such that,

(i) $\mu(\emptyset) = 0$,
(ii) If $\{A_i \in \Sigma : i \in N\}$ is a countable disjoint collection of subsets in Σ, then
$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ (σ-additivity property)

Therefore the space (R^n, τ, Σ, μ) is called a topological measure space.

Definition 3.8: Compact Support of a function [46][53][59][60]
Compact support of a function f is the closure of the set $\{x \in R^n : f(x) \neq 0\}$. It is denoted by suppf.

Definition 3.9: Locally Finite Measure on (R^n, τ, Σ) [46][53][59][60]
The measure μ is locally finite if every point x of (R^n, τ, Σ, μ) has a neighbourhood A of x for which $\mu(A) < \infty$.

Definition 3.10: Inner Regular Measure on (R^n, τ, Σ) [46][53][59][60]
Let $\mu : \Sigma \longrightarrow [0, \infty]$ be a measure on (R^n, τ, Σ). The measure μ is called inner regular if, for any Borel open set A, $\mu(A) = \sup\{ \mu(K_i) : i \in I; K_i \subseteq A; K_i \text{ compact}\}$.

Definition 3.11: Outer Regular Measure on (R^n, τ, Σ) [46][53][59][60]
Let $\mu : \Sigma \longrightarrow [0, \infty]$ be a measure on (R^n, τ, Σ). The measure μ is called outer regular if, for any Borel open set A, $\mu(A) = \inf\{ \mu(U_i) : i \in I; U_i \supseteq A; U_i \text{ open}\}$.

Definition 3.12: Regular Measure on (R^n, τ, Σ) [46][53][59][60]
A regular measure on (R^n, τ, Σ) is a measure μ for which every Borel subset can be approximated from above by an open measurable subsets and from below by a compact measurable subset, that is, a measure is called regular if it is outer regular and inner regular.

Definition 3.13: Borel measure [46][53][59][60]
A Borel measure on (R^n, τ, Σ) is a measure μ which is defined on $\mathcal{B}(R^n)$ and satisfies $\{\forall\ K \subset R^n \text{ compact}: \mu(K) < \infty\}$.

Definition 3.14: Radon Measure on (R^n, τ, Σ) [46][53][59][60]
A Radon Measure μ_R on a measurable space (R^n, τ, Σ) is a positive Borel measure $\mu_R : \mathcal{B} \longrightarrow [0, \infty]$ which is finite on compact Borel subsets and is inner regular in the sense that for every Borel subsets $A \subset (R^n, \tau, \Sigma)$ we have,

(i) $\mu_R(A) = \sup\{ \mu_R(K) : K \subseteq A ; K \in \mathcal{K}\}$, (1)
where $\mathcal{K}$ denote the family of all compact Borel subsets and μ_R is outer regular if for every $A \subset (R^n, \tau, \Sigma)$ we have,

(ii) $\mu_R(A) = \inf\{ \mu_R(O) : O \supseteq A ; O \in \mathcal{O}\}$, (2)
where $\mathcal{O}$ denote the family of all open Borel subsets.

Example 3.1: Lebesgue measure on R^n is a Radon measure.

Definition 3.15: Locally compact topological Space [19][20]
A topological space (R^n, τ) is locally compact if for all $x \in R^n$ there exists an open neighbourhood $V \subset R^n$ of x such that V is compact. (Alternatively, this is equivalent to requiring that to each $x \in R^n$ there exists a compact neighborhood N_x of x.

Definition 3.16: Locally compact Hausdorff topological space [19][20]
A topological space is locally compact Hausdorff, if for all $p \neq q$, $\exists$ open neighbourhoods A and B belonging to (R^n, τ), $A \subset \bar{A}$ and $B \subset \bar{B}$ where $\bar{A}$ and $\bar{B}$are compact subsets belonging to (R^n, τ), such that $p \in \bar{A}$, $q \in \bar{B}$ and $\bar{A} \cap \bar{B} = \emptyset$.

Definition 3.17: Locally compact Hausdorff regular measurable space [35][36]
A topological space is locally compact Hausdorff regular measurable space, if for all $p \in \bar{A}$, $\exists$ open Borel neighbourhoods A and B belonging to (R^n, τ, Σ) such that $A \subset \bar{A}$ and $B \subset \bar{B}$ where $\bar{A}$ and $\bar{B}$ are compact subsets belonging to (R^n, τ, Σ), and $\exists$ closed Borel subset $F \subset \bar{B}$ such that $\bar{A} \cap \bar{B} = \emptyset$.

Definition 3.18: Measurable Normal space (e-Normal)[35][36][40]
A measure space (R^n, τ, Σ, μ) is said to be e-normal measure space (or e-normal) if each pair of disjoint $F_\sigma - sets\ A$ and B in (R^n, τ, Σ, μ), $\exists$ a pair of disjoint $G_\delta - sets\ U$ and V such that $A \subset U$, $B \subset V$.

Proposition 3.1: [19][20]
Suppose that (R^n, τ) is a Hausdorff topological space, $K \subset (R^n, \tau)$ and $x \in K^c$. Then there exists $U, V \in (R^n, \tau)$ such that $U \cap V = \emptyset$, $x \in U$ and $K \subset V$. In particular, K is closed (so compact subsets of Hausdorff topological spaces are closed). More generally, if $K$and F are two disjoint compact subsets of (R^n, τ), there exists disjoint open sets $U, V \in (R^n, \tau)$ such that $K \subset V$ and $F \subset U$.

Definition 3.19: Measurable Functions [46][53][59][60]
Let (R^n, τ_1, Σ_1) and (R^m, τ_2, Σ_2) be measurable spaces. A map $F : (R^n, \tau_1, \Sigma_1) \longrightarrow (R^m, \tau_2, \Sigma_2)$ is measurable if for all $B \in \Sigma_2$, the set $F^{-1}(B) \in \Sigma_1$.

Definition 3.20: Invariant Measure [46][53][59][60]
Let (X, Σ) be a measurable space and let f be a measurable function from X to itself. A measure μ on (X, Σ) is said to be invariant under f if, for every measurable set A in Σ, $\mu\left(f^{-1}(A)\right) = \mu(A)$.

Theorem 3.1: [46][53][59][60]
The composition of two measurable functions is measurable.

Proposition 3.2: [19][20]
Let $(R^n, \tau_1, \Sigma_1, \mu_1)$ be a complete locally determined measure space, $(R^m, \tau_2, \Sigma_2, \mu_2)$ be a measure space and $f : R^n \longrightarrow R^m$ be a function. Suppose that $K \subseteq \Sigma_2$ such that,

(i) μ_2 is inner regular with respect to $\mathcal{K}$,
(ii) $f^{-1}(K) \in \Sigma_1$ and $\mu_1 f^{-1}(K) = \mu_2(K)$ for every $K \in \mathcal{K}$;
(iii) whenever $E \in \Sigma_1$ and $\mu(E) > 0$ there is a $K \in \mathcal{K}$ such that $\mu_2(K) < \infty$ and $\mu_1\left(E \cap f^{-1}(K)\right) > 0$.

Then f is inverse-measure-preserving for μ_1 and μ_2.

Definition 3.21: Lebesgue measure [46][53][59][60]
Let (R^n, τ, Σ) be a measurable space. There is the Lebesgue measure 'λ' which assigns to the rectangle its usual n-dimensional volume:

$$\lambda([a_1, b_1] \times \ldots \times [a_n, b_n]) = (b_1 - a_1) \ldots (b_n - a_n)$$

This measure 'λ' should also assign the correct volumes to the usual geometric figures, as well as for all the other subsets in Σ. Volume of any measurable subset can be approximated by the volume of many small rectangles.

Example 3.2: [46][53][59][60]
Any closed interval $[a,\mathrm{b}]$ of real numbers is Lebesgue measurable.

Definition 3.22: Regular Measure on $(\boldsymbol{R^n},\boldsymbol{\tau},\boldsymbol{\Sigma})$ [46][53][59][60]
A regular measure on (R^n,τ,Σ) is a measure for which every Borel subset can be approximated from above by an open measurable subset and from below by a compact measurable subset , that is, a measure is called regular if it is outer regular and inner regular.

Definition 3.23: Borel regular measure [46][53][59][60]
A Borel measure on (R^n,τ,Σ) is said to be regular if $\mu(E) = \sup\{\mu(K) : K \subset E, K\ compact\}$ for all Borel subsets E.

Example 3.3: [46][53][59][60]
The Lebesgue outer measure on (R^n,τ,Σ) is an example of a Borel regular measure.

Definition 3.24: Counting measure [46][53][59][60]
Let (R^n,τ,Σ) be a measurable space. A measure $\mu : \Sigma \longrightarrow [0,\infty]$ defined by

$$\mu(A) = \begin{cases} |A|, if A\ is\ a\ finite\ subset, \\ \infty, if\ A\ is\ an\ infinite\ subset, \end{cases}$$

is called the counting measure.

Definition 3.25: Quasi-Radon measure [46][53][59][60]
A quasi-Radon measure space is a topological measure space (R^n,τ,Σ,μ) such that,

(i) (R^n,τ,Σ,μ) is complete and locally determined,

(ii) μ is τ- additive, inner regular with respect to the closed subsets and effectively locally finite.

Remark 3.1: The Borel measure agrees with the Lebesgue measure on those subsets for which it is defined.

Definition 3.26: Push forward Measure [46][53][59][60]
Let $(R^n,\tau_1,\Sigma_1,\mu_1)$ be a measure space and (R^m,τ_2,Σ_2) be a measurable space, and $f: R^n \longrightarrow R^m$ be a measurable map. Then the following function μ_2 on Σ_2 is a measure such that $\mu_2(B) = \mu_1\big(f^{-1}(B)\big)\ for\ B \in \Sigma_2$.

Definition 3.27: Complete Measure [46][53][59][60]
Let (R^n,τ,Σ) be a measurable space. A complete measure is a measure space in which every subset of every null set is measurable (having measure zero). More formally (R^n,τ,Σ,μ) is complete if and only if $S \subseteq N \in \Sigma$ and $\mu(N) = 0$ implies $S \in \Sigma$.

Definition 3.28. Trivial measure [46][53][59][60]
The trivial measure on any measurable space (R^n,τ,Σ,μ) is the measure μ which assign zero measure to every measurable subset. That is, $\mu(A) = 0$ for all $A \in \Sigma$.

Definition 3.29: Hausdorff measure [46][53][59][60]
A Hausdorff measure is a type of outer measure that assigns a number in $[0,\infty]$ to each Borel subset in (R^n,τ,Σ,μ) or more generally in any metric measure space. Let (R^n,ρ,Σ) be a metric measure space. For any Borel subset $U \subset (R^n,\rho,\Sigma)$, let $diamU$ denote its diameter, that is, $diamU = \sup\{\rho(x,y)/x,y \in U\}$, $diam\emptyset = 0$.

Let S be any Borel subset of (R^n,ρ,Σ), and $\rho > 0$ is a real number then define

$$H_\delta^d(S) = \inf\{\textstyle\sum_{i=1}^{\infty}(diamU_i)^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, diamU_i < \delta\}.$$

The infimum is over all countable covers of S by Borel subsets $U_i \subset (R^n,\rho,\Sigma)$ satisfying $diamU_i < \delta$.

Definition 3.30: Tangent measure [46][53][59][60]
Tangent measures are used to study the local behavior of Radon measures, in much the same way as tangent spaces are used to study the local behavior of differentiable manifolds.

Consider a Radon measure μ_R defined on Borel subset A of measurable space (R^n,τ,Σ) and let $'a'$ be any arbitrary point in A. Let us consider a Borel open ball of radius r around a, $B_r(a)$ via the transformation $T_{a,r}(x) = \frac{x-a}{r}$ which enlarges the ball of radius r about a to a ball of radius 1 centered at 0.

Now, μ behaves on $B_r(a)$ by push-forward measure defined by,

$$T_{a,r\#}\mu(A) = \mu(a + rA) \text{ where } a + rA = \{a + rx : x \in A\}.$$

Definition 3.31: Haar measure [46][53][59][60]
Haar measure is a way to assign an "invariant volume" to subsets of locally compact topological groups and subsequently define an integral for functions on those groups. Let G be a topological group. A left Haar measure (resp. right Haar measure) on G is a non zero regular Borel measure μ on G such that $\mu(gA) = \mu(A)$ (resp. $\mu(Ag) = \mu(A)$) for all $g \in G$ and all measurable subsets A of G.

In the similar way other types of measure structures are studied on different patterns of Borel subsets on different measurable spaces.

Definition 3.32: Quantum measure space/q-measure space [26][27][28][57][58]
A measurable space is a pair $(X, \mathcal{A})$ where X is a nonempty set and $\mathcal{A}$ is a σ-algebra of subsets of $\mathcal{A}$. A (finite) measure on $\mathcal{A}$ is a map μ: $\mathcal{A} \to R^+$ satisfying the following conditions:

(1) $\mu(A \cup B) = \mu(A) + \mu(B)$ (additivity)
(2) If $A_i \in \mathcal{A}$ is an increasing sequence, then
$\mu(\cup A_i) = \lim \mu(A_i)$ (continuity)
Conditions (1) and (2) together are equivalent to
$\mu(\cup A_i) = \sum \mu(A_i)$ (σ-additivity)
It follows from (2) that
(3) If $A_i \in \mathcal{A}$ is a decreasing sequence, then $\mu(\cap A_i) = \lim \mu(A_i)$
(4) $\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(A \cup C) + \mu(B \cup C) - \mu(A) - \mu(B) - \mu(C)$.
condition (4) is called grade-2 additivity and condition (1) is called grade-1 additivity.

If μ: $\mathcal{A} \to R^+$ is grade-2 additive and satisfies conditions (2) and (3), we call μ a quantum measure/q-measure. If μ is a q-measure on $\mathcal{A}$, then $(X, \mathcal{A}, \mu)$ is a quantum measure space/q-measure space.

4. INTER-RELATIONSHIP BETWEEN DIFFERENT TYPES OF MEASURES ON MEASURABLE SPACE

A Borel measure μ is inner regular if and only if for every $\varepsilon > 0$ there is a compact set K_ε such that $|\mu(X \backslash K_\varepsilon| < \varepsilon$. It is clear that any measure on a compact space is inner regular. If (R^n, τ, Σ, μ) is a metric measure space, then every Borel measure μ on (R^n, τ, Σ, μ) is regular. If (R^n, τ, Σ, μ) is compact and separable, then the measure μ is Radon. Thus, Borel measure which is positive gives raise to a unique extension to a Radon measure. Every Radon measure on a measurable Hausdorf topological space (R^n, τ, Σ, μ) is regular and inner regular. If a Borel measure is regular and inner regular then it is Radon measure, since the intersection of a compact set and a closed set is compact. Borel measure has a unique extension to a Radon measure. On complete separable metric space all Borel measures are Radon. Radon measures are the important class of measures for applications. A Radon measure is by definition a regular Borel measure. Further Radon measure is extended to a Haar measure if the space is locally compact topological group [1] [6] [7] [8] [9] [18] [19] [20] [23].

5. CONCLUSION

Recently the measure structure has been introduced in 2014 by S. C. P. Halakatti on measure manifolds [29]-[45]. Also the Radon measure structure on measure manifold has been introduced and studied extensively in [35]-[39]. This study is potential enough to study the Radon measure manifold, Quotient Radon measure manifold and Network Radon measure manifold and its generated different categories. It has lot of applications in field of engineering, life science, cosmology, quantum physics and brain science.

Different measures as discussed above can be used to generate different measure structures on measure manifolds. Such a study opens a new branch in the field of applications. In our study the measure functions are not only used to quantify but they induce measure structures [29]-[45] on measure manifolds that describe the real world situations.

REFERENCES

1. William Arveson, Notes on Measure and Integration in Locally Compact Spaces, Department of Mathematics, University of California, Berkeley, (1996).
2. A. G. Babiker, Lebesgue Measures on Topological Spaces, Mathematika, 24, (1977), 52-59.
3. A. G. Babiker, Lifting Properties and Uniform Regularity of Lebesgue Measures on Topological Spaces, Mathematika, 28, (1981), 198-205.
4. Heinz Bauer, Measure and Integration Theory, Walter Gruyter, Berlin, (2001).
5. R. Berlanga and D. B. A. Epstein, Measures on Sigma-Compact Manifolds and Their Equivalence Under Homeomorphism, J. London Math. Soc. 27(2), (1983), 63-74.
6. V. I. Bogachev, Measures on Topological Spaces, Journal of Mathematical Sciences, 91(4), (1998).
7. V. I. Bogachev, Measure Theory, Volume II, Springer, (2006).

8. P. Cannarsa. and T. D. Aprile, Lecture Notes on Measure Theory and Functional Analysis, Dipartimento di Matematica, Universita di Roma, Tor Vergata, (2016).
9. T. C. Dorlas, Remainder Notes for the Course on Measure on Topological Spaces, Dublin Institute for Advanced Studies, School of theoretical Physics, 10, Dublin 4, Ireland, (2010).
10. Tadeusz Dobrowolski, Katsuro Sakai, Spaces of measures on metrizable space, Topology and Its Applications, Elsevier, 72, (1996), 215-258.
11. Yael Naim Dowker, On Measurable Transformations in Finite Measure Spaces, Annals of Mathematics, Second Series, 62, 3, (1955), 504-516.
12. Lester Dubins and David Freedman, Measurable Sets of Measures, Pacific Journal of Mathematics, 14(4), (1964), 1211-1222.
13. Fay Dowker, Steven Johnston and Sumati Surya, On extending the Quantum Measure, J. Phys. A: Math. Theor., vol. 43, 2010, PP 01-20.
14. Lawrence C. Evans, Ronald F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, (1992).
15. A. Fathi, Structures of the group of homeomorphisms preserving a good measure on a Compact Manifold}, Ann. Scient. Ec. Norm. Sup, 13(4), (1980), 45-93.
16. F. Filbir, H. N. Mhaskar, Marcinkiewicz - Zygmund measures on manifold, Journal of Complexity, 27(6), (2011), 568-596.
17. Liviu C. Florescu, Christiane Godet-Thobie, Young Measures And Compactness In Measure Spaces, De Gruyter,(2012).
18. B. Folland and Gerald, Real Analysis, Modern Techniques and Their Applications, Second Edition, Wiley Interscience Publication, (1999), 22-36.
19. D. H. Fremlin, Radon measures, Acta Universitatis Carolinae Mathematica et. Physica, 45(2), (2004), 23-34.
20. D. H. Fremlin, Measure Theory, University of Essex, 4,(2007).
21. P. M. Gadea and J. Munoz Masque, Analysis and Algebra on Differentiable Manifolds: A work book for students and teachers, Springer, (2009).
22. Peter Ganssler, Compactness and Sequentially Compactness in Spaces of Measures, Z. Wahrscheinlinchkeitheorie verw Geb 17, Springer Verlag, (1971), 124-146.
23. Chandra Gowrisankaran, Quasi-Invariant Radon Measures on Groups, American Mathematical Society, 35(2), (1972), 503-506.
24. Chandra Gowrisankaran, Semi groups with Invariant Radon measures, Proceedings of American Mathematical Society, 38(2), (1973).
25. Casper Goffman and George Pedrick, A Proof of the Homeomorphism of Lebesgue Stieltjes Measure With Lebesgue Measure, Proceeding of American Mathematical Society, 52, (1975), 196-198.
26. Stan Gudder, Finite Quantum Measure Space, The American Mathematical Monthly, vol. 117, No. 6, 2010, PP 512-527.
27. Stan Gudder, Quantum Measure Theory, Math. Slovaca, vol. 60, No. 5, 2010, PP 681-700.
28. Stan Gudder, Quantum Measures and Integrals, Reports of Mathematical Physics, vol. 69, 2012, PP 87-101.
29. S.C.P. Halakatti, Interpretation of the Theory of Measure Manifolds - Analysis and its applications, (communicated).
30. S.C.P. Halakatti and H. G. Haloli, Some Topological Properties on Differentiable Manifold, International Journal of Mathematical Archive, Volume 6, Issue 1, (2015), 93-101.
31. S.C.P. Halakatti and H. G. Haloli, Introducing the Concept of Measure Manifold (M, τ, Σ, μ), IOSR Journal of Mathematics(IOSR-JM), Volume 12, Issue 3 Ver. VI, (2016), 21-31.
32. S.C.P. Halakatti and H. G. Haloli, Extended Topological Properties on Measure Manifold, International Journal of Mathematical Archive (IJMA), Volume 05, Issue 1, (2014), 01-18.
33. S.C.P. Halakatti and H. G. Haloli, Convergence and Connectedness on Complete Measure Manifold, International Journal of Engineering Research and Tecnology (IJERT), Volume 04, Issue 3, (2015), 225-232.
34. S.C.P. Halakatti and H. G. Haloli and Soubhagya Baddi, Space-Time as a Complete Measure Manifold of dimension-4, International Journal of Engineering Research and Technology (IJERT), Volume 04, Issue 3, (2015), 1036-1041.
35. S.C.P. Halakatti and Soubhagya Baddi, Radon Measure on Compact Topological Measurable Space, IOSR Journal of Mathematics(IOSR-JM), Volume 11, Issue 2, Ver. III, (2015), 10-14.
36. S.C.P. Halakatti and Soubhagya Baddi, Radon Measure on Compact Measurable Manifold with Some Measurable Properties, International Journal of Mathematical Archive (IJMA), Volume 06, Issue 7, (2015), 01-12.
37. S.C.P. Halakatti and Soubhagya Baddi, On measurable Separable-Normal Radon Measure Manifold, Global Journal of Pure and Applied Mathematics,(GJPAM), Volume 12, No. 1, (2016), 1095-1106.
38. S.C.P. Halakatti and Soubhagya Baddi, Radon Measure on Measure Manifold, International Journal of Mathematical Trends and Technology, (IJMTT), Volume 30, No. 1, (2016), 53-60.
39. S.C.P. Halakatti and Soubhagya Baddi, Proposed methods to construct different categories of Radon measure manifolds - An analytical approach, Global Journal of Pure and Applied Mathematics,(GJPAM), (accepted).
40. S.C.P. Halakatti, Akshata Kengangutti and Soubhagya Baddi, Generating A Measure Manifold, IOSR Journal of Mathematics (IOSR-JM), Volume 10, Issue 5, Ver. III, (2014), 18-29.

41. S.C.P. Halakatti and Akshata Kengangutti, Introducing Atomic Separation Axioms on Atomic Measure Space- An Advanced Study, Global Journal of Pure and Applied Mathematics,(GJPAM), Volume 12, No. 1, (2016), 1085-1094.
42. S.C.P. Halakatti and Akshata Kengangutti, On Complete Network Measure Manifold, Global Journal of Pure and Applied Mathematics,(GJPAM), Volume 12, No. 1, (2016), 1085-1094.
43. S.C.P. Halakatti and Akshata Kengangutti, A Study of Measurable Hausdorff, Regular and Normal Properties on the Measure Manifold, Journal of Advanced Studies in Topology, 7(4) (2016), 295 – 300.
44. S.C.P. Halakatti and Akshata Kengangutti, Generating a Quotient Measure Manifold and Measurable Fibre Bundle with Two Group Structures, (communicated).
45. S.C.P. Halakatti and Soubhagya Baddi, On semi-compact and semi-Lindelof Quotient Radon measure manifolds and their intrinsic structures, (communicated).
46. John.K. Hunter, Measure Theory, Lecture notes, 2-11, 20, 33, University of California at Davis, (2011).
47. Gregg Jaeger, Measurement and Fundamental Process in Quantum Mechanics, Found. Phys., vol. 45, 2015, PP 806-819.
48. H. Kato, K. Kawamura, H. M. Tuncali, E. D. Tymchatyn, Measures and Topological dynamics on Menger Manifolds, Topology and Its Applications, Elsevier, 103, (2000), 249-282.
49. W. Moran, Measures and Mappings on Topological Spaces, Proceedings of London Mathematical Society, 3 (19), (1969), 493-508.
50. Togo Nishiura, Absolute Measurable Spaces, Cambridge, (2008).
51. J. Oxtoby and S. Ulam, Measure preserving homeomorphisms and metrical transitivity, Ann. of Math., 42, (1941), 874-920.
52. K. P. S. Bhaskara Rao and M. Bhaskara Rao, Mixtures of Non atomic Measures, Proc. of The American Math. Soc., 33(2), (1972), 507-510.
53. M. M. Rao, Measure Theory and Integration, Second Edition, Marcel Dekkar, INC., New York-Basel, (2004).
54. Kenneth A. Ross and Karl Stromberg, Baire sets and Baire measures, 6(2), (1965), 151-160.
55. Maurice Sion, History of Measure Theory in the Twentieth Century, Encyclopedia Italiana, Italian translations, (1990).
56. Florian Stampfer, Pullback of measures on Riemannian manifolds, J. Math. Anal. Appl., 381 (2011), 812-820.
57. Sumati Surya, Petros Wallden, Quantum Covers in Quantum Measure Theory, Found. Phys., vol. 40, 2010, PP 585-606.
58. Sumati Surya, In search of a covariant Quantum Measure, Journal of Physics, conference series, IOP Publishing, vol. 306, 2011, PP 1-6.
59. Eugene Michael Taylor, Measure Theory and Integration, American Mathematical Society, (2006).
60. Terence Tao, An Introduction to Measure Theory, American Mathematical Society, (2010).
61. V. S. Varadarajan, Weak Convergence of Measures on Separable Metric Spaces, Sankhya, The Indian Journal of Statistics, 19 (1/2), (1958), 15-22.
62. Yongjian Xie-Aili Yang, Fang Ren, Super Quantum Measures on Finite Spaces, Found. Phys., vol. 43, 2013, PP 1039-1065.
63. Yongjian Xie, Aili Yang and Fang Ren, Super Quantum Measures on Effect Algebras with the Riez Decomposition properties, Journal of Math. Physics, AIP Publishing, vol. 56, 2015, PP 01-23.
64. Tatsuhiko Yagasaki, Groups of Measure-Preserving Homeomorphisms of Non-Compact 2-manifolds, Topology and Its Applications, Elsevier, 154, (2007), 1521-1531.
65. Tatsuhiko Yagasaki, Groups of Measure-Preserving Homeomorphisms and Volume-Preserving Diffeomorphisms of Non-Compact manifolds and Mass Flow Towards Ends, arXiv math. GT/ 0512231, (2007), 10-19.
66. Tatsuhiko Yagasaki, Groups of Volume-Preserving Diffeomorphisms of Non-Compact manifolds and Mass Flow Towards Ends, Transactions of the American Mathematical Society, 362(11), (2010), 5745-5770.
67. Tatsuhiko Yagasaki, Measure Preserving Homeomorphisms of non-compact Manifolds and Mass Flow Towards Ends, Fundamenta Mathematicae, 197, (2007), 271-287.
68. Aili Yang, Yongjian Xie, Quantum Measures on Finite Effect Algebras with the Riez Decomposition properties, Found. Phys., vol. 44, 2014, PP 1009-1037.

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