ON WEAKLY SEMI CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this research paper, a new class of closed sets called weakly semi closed sets (ws-closed sets) in topological spaces are introduced and studied. A subset \( A \) of a topological space \( (X, \tau) \) is called ws-closed set if \( U \) contains semi closure of \( A \) whenever \( U \) contains \( A \) and \( U \) is w-open set in \( (X, \tau) \). This new class of sets lies between the class of all semi-closed sets and generalized semi-pre regular closed sets in topological spaces. Also some of their properties have been investigated.

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1. INTRODUCTION

In 1970 N. Levine [18], first introduced the concept of generalized closed sets were defined and investigated. In 2000 M. Sheik John [33], introduced and studied w-closed sets in topological space \( X \). Throughout this paper \( X \) or \( (X, \tau) \) represent non-empty topological space. Let \( A \) be subset of a topological space \( X \). \( cl(A), int(A), scl(A), \alpha cl(A) \) and \( spcl(A) \) denote the closure of \( A \), the interior of \( A \), the semi-closure of \( A \), the \( \alpha \)-closure of \( A \) and the semi pre closure of \( A \) in \( X \) respectively.

2. PRELIMINARIES

Definition 2.1: A subset \( A \) of a topological space \( (X, \tau) \) is called a

i. Regular open set [32] if \( A = int(cl(A)) \) and regular closed if \( A = cl(int(A)) \)

ii. Semi-open set [19] if \( A \subset cl(int(A)) \) and a semi-closed set if \( int(cl(A)) \subset A \).

iii. \( \alpha \)-open set [20] if \( A \subset int(cl(int(A))) \) and a \( \alpha \)-closed set if \( cl(int(cl(A))) \subset A \).

iv. Generalized semi pre closed set (gspr-closed) [8] if \( spcl(A) \subset U \) whenever \( A \subset U \) and \( U \) is open in \( (X, \tau) \).

v. w-closed set [33] if \( cl(A) \subset U \) whenever \( A \subset U \) and \( U \) is semi-open in \( (X, \tau) \).

vi. gspr-closed set [10] if \( spcl(A) \subset U \) whenever \( A \subset U \) and \( U \) is regular-open in \( (X, \tau) \).

vii. g\#s-closed set [40] if \( scl(A) \subset U \) whenever \( A \subset U \) and \( U \) is g\#s-open in \( (X, \tau) \).

viii. rb-closed set [24] if \( cl(A) \subset U \) whenever \( A \subset U \) and \( U \) is b-open in \( (X, \tau) \).

ix. g\#s-closed set [41] if \( cl(A) \subset U \) whenever \( A \subset U \) and \( U \) is g\#s-open in \( (X, \tau) \).

x. g\#s-closed set [40] if \( scl(A) \subset U \) whenever \( A \subset U \) and \( U \) is g\#s-open in \( (X, \tau) \).

xi. g\#s-closed set [17] if \( \alpha cl(A) \subset U \) whenever \( A \subset U \) and \( U \) is g\#s-open in \( (X, \tau) \).

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3. BASIC PROPERTIES OF WS-CLOSED SETS IN TOPOLOGICAL SPACE

Definition 3.1: A subset $A$ of a topological space $(X, \tau)$ is called weakly semi closed (ws-closed) set if $\text{scl}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is w-open set in $(X, \tau)$. The family of all ws–closed sets $X$ is denoted by WSC$(X)$. The compliment of ws–closed set is called ws-open set in $(X, \tau)$. The family of all ws-open sets in $X$ is denoted by WSO$(X)$.

Example 3.2: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then closed sets in $(X, \tau)$ are $X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$.

Semi-closed sets in $(X, \tau)$ are $X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$.

W-closed sets in $(X, \tau)$ are $X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$.

ws-closed sets in $(X, \tau)$ are $X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$.

ws-open sets in $(X, \tau)$ are $X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$.

We prove that the class of ws-closed sets are properly lies between the class of all semi-closed sets and generalised semi-pre regular closed sets in topological spaces.

Theorem 3.3: Every semi-closed [19] set in $X$ is ws-closed set in $X$.

Proof: Let $A$ be a semi-closed set in $X$. Let $U$ be any $w$-open set in $X$ such that $A \subseteq U$, we have $\text{scl}(A) \subseteq U$. Since $A$ is semi-closed, we have $\text{scl}(A) = A \subseteq U$. Hence $A$ is ws-closed set in $X$.

Remark 3.4: The converse of the above Theorem 3.3 need not be true as seen from the following Example 3.5.

Example 3.5: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ then the set $A = \{a, b, d\}$ is ws-closed set but not semi-closed in $X$.

Corollary 3.6: In a topological space $(X, \tau)$,


ii) Every closed set in $X$ is ws-closed set in $X$.

iii) Every $\alpha$-closed [20] set in $X$ is ws-closed set in $X$.

iv) Every $g^s$-closed [37] set in $X$ is ws-closed set in $X$.

v) Every $^*g\alpha$-closed [41] set in $X$ is ws-closed set in $X$.

vi) Every $g^s$-$\alpha$-closed [40] set in $X$ is ws-closed set in $X$.


viii) Every $\tilde{g}$-closed set in $X$ is ws-closed set in $X$.

ix) Every $g\xi^*$-closed [17] set in $X$ is ws-closed set in $X$.


Proof:

i) In view of the fact that every regular closed is semi-closed, therefore by 3.3 every regular closed is ws-closed set.

ii) In view of the fact that every closed set is semi-closed, therefore by 3.3 every closed set is ws-closed set.

iii) In view of the fact that every $\alpha$-closed is semi-closed, therefore by 3.3 every $\alpha$-closed is ws-closed set.

iv) Let $A$ be $g^s$-closed set in $X$. Let $U$ be any $w$-open set in $X$ s.t $A \subseteq U$. Since $A$ is $g^s$-closed, we have $\text{cl}(A) = A \subseteq U$, we have $\text{scl}(A) \subseteq U$. Hence $A$ is ws-closed set in $X$.

v) Let $A$ be $^*g\alpha$-closed set in $X$. Let $U$ be any $w$-open set in $X$ s.t $A \subseteq U$. Since $A$ is $^*g\alpha$-closed, we have $\text{cl}(A) = A \subseteq U$, we have $\text{scl}(A) \subseteq U$. Hence $A$ is ws-closed set in $X$.

vi) Let $A$ be $g^s$-$\alpha$-closed set in $X$. Let $U$ be any $w$-open set in $X$ s.t $A \subseteq U$. Since $A$ is $g^s$-$\alpha$-closed, we have $\text{cl}(A) = A \subseteq U$, we have $\text{scl}(A) \subseteq U$. Hence $A$ is ws-closed set in $X$.

vii) Let $A$ be $rb$-closed set in $X$. Let $U$ be any $w$-open set in $X$ s.t $A \subseteq U$. Since $A$ is $rb$-closed, we have $\text{cl}(A) = A \subseteq U$, we have $\text{scl}(A) \subseteq U$. Hence $A$ is ws-closed set in $X$.

viii) Let $A$ be $\tilde{g}$-closed set in $X$. Let $U$ be any $w$-open set in $X$ s.t $A \subseteq U$. Since $A$ is $\tilde{g}$-closed, we have $\text{cl}(A) = A \subseteq U$, we have $\text{scl}(A) \subseteq U$. Hence $A$ is ws-closed set in $X$.

ix) Let $A$ be $g\xi^*$-closed set in $X$. Let $U$ be any $w$-open set in $X$ s.t $A \subseteq U$. Since $A$ is $g\xi^*$-closed, we have $\text{cl}(A) = A \subseteq U$, we have $\text{scl}(A) \subseteq U$. Hence $A$ is ws-closed set in $X$.

x) Let $A$ be $agp$-closed set in $X$. Let $U$ be any $w$-open set in $X$ s.t $A \subseteq U$. Since $A$ is $agp$-closed, we have $\text{cl}(A) = A \subseteq U$, we have $\text{scl}(A) \subseteq U$. Hence $A$ is ws-closed set in $X$.

Remark 3.7: The converse of the above Corollary 3.6 need not be true as seen from the following Example 3.8.
Example 3.8: Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ then the sets

i. regular-closed sets in $(X, \tau)$ are $X, \phi, \{a, c, d\}, \{b, c, d\}$.

ii. closed sets in $(X, \tau)$ are $X, \phi, \{a\}, \{c, d\}, \{a, c, d\}$.

iii. $\alpha^\#$ -closed sets in $(X, \tau)$ are $X, \phi, \{c\}, \{a, d\}, \{a, c, d\}$.

iv. $g^*$ -closed sets in $(X, \tau)$ are $X, \phi, \{d\}, \{a, c, d\}, \{b, c, d\}$.

v. $*g\alpha$ -closed sets in $(X, \tau)$ are $X, \phi, \{c\}, \{a, c, d\}$.

vi. $g^\#s$ -closed sets in $(X, \tau)$ are $X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}$.

vii. $rb$ -closed sets in $(X, \tau)$ are $X, \phi, \{c, d\}, \{a, c, d\}, \{b, c, d\}$.

viii. $g\xi*$ -closed sets in $(X, \tau)$ are $X, \phi, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}$.

ix. $g^\# -closed sets in (X, \tau)$ are $X, \phi, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}$.

x. $wgs$ -closed sets in $(X, \tau)$ are $X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

It is observed that set $A = \{a, b, d\}$ is ws-closed set but not regular closed (closed, $\alpha^\#$ -closed, $g^*$ -closed, $*g\alpha$ –closed, $g^\#s$ –closed, $rb$ -closed, $g\xi*$ -closed, $g^\# -closed$ sets) in $X$.


**Proof:** Let $A$ be a ws-closed set in $X$. Let $U$ be any regular open set in $X$ such that $A \subseteq U$. Since every regular open set is w- open set and $A$ is ws-closed set, we have $sc(A) \subseteq U$. Therefore $sc(A) \subseteq U$. Hence $A$ is regular open in $X$.

**Remark 3.10:** The converse of the above Theorem 3.9 need not be true as seen from the following Example 3.11.

Example 3.11: Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then the set $A = \{b\}$ is $gspr$ -closed set but not ws-closed set in $X$.

Corollary 3.12:


ii) Every ws-closed set is rgb-closed [22] set in $X$.

**Proof:**

i) Follow from Govindappa Navalagi et al.[8], every $gspr$-closed set is gsp-closed set and then follows from Theorem 3.9.

ii) Let $A$ be a ws-closed set in $X$. Let $U$ be any regular open set in $X$ such that $A \subseteq U$. Since every regular open set is w- open set and $A$ is ws-closed set, we have $sc(A) \subseteq U$. Therefore $sc(A) \subseteq U$. Hence $A$ is regular open in $X$. Hence $A$ is rgb -closed in $X$.

The converse of the Corollary 3.12 is need not be true in general as seen from the following Example 3.13.

Example 3.13: Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{c\}\}$. Then the set $A = \{b\}$ is $gspr$ (rgb) -closed set but not ws-closed set in $X$.


Example 3.15: Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ Then

i) closed sets in $(X, \tau)$ are $X, \phi, \{a\}, \{c, d\}, \{a, c, d\}$.

ii) ws-closed sets in $(X, \tau)$ are $X, \phi, \{a\}, \{c, d\}, \{a, c\}, \{a, c, d\}$.

iii) $\alpha^\#$ -closed sets in $(X, \tau)$ are $X, \phi, \{c\}, \{a, c\}$.

iv) $g^*$ -closed sets in $(X, \tau)$ are $X, \phi, \{c\}, \{a, c\}$.

v) $wgr\alpha$ -closed sets in $(X, \tau)$ are $X, \phi, \{c\}, \{a, c\}$.

vi) $pg\alpha$ -closed sets in $(X, \tau)$ are $X, \phi, \{c\}, \{a, c\}$.
vi) \( \text{rg} - \)closed sets in \((X, \tau)\) are \(X, \phi, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\).

vii) \( \text{gprw} - \)closed sets in \((X, \tau)\) are \(X, \phi, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\).

viii) \( \text{rgw} - \)closed sets in \((X, \tau)\) are \(X, \phi, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\).

ix) \( \text{rw} - \)closed sets in \((X, \tau)\) are \(X, \phi, \{d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\).

x) \( \text{rga} - \)closed sets in \((X, \tau)\) are \(X, \phi, \{c\}, \{d\}, \{a, b\}, \{a, c, d\}, \{a, b, d\}, \{b, c\}, \{a, d\}, \{b, c, d\}\).

xi) \( \text{βwg} - \)closed sets in \((X, \tau)\) are \(X, \phi, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}\).

Therefore \(\{a\}\) is ws-closed in \(X\) but not \(\text{gpr-closed}\) (resp. \(\text{wgra-closed}, \text{rgw-closed}, \text{rga-closed}, \text{βwg-closed}\)) set in \(X\).

**Remark 3.16:** The following Example 3.17 shows that ws-closed sets are independent of \(\text{gαb-closed}[39]\) sets, \(\text{βwg-closed}[7]\) sets, \(\text{gα-closed}[41]\) sets, \(\text{Gα-closed}[38]\) sets, \(\text{Gα-closed}[14]\) sets, \(\text{gα-closed}[6]\) sets, \(\text{gα-closed}[28]\) sets, \(\text{gα-closed}[21]\) sets, \(\text{sgb-closed}[13]\) sets, \(\text{pgpr-closed}[12]\) sets, \(\text{gα-closed}[42]\) sets and \(\text{rgα-closed}[34]\) sets.

**Example 3.17:** Let \(X = \{a, b, c, d\}\), \(\tau_1 = \{X, \phi, \{a\}, \{a, b\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}\). Then

i) \(\text{closed sets in } (X, \tau_1)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

ii) \(\text{ws-closed sets in } (X, \tau_1)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

iii) \(\text{gα-closed sets in } (X, \tau_1)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

iv) \(\text{gα-closed sets in } (X, \tau_2)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

v) \(\text{βwg-closed sets in } (X, \tau_1)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

vi) \(\text{βwg-closed sets in } (X, \tau_2)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

vii) \(\text{gα-closed sets in } (X, \tau_1)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

viii) \(\text{gα-closed sets in } (X, \tau_2)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

ix) \(\text{βwg-closed sets in } (X, \tau_1)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

x) \(\text{βwg-closed sets in } (X, \tau_2)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

xi) \(\text{βwg-closed sets in } (X, \tau_1)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).

Therefore \(\{b\}\) is ws-closed in \((X, \tau_1)\) but not in \(\text{gα-closed}\) (resp., \(\text{gα-closed}, \text{gα-closed}, \text{βwg-closed}, \text{βwg-closed}, \text{βwg-closed}\) set in \((X, \tau_1)\).

Meanwhile \(\{b\}\) in \(\text{gα-closed}\) (resp., \(\text{gα-closed}, \text{gα-closed}, \text{βwg-closed}, \text{βwg-closed}, \text{βwg-closed}\) set in \((X, \tau_2)\) but not \(\text{ws-closed set in } (X, \tau_2)\).

**Remark 3.18:** The following Example 3.19 shows that \(\text{ws-closed sets are independent of } \text{sets } g\text{-closed}[18] \text{ sets, } sg\text{-closed}[14] \text{ sets, } ga\text{-closed}[21] \text{ sets, } sgb\text{-closed}[13] \text{ sets, } rg\text{-closed}[12] \text{ sets, } pgr\text{-closed}[1] \text{ sets, } gab\text{-closed}[42] \text{ sets and } rps\text{-closed}[34] \text{ sets.}

**Example 3.19:** Let \(X = \{a, b, c, d\}\), \(\tau_1 = \{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}\). Then

i) \(\text{closed sets in } (X, \tau_1)\) are \(X, \phi, \{d\}, \{c, d\}, \{b, c, d\}\).

ii) \(\text{ws-closed sets in } (X, \tau_1)\) are \(X, \phi, \{a\}, \{b\}, \{c\}, \{d\}\).
iii) $g$-closed sets in $(X, \tau_1)$ are $X$, $\phi$, $\{a, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c\}$.
iv) $sg$-closed sets in $(X, \tau_1)$ are $X$, $\phi$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

v) $g\alpha$-closed sets in $(X, \tau_1)$ are $X$, $\phi$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$.
vi) $sg\beta$-closed sets in $(X, \tau_1)$ are $X$, $\phi$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

vii) $rg\ast b$-closed sets in $(X, \tau_1)$ are $X$, $\phi$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

viii) $pgpr$-closed sets in $(X, \tau_1)$ are $X$, $\phi$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.
ix) $g\alpha b$-closed sets in $(X, \tau_1)$ are $X$, $\phi$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

x) $rps$-closed sets in $(X, \tau_1)$ are $X$, $\phi$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ and also $\{a\}$.

xi) $g\ast$-closed sets in $(X, \tau_2)$ are $X$, $\phi$, $\{a, b\}$, $\{c, d\}$.

xii) $ws$-closed sets in $(X, \tau_2)$ are $X$, $\phi$, $\{a, b\}$, $\{c, d\}$.

xiii) $g$-closed sets in $(X, \tau_2)$ are $X$, $\phi$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

xiv) $sg$-closed sets in $(X, \tau_2)$ are $X$, $\phi$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

xv) $g\alpha$-closed sets in $(X, \tau_2)$ are $X$, $\phi$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

xvi) $sg\beta$-closed sets in $(X, \tau_2)$ are $X$, $\phi$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

xvii) $rg\ast b$-closed sets in $(X, \tau_2)$ are $X$, $\phi$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

xviii) $pgpr$-closed sets in $(X, \tau_2)$ are $X$, $\phi$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

Therefore $\{a\}$ is $ws$-closed set in $(X, \tau_1)$ but not $g$-closed (resp. $sg$-closed, $g\alpha$-closed, $sg\beta$-closed, $rg\ast b$-closed, $pgpr$-closed, $g\alpha b$-closed, $rps$-closed) set in $(X, \tau_1)$.

Meanwhile $\{a\}$ is $g$-closed (resp. $sg$-closed, $g\alpha$-closed, $sg\beta$-closed, $rg\ast b$-closed, $pgpr$-closed, $g\alpha b$-closed, $rps$-closed) set in $(X, \tau_2)$ but not $ws$-closed set in $(X, \tau_2)$.


**Example 3.21:** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}\}$. Then

i) closed sets in $(X, \tau)$ are $X$, $\phi$, $\{c\}$, $\{a, c\}$, $\{b, c\}$.
ii) $ws$-closed sets in $(X, \tau)$ are $X$, $\phi$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, c\}$.
iii) $R^*$-closed sets in $(X, \tau)$ are $X$, $\phi$, $\{c\}$, $\{a\}$, $\{b\}$, $\{a, b\}$.
iv) $rg\beta$-closed sets in $(X, \tau)$ are $X$, $\phi$, $\{c\}$, $\{a\}$, $\{b\}$, $\{a, b\}$.
v) $pgpr$-closed sets in $(X, \tau)$ are $X$, $\phi$, $\{c\}$, $\{a\}$, $\{b\}$, $\{a, c\}$
vi) $rgw$-closed sets in $(X, \tau)$ are $X$, $\phi$, $\{c\}$, $\{a\}$, $\{b\}$, $\{a, c\}$

Therefore $\{a\}$ is $ws$-closed set in $X$ but not $R^*$-closed (resp. $rg\beta$-closed, $pgpr$-closed, $rgw$-closed, $gprw$-closed) set in $X$.

**Remark 3.22:** From the above discussion and results we have the following implications.
A $\Rightarrow$ B means A implies B, but converse is not true.

A $\Leftrightarrow$ B means A and B are independent of each other

**Theorem 3.23:** The intersection of two ws-closed subsets of X is ws-closed set in X.

**Proof:** Let A and B be are ws-closed sets in X. Let U be any semiopen set in X such that $(A \cap B) \subseteq U$ that is $A \subseteq U$ and $B \subseteq U$. Since A and B are ws-closed sets then $\text{scl}(A) \subseteq U$ and $\text{scl}(B) \subseteq U$ and we know that $(\text{scl}(A) \cap \text{scl}(B)) = \text{scl}(A \cap B) \subseteq U$. Therefore $A \cap B$ is ws-closed set in X.

**Remark 3.24:** The union of two ws-closed sets in X is generally not a ws-closed set in X.

**Example 3.25:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ then the sets $A = \{a\}$ and $B = \{b\}$ are ws-closed sets in X but $A \cup B = \{a, b\}$ is not a ws-closed set in X.

**Theorem 3.26:** If a subset A of a topological space X is ws-closed set in X then $\text{scl}(A) - A$ does not contain any non-empty open set in X but converse is not true.

**Proof:** Let A is an ws-closed set in X and suppose F be an non empty w-closed subset of $\text{scl}(A) - A$. 

$F \subseteq \text{scl}(A) - A \Rightarrow F \subseteq \text{scl}(A) \cap (X - A) \Rightarrow F \subseteq \text{scl}(A) \Rightarrow \text{FGX} - A \\
\Rightarrow A \subseteq X - F$ and $X - F$ is w-open set and A is a ws-closed set, $\text{scl}(A) \subseteq X - F$ \\
$\Rightarrow F \subseteq X - \text{scl}(A)$ from equations (1) and (2) we get $F \subseteq \text{scl}(A) \cap (X - \text{scl}(A)) = \emptyset$ \\
$\Rightarrow F = \emptyset$ thus $\text{scl}(A) - A$ does not contain any non-empty w-closed set in X.

**Remark 3.27:** The converse of the above Theorem need not be true as seen from the following Example 3.28.

**Example 3.28:** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}\}$ then the set $A = \{b\}$, $\text{scl}(\{b\}) = \{b\}$, $\text{scl}(\{A\} - A) = \{b\}$ does not contain any non-empty w-closed set in X but A is not ws-closed set.
Theorem 3.29: If $A$ is a ws-closed set in $X$ and $A \subseteq B \subseteq \text{scl}(A)$ then $B$ is also ws-closed set in $X$.

Proof: Let $A$ be a ws-closed set in $X$ such that $B \subseteq \text{scl}(A)$. Let $U$ be a w-open set of $X$ such that $B \subseteq U$ then $A \subseteq U$. Since $A$ is ws-closed set, we have $\text{scl}(A) \subseteq U$ and $A \subseteq U$. Now $B \subseteq \text{scl}(A) \Rightarrow \text{scl}(B) \subseteq \text{scl}(\text{scl}(A)) = \text{scl}(A) \subseteq U$. That is $\text{scl}(B) \subseteq U$. Therefore $B$ is a ws-closed set in $X$.

Remark 3.30: The converse of the above Theorem 3.29 is need not be true as seen from the following Example 3.31.

Example 3.31: Let $X = \{a, b, c \}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, then the set $A = \{a\}, B = \{a, c\}$ such that $A$ and $B$ are ws-closed sets in $X$ but $A \subseteq B \nsubseteq \text{scl}(A)$ because $\text{scl}(A) = \{a\}$.

Theorem 3.32: Let $(X, \tau)$ be a topological space then for each $x \in X$ the set $X - \{x\}$ is ws-closed or semi open.

Proof: Let $x \in X$. Suppose $X - \{x\}$ is not a semiopen set. Then there is no semiopen set containing $X - \{x\}$, that is $X - \{x\} \subseteq X \Rightarrow \text{cl}(X - \{x\}) \subseteq \text{cl}(X) \Rightarrow \text{cl}(X - \{x\}) \subseteq X$. Therefore $X - \{x\}$ is ws-closed set in $X$.

Theorem 3.33: Let $X$ and $Y$ are topological spaces and $A \subseteq Y \subseteq X$. Suppose that $A$ is ws-closed set in $X$ then $A$ is ws-closed relative to $Y$.

Proof: Let $A \subseteq Y \cap G$, where $G$ is a w-open. Since $A$ is a ws-closed set in $X$, then $A \subseteq G$ and $\text{scl}(A) \subseteq G$. This implies that $Y \cap \text{scl}(A) \subseteq Y \cap G$ where $Y \cap \text{scl}(A)$ is closed set of $A$ in $Y$. Thus $A$ is a ws-closed set in $Y$.

Theorem 3.34: In a topological space $X$ if $\text{SO}(X) = \{X, \emptyset\}$ then every subset of $X$ is a ws-closed set.

Proof: Let $X$ be a topological space and $\text{SO}(X) = \{X, \emptyset\}$. Let $A$ be any subset of $X$. Suppose $A = \emptyset$. Then $\emptyset$ is ws-closed set. Suppose $A \neq \emptyset$. Then $X$ is the only semiopen set containing $A$ and so $\text{scl}(A) \subseteq X$. Hence $A$ is a ws-closed set in $X$.

Remark 3.35: The converse of the above Theorem need not be true in general as seen from the following Example 3.36.

Example 3.36: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then every subset of $(X, \tau)$ is a ws-closed set in $X$ but $\text{SO} = \{\emptyset, X, \{a\}, \{b, c\}\}$.

Theorem 3.37: If $A$ is regular open and gspr-closed set in $X$ then $A$ is ws-closed set in $X$.

Proof: Let $A$ be a regular open and gspr-closed in $X$. Let $U$ be any w-open set in $X$ such that $A \subseteq U$. Since $A$ is regular open and gspr-closed set in $X$, by definition, $\text{scl}(A) \subseteq A$ then $\text{scl}(A) \subseteq A \subseteq U$. Hence $A$ is ws-closed set in $X$.

Theorem 3.38: If $A$ is regular open and rgb-closed set in $X$ then $A$ is ws-closed set in $X$.

Proof: Let $A$ be a regular open and rgb-closed in $X$. Let $U$ be any w-open set in $X$ such that $A \subseteq U$. Since $A$ is regular open and rgb-closed in $X$, by definition, $\text{scl}(A) \subseteq A$ then $\text{scl}(A) \subseteq A \subseteq U$. Hence $A$ is ws-closed set in $X$.

Theorem 3.39: If $A$ is semiopen and swg*-closed then $A$ is ws-closed set in $X$.

Proof: Let $A$ be a semiopen and swg*-closed in $X$. Let $U$ be any w-open set in $X$ such that $A \subseteq U$. Since $A$ is semiopen and swg*-closed in $X$, by definition, $\text{scl}(A) \subseteq A$ then $\text{scl}(A) \subseteq A \subseteq U$. Hence $A$ is ws-closed set in $X$.

Theorem 3.40: If $A$ is semiopen and swg-closed then $A$ is ws-closed set in $X$.

Proof: Let $A$ be a semiopen and swg-closed in $X$. Let $U$ be any w-open set in $X$ such that $A \subseteq U$. Since $A$ is semiopen and swg-closed in $X$, by definition, $\text{scl}(A) \subseteq A$ then $\text{scl}(A) \subseteq A \subseteq U$. Hence $A$ is ws-closed set in $X$.

Theorem 3.41: If $A$ is semiopen and sg-closed then $A$ is ws-closed set in $X$.

Proof: Let $A$ be a semiopen and sg-closed in $X$. Let $U$ be any w-open set in $X$ such that $A \subseteq U$. Since $A$ is semiopen and sg-closed in $X$, by definition, $\text{scl}(A) \subseteq A$ then $\text{scl}(A) \subseteq A \subseteq U$. Hence $A$ is ws-closed set in $X$.

Theorem 3.42: If $A$ is semiopen and sgb-closed then $A$ is ws-closed set in $X$.

Proof: Let $A$ be a semiopen and sgb-closed in $X$. Let $U$ be any w-open set in $X$ such that $A \subseteq U$. Since $A$ is semiopen and sgb-closed in $X$, by definition, $\text{scl}(A) \subseteq A$ then $\text{scl}(A) \subseteq A \subseteq U$. Hence $A$ is ws-closed set in $X$. 
Theorem 3.43: If A is semiopen and \(\alpha_{gs}\)-closed then A is ws-closed set in X.

Proof: Let A be a semiopen and \(\alpha_{gs}\)-closed in X. Let U be any w-open set in X such that \(A \subseteq U\). Since A is semiopen and \(\alpha_{gs}\)-closed in X, by definition, \(scl(A) \subseteq A\) then \(scl(A) \subseteq A \subseteq U\). Hence A is ws-closed set in X.

Theorem 3.44: If A is \(\beta\)-open and \(\beta wg^*\)-closed then A is ws-closed set in X.

Proof: Let A be a \(\beta\)-open and \(\beta wg^*\)-closed in X. Let U be any regular semiopen set in X such that \(A \subseteq U\). Since A is \(\beta\)-open and \(\beta wg^*\)-closed in X, by definition, \(gcl(A) \subseteq A\) then \(gcl(A) \subseteq A \subseteq U\). Hence A is ws-closed set in X.

Theorem 3.45: If A is both open and g-closed then A is ws-closed set in X.

Proof: Let A be open and g-closed set in X. Let U be any regular open set in X such that \(A \subseteq U\). By definition, \(cl(A) \subseteq A \subseteq U\) and \(gcl(A) = A\). This implies that \(cl(A) \subseteq gcl(A) \subseteq A \subseteq U\). Hence A is ws-closed set.

Theorem 3.46: If A is regular semiopen and \(rw\)-closed then A is ws-closed set in X.

Proof: Let A be a regular semiopen and \(rw\)-closed set in X. Let U be any w-open set in X such that \(A \subseteq U\). Now \(A \subseteq A\) by hypothesis \(cl(A) \subseteq A\) then we know that \(cl(A) \subseteq scl(A) \subseteq A\). Hence \(scl(A) \subseteq U\) therefore A is ws-closed set in X.

Theorem 3.47: If A is regular semiopen and \(R^*\)-closed then A is ws-closed set in X.

Proof: Let A be a regular semiopen and \(R^*\)-closed set in X. Let U be any w-open set in X such that \(A \subseteq U\). Now \(A \subseteq A\) by hypothesis \(cl(A) \subseteq A\) then we know that \(cl(A) \subseteq scl(A) \subseteq A\). Hence \(scl(A) \subseteq U\) therefore A is ws-closed set in X.

Theorem 3.48: If A is regular semiopen and \(gprw\)-closed then A is ws-closed set in X.

Proof: Let A be a regular semiopen and \(gprw\)-closed set in X. Let U be any w-open set in X such that \(A \subseteq U\). Now \(A \subseteq A\) by hypothesis \(cl(A) \subseteq A\) then we know that \(cl(A) \subseteq scl(A) \subseteq A\). Hence \(scl(A) \subseteq U\) therefore A is ws-closed set in X.

Theorem 3.49: If A is regular semiopen and \(rgw\)-closed then A is ws-closed set in X.

Proof: Let A be a regular semiopen and \(rgw\)-closed set in X. Let U be any w-open set in X such that \(A \subseteq U\). Now \(A \subseteq A\) by hypothesis \(cl(A) \subseteq A\) then we know that \(cl(A) \subseteq scl(A) \subseteq A\). Hence \(scl(A) \subseteq U\) therefore A is ws-closed set in X.

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