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FIXED POINT RESULTS IN SOFT G-METRIC SPACES

SHINJINI SOLANKI[1], BASANT KUMAR SINGH[1], RAMAKANT BHARDWAJ*[2]
AND ANURAG CHOUBEY[3]

[1]Research Scholar of Mathematics, [1*]Department of Mathematics,
AISECT University Bhopal – (M.P.), India.

[2]Department of Mathematics, [3]Department of Computer Science,
TIT Group of Institutes Bhopal – (M.P.), India.

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ABSTRACT

In the present chapter, we prove fixed point results of mapping defined on soft G-metric space which generalize many well known results.

2. INTRODUCTION & PRELIMINARIES

In the year 1999, Molodtsov [8] initiated a novel concept of soft sets theory as a new mathematical tool for dealing with uncertainties. A soft set is a collection of approximate descriptions of an object. Soft systems provide a very general framework with the involvement of parameters. Since soft set theory has a rich potential, applications of soft set theory in other disciplines and real life problems are progressing rapidly. Maji *et al.* [5, 6] worked on soft set theory and presented an application of soft sets in decision making problems.

Guler *et. al.* [4] introduced the concept of soft G-metric space according to a soft element and obtained some of its properties. Then, they defined soft G-convergence and soft G-continuity, they proved existence and uniqueness of fixed pints in soft G-metric spaces.

Our aim of this article is to present a fixed point theorems in soft G-metric space satisfying a new rational contractive condition.

Definition 2.1: Let X be an initial universe set and E be a set of parameters. A pair (F, E) is called a soft set over X if and only if X is a mapping from E into the set of all subsets of the set $X, i.e. F: E \to P(X)$, where $P(X)$ is the power set of X.

Definition 2.2: The intersection of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cap B$ and $\forall \varepsilon \in C, H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$.This is denoted by $(F, A) \cap (G, B) = (H, C)$.

Definition 2.3: The union of two soft sets (F, A) and (G, B) over X is the soft set, where $C = A \cup B$ and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & if\ \varepsilon \in A - B \\ G(\varepsilon), & if\ \varepsilon \in B - A \\ F(\varepsilon) \cup G(\varepsilon), & \varepsilon \in A \cap B \end{cases}$$

This relationship is denoted by $(F, A) \cup (G, B) = (H, C)$.

Definition 2.4: The soft set (F, A) over X is said to be a null soft set denoted by Φ if for all $\varepsilon \in A, F(\varepsilon) = \phi$ (null set)

Definition 2.5: A soft set (F, A) over X is said to be an absolute soft set, if for all $\varepsilon \in A, F(\varepsilon) = X$.

Definition 2.6: The difference (H, E) of two soft sets (H, E) and (H, E) over X denoted by $(H, E)\backslash(H, E)$, is defined as $H(e) = F(e)\backslash G(e)$ for all $e \in E$.

Corresponding Author: Dr. Ramakant Bhardwaj[2]*
[2]Department of Mathematics, TIT Group of Institutes Bhopal – (M.P.), India.

Definition 2.7: The complement of a soft set (F,A) is denoted by $(F,A)^c$ and is defined by $(F,A)^c = (F^c, A)$ where $F^c: A \to P(X)$ is mapping given by $F^c(\alpha) = X - F(\alpha), \forall \alpha \in A$.

Definition 2.8: Let $\mathfrak{R}$ be the set of real numbers and $B(\mathfrak{R})$ be the collection of all nonempty bounded subsets of $\mathfrak{R}$ and E taken as a set of parameters. Then a mapping $F: E \to B(\mathfrak{R})$ is called a soft real set. It is denoted by (F,E). If specifically (F,E) is a singleton soft set, then identifying (F,E) with the corresponding soft element, it will be called a soft real number and denoted $\tilde{r}, \tilde{s}, \tilde{t}$ etc.

$\overline{0}, \overline{1}$ are the soft real numbers where $\overline{0}(e) = 0, \overline{1}(e) = 1$ for all $e \in E$, respectively.

Definition 2.9: For two soft real numbers

(i) $\tilde{r} \le \tilde{s}$, if $\tilde{r}(e) \le \tilde{s}(e)$, for all $e \in E$.
(ii) $\tilde{r} \ge \tilde{s}$, if $\tilde{r}(e) \ge \tilde{s}(e)$, for all $e \in E$.
(iii) $\tilde{r} < \tilde{s}$, if $\tilde{r}(e) < \tilde{s}(e)$, for all $e \in E$.
(iv) $\tilde{r} > \tilde{s}$, if $\tilde{r}(e) > \tilde{s}(e)$, for all $e \in E$.

Definition 2.10: A soft set over X is said to be a soft point if there is exactly one $e \in E$, such that $P(e) = \{x\}$ for some $x \in X$ and $P(e') = \phi, \forall e' \in E \backslash \{e\}$. It will be denoted by $\tilde{x}_e$.

Definition 2.11: Two soft points $\tilde{x}_e, \tilde{y}_e$ are said to be equal if $e = e'$ and $P(e) = P(e')$ i.e. $x = y$. Thus
$\tilde{x}_e \ne \tilde{y}_e \Leftrightarrow x \ne y$ or $e \ne e'$.

Definition 2.12: A mapping $\tilde{d}: SP(\tilde{X}) \times SP(\tilde{X}) \to \mathbb{R}(E)^*$, is said to be a soft metric on the soft set $\tilde{X}$ if d satisfies the following conditions:

(M1) $\tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) \geqq \overline{0}$ for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \tilde{\in} \tilde{X}$,
(M2) $\tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) = \overline{0}$ if and only if $\tilde{x}_{e_1} = \tilde{y}_{e_2}$,
(M3) $\tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) \geqq \tilde{d}(\tilde{y}_{e_2}, \tilde{x}_{e_1})$ for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \tilde{\in} \tilde{X}$,
(M4) $\tilde{d}(\tilde{x}_{e_1}, \tilde{z}_{e_3}) \leqq \tilde{d}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) + \tilde{d}(\tilde{y}_{e_2}, \tilde{z}_{e_3})$ for all $\tilde{x}_{e_1}, \tilde{y}_{e_2}, \tilde{z}_{e_3} \tilde{\in} \tilde{X}$.

The soft set $\tilde{X}$ with a soft metric $\tilde{d}$ on $\tilde{X}$ is called a soft metric space and denoted by $(\tilde{X}, \tilde{d}, E)$.

Definition 2.13 (Cauchy Sequence): A sequence $\{\tilde{x}_{\lambda,n}\}_n$ of soft points in $(\tilde{X}, \tilde{d}, E)$ is considered as a Cauchy sequence in $\tilde{X}$ if corresponding to every $\tilde{\varepsilon} \geqq \overline{0}, \exists m \in N$ such that $d(\tilde{x}_{\lambda,i}, \tilde{x}_{\lambda,j}) \leqq \tilde{\varepsilon}, \forall\, i,j \ge m$, i.e. $d(\tilde{x}_{\lambda,i}, \tilde{x}_{\lambda,j}) \to \overline{0}$, as $i,j \to \infty$.

Definition 2.14 (Soft Complete Metric Space): A soft metric space $(\tilde{X}, \tilde{d}, E)$ is called complete, if every Cauchy Sequence in $\tilde{X}$ converges to some point of $\tilde{X}$.

Definition 2.15[4]: Let X be a nonempty set and E be the nonempty set of parameters.

Let $\tilde{G}: SE(\tilde{X}) \times SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(E)^*$ be a function satisfying the following axioms:

$(\tilde{G}_1)$ $\tilde{G}(\tilde{x}, \tilde{y}, \tilde{z}) = 0$ if $\tilde{x} = \tilde{y} = \tilde{z}$
$(\tilde{G}_2)$ $\tilde{G}(\tilde{x}, \tilde{x}, \tilde{y}) > 0$ for all $\tilde{x} = \tilde{y} \in SE(\tilde{X})$ with $\tilde{x} \ne \tilde{y}$
$(\tilde{G}_3)$ $\tilde{G}(\tilde{x}, \tilde{x}, \tilde{y}) \le \tilde{G}(\tilde{x}, \tilde{y}, \tilde{z})$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in SE(\tilde{X})$ with $\tilde{y} \ne z$
$(\tilde{G}_4)$ $\tilde{G}(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{G}(\tilde{x}, \tilde{z}, \tilde{y}) = \tilde{G}(\tilde{y}, \tilde{z}, x) = \cdots$ (Symmetry in all three variables)
$(\tilde{G}_5)$ $\tilde{G}(\tilde{x}, \tilde{y}, \tilde{z}) \le \tilde{G}(\tilde{x}, a, a) + \tilde{G}(a, \tilde{y}, z)$ for all $\tilde{x}, \tilde{y}, \tilde{z}, a \in X$ (Rectangle inequality)

Then the function $\tilde{G}$ is called a soft generalized metric or soft G-metric on $\tilde{X}$ and $(\tilde{X}, \tilde{G}, E)$ is called a soft G-metric space.

Definition 2.16: Let $(\tilde{X}, \tilde{G}, E)$ be a soft G-metric space, let $\{\tilde{x}_n\}$be a sequence of soft points of $\tilde{X}$, a soft point $\tilde{x} \in \tilde{X}$ is said to the limit of the sequence $\{\tilde{x}_n\}$, if $lim_{n\to\infty} G(\tilde{x}, \tilde{x}_n, \tilde{x}_m) = 0$. Then $\{\tilde{x}_n\}$ is G-convergent to $\tilde{X}$.

Proposition 2.17[4]: Let $(\tilde{X}, \tilde{G}, E)$ be a soft G-metric space, then for a sequence $\{\tilde{x}_n\} \subseteq \tilde{X}$ and a soft point $\tilde{x} \in \tilde{X}$. The following are equivalent

(i) $\{\tilde{x}_n\}$ is soft G-convergent to $\tilde{x}$.
(ii) $\tilde{G}(\tilde{x}_n, \tilde{x}_n, \tilde{x}) \to 0$ as $n \to \infty$
(iii) $\tilde{G}(\tilde{x}_n, \tilde{x}, \tilde{x}) \to 0$ as $n \to \infty$
(iv) $\tilde{G}(\tilde{x}_m, \tilde{x}_n, \tilde{x}) \to 0$ as $m, n \to \infty$.

Definition 2.18: Let $(\tilde{X}, \tilde{G}, E)$ be a soft G-metric space, then the sequence $\{\tilde{x}_n\}$ is said to be soft G-Cauchy if for every $\varepsilon > 0$ there exists a positive integer N such that $\tilde{G}(\tilde{x}_n, \tilde{x}_m, \tilde{x}_l) < \varepsilon$ for all $n, m, l \geq N$ i.e. $\tilde{G}(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 2.19: A soft G-metric space $(\tilde{X}, \tilde{G}, E)$ is said to be soft G-complete space if every soft G-Cauchy sequence in $(\tilde{X}, \tilde{G}, E)$ is G-convergent in $(\tilde{X}, \tilde{G}, E)$.

Proposition 2.20[4]: Let $(\tilde{X}, \tilde{G}, E), (\tilde{X}', \tilde{G}', E')$ be two soft G-metric spaces, then a function $f: \tilde{X} \to \tilde{X}'$ is soft G-continuous at a soft point $\tilde{x} \in SE(\tilde{X})$ if and only if it is soft G-sequentially continuous at $\tilde{x} \in SE(\tilde{X})$; i.e. whenever $\{\tilde{x}_n\}$ is soft G-convergent to $\tilde{x}$, $\{f(\tilde{x}_n)\}$ is soft G-convergent to $f(\tilde{x})$.

3 MAIN RESULTS

Our main results of this article are as follows.

Theorem 3.1: Let $(\tilde{X}, \tilde{G}, E)$ be a soft G-metric space and $T: (\tilde{X}, \tilde{G}, E) \to (\tilde{X}, \tilde{G}, E)$ be a mapping that satisfies the following condition

$$\tilde{G}(T\tilde{x}, T\tilde{y}, T\tilde{z}) \leq \frac{a_1\tilde{G}(\tilde{x}, T\tilde{x}, T\tilde{x})\tilde{G}(\tilde{x}, T\tilde{y}, T\tilde{y})\tilde{G}(\tilde{y}, T\tilde{z}, T\tilde{z}) + a_2\tilde{G}(\tilde{y}, T\tilde{y}, T\tilde{y})\tilde{G}(\tilde{y}, T\tilde{x}, T\tilde{x})\tilde{G}(\tilde{x}, T\tilde{z}, T\tilde{z})}{\tilde{G}(\tilde{x}, T\tilde{y}, T\tilde{y})\tilde{G}(\tilde{y}, T\tilde{z}, T\tilde{z}) + \tilde{G}(\tilde{y}, T\tilde{x}, T\tilde{x})\tilde{G}(\tilde{x}, T\tilde{z}, T\tilde{z})} + \frac{b_1\tilde{G}(\tilde{x}, T\tilde{x}, T\tilde{x})\tilde{G}(\tilde{y}, T\tilde{x}, T\tilde{x})\tilde{G}(\tilde{z}, T\tilde{y}, T\tilde{y}) + b_2\tilde{G}(\tilde{y}, T\tilde{y}, T\tilde{y})\tilde{G}(\tilde{x}, T\tilde{y}, T\tilde{y})\tilde{G}(\tilde{z}, T\tilde{x}, T\tilde{x})}{\tilde{G}(\tilde{y}, T\tilde{x}, T\tilde{x})\tilde{G}(\tilde{z}, T\tilde{y}, T\tilde{y}) + \tilde{G}(\tilde{x}, T\tilde{y}, T\tilde{y})\tilde{G}(\tilde{z}, T\tilde{x}, T\tilde{x})} \tag{3.1.1}$$

For all $\tilde{x}, \tilde{y}, \tilde{z} \in SE(\tilde{X})$

$\tilde{G}(\tilde{x}, T\tilde{y}, T\tilde{y})\tilde{G}(\tilde{y}, T\tilde{z}, T\tilde{z}) + \tilde{G}(\tilde{y}, T\tilde{x}, T\tilde{x})\tilde{G}(\tilde{x}, T\tilde{z}, T\tilde{z}) \neq 0$ and

$\tilde{G}(\tilde{y}, T\tilde{x}, T\tilde{x})\tilde{G}(\tilde{z}, T\tilde{y}, T\tilde{y}) + \tilde{G}(\tilde{x}, T\tilde{y}, T\tilde{y})\tilde{G}(\tilde{z}, T\tilde{x}, T\tilde{x}) \neq 0$

Where $a_i, b_i \geq 0$ $(i = 1,2)$ and $a_1 + a_2 + b_1 + b_2 < 1$. Then T has a unique fixed point $\tilde{u}$ and T is G-continuous at $\tilde{u}$.

Proof: Let $\tilde{x}_0 \in SE(\tilde{X})$ be an arbitrary soft element and define the sequence $\{\tilde{x}_n\}$ by

$$T\tilde{x}_0 = \tilde{x}_1, T\tilde{x}_1 = \tilde{x}_2, T\tilde{x}_2 = \tilde{x}_3, \dots\dots\dots\dots T\tilde{x}_n = \tilde{x}_{n+1}$$

Here we may assume that $\tilde{x}_n \neq \tilde{x}_{n+1}$ for each $n \in N \cup \{0\}$.

Consider,

$$\begin{aligned}
\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) &= \tilde{G}(T\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n) \\
&\leq \frac{a_1\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n)\tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n) + a_2\tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n)\tilde{G}(\tilde{x}_n, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n)}{\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n)\tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n) + \tilde{G}(\tilde{x}_n, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n)} \\
&+ \frac{b_1\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})\tilde{G}(\tilde{x}_n, T\tilde{x}_{n-1} T\tilde{x}_{n-1})\tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n) + b_2\tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n)\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n)\tilde{G}(\tilde{x}_n, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})}{\tilde{G}(\tilde{x}_n, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})\tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n) + \tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n)\tilde{G}(\tilde{x}_n, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})} \\
&\leq \frac{a_1\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) + a_2\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})\tilde{G}(\tilde{x}_n, \tilde{x}_n, \tilde{x}_n)\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})}{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) + \tilde{G}(\tilde{x}_n, \tilde{x}_n, \tilde{x}_n)\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})} \\
&+ \frac{b_1\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)\tilde{G}(\tilde{x}_n, \tilde{x}_n, \tilde{x}_n)\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) + b_2\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})\tilde{G}(\tilde{x}_n, \tilde{x}_n, \tilde{x}_n)}{\tilde{G}(\tilde{x}_n, \tilde{x}_n, \tilde{x}_n)\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) + \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})\tilde{G}(\tilde{x}_n, \tilde{x}_n, \tilde{x}_n)}
\end{aligned}$$

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq a_1\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) \tag{3.1.2}$$

On further decomposing we can write

$$\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) \leq a_1\tilde{G}(\tilde{x}_{n-2}, \tilde{x}_{n-1}, \tilde{x}_{n-1}) \tag{3.1.3}$$

By combination of (3.1.2) and (3.1.3) we have

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq a_1^2\tilde{G}(\tilde{x}_{n-2}, \tilde{x}_{n-1}, \tilde{x}_{n-1})$$

On continuing this process *n* times

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq a_1^n\tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)$$

Then, for all $n, m \in N, n < m$ we have

$$\begin{aligned}
\tilde{G}(\tilde{x}_n, \tilde{x}_m, \tilde{x}_m) &\leq \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) + \tilde{G}(\tilde{x}_{n+1}, \tilde{x}_{n+2}, \tilde{x}_{n+2}) + \cdots + \tilde{G}(\tilde{x}_{m-1}, \tilde{x}_m, \tilde{x}_m) \\
&\leq (a_1^n + a_1^{n+1} + \cdots + a_1^{m-1})\tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1) \\
&\leq \frac{a_1^n}{1 - a_1}\tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)
\end{aligned}$$

Therefore $\{\tilde{x}_n\}$ is soft G-Cauchy sequence. Since $(\tilde{X}, \tilde{G}, E)$ is soft G-complete, there exists $\tilde{u} \in SE(\tilde{X})$ such that $\{\tilde{x}_n\}$ soft G-converges to $\tilde{u}$.

Form (3.1.1) we have

$$\tilde{G}(\tilde{x}_n, T\tilde{u}, T\tilde{u}) = \tilde{G}(T\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u})$$
$$\leq \frac{a_1\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_{n-1},T\tilde{x}_{n-1})\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})+a_2\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{x}_{n-1},T\tilde{x}_{n-1})\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})}{\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})+\tilde{G}(\tilde{u},T\tilde{x}_{n-1},T\tilde{x}_{n-1})\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})}$$
$$+\frac{b_1\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_{n-1},T\tilde{x}_{n-1})\tilde{G}(\tilde{u},T\tilde{x}_{n-1},T\tilde{x}_{n-1})\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})+b_2\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{x}_{n-1},T\tilde{x}_{n-1})}{\tilde{G}(\tilde{u},T\tilde{x}_{n-1},T\tilde{x}_{n-1})\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})+\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{x}_{n-1},T\tilde{x}_{n-1})}$$

Taking the limit of both sides of above as $n \to \infty$ yields

$$\tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u}) \leq 0$$

Which implies that

$$\tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u}) = 0$$

and hence $\tilde{u} = T\tilde{u}$

To prove uniqueness: suppose that $\tilde{u}$ and $\tilde{v}$ are two fixed point for T. Then

$$G(\tilde{u}, \tilde{v}, \tilde{v}) = G(T\tilde{u}, T\tilde{v}, T\tilde{v})$$
$$\leq \frac{a_1\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})\tilde{G}(\tilde{v},T\tilde{v},T\tilde{v})+a_2\tilde{G}(\tilde{v},T\tilde{v},T\tilde{v})\tilde{G}(\tilde{v},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})}{\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})\tilde{G}(\tilde{v},T\tilde{v},T\tilde{v})+\tilde{G}(\tilde{v},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})}$$
$$+\frac{b_1\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{v},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{v},T\tilde{v},T\tilde{v})+b_2\tilde{G}(\tilde{v},T\tilde{v},T\tilde{v})\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})\tilde{G}(\tilde{v},T\tilde{u},T\tilde{u})}{\tilde{G}(\tilde{v},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{v},T\tilde{v},T\tilde{v})+\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})\tilde{G}(\tilde{v},T\tilde{u},T\tilde{u})}$$

$$G(\tilde{u}, \tilde{v}, \tilde{v}) \leq 0$$

$$\Rightarrow G(\tilde{u}, \tilde{v}, \tilde{v}) = 0$$

$$\Rightarrow \tilde{u} = \tilde{v}$$

To show that T is soft G-continuous at $\tilde{u}$. Let $\{\tilde{y}_n\}$ be a sequence of soft elements in $\tilde{X}$ such that $\{\tilde{y}_n\} \to \tilde{u}$ then we can deduce that

$$G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) = G(T\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)$$
$$\leq \frac{a_1\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n)+a_2\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n)\tilde{G}(\tilde{y}_n,T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)}{\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n)+\tilde{G}(\tilde{y}_n,T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)}$$
$$+\frac{b_1\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{y}_n,T\tilde{u},T\tilde{u})\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n)+b_2\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n)\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\tilde{G}(\tilde{y}_n,T\tilde{u},T\tilde{u})}{\tilde{G}(\tilde{y}_n,T\tilde{u},T\tilde{u})\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n)+\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\tilde{G}(\tilde{y}_n,T\tilde{u},T\tilde{u})}$$

Taking the limit as $n \to \infty$ from which we see that $\tilde{G}(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) \to 0$ and so, by proposition (2.17) we have that the sequence $T\tilde{y}_n$ is G – convergent to $T\tilde{u} = \tilde{u}$ therefore proposition (2.20) implies that T is G-continuous at $\tilde{u}$.

Theorem 3.2: Let $(\tilde{X}, \tilde{G}, E)$ be a soft G-metric space and $T: (\tilde{X}, \tilde{G}, E) \to (\tilde{X}, \tilde{G}, E)$ be a mapping that satisfies the following condition for all $\tilde{x}, \tilde{y}, \tilde{z} \in SE(\tilde{X})$

$$\begin{aligned}\tilde{G}(T\tilde{x}, T\tilde{y}, T\tilde{z}) \leq\; & a_1\tilde{G}(\tilde{x}, \tilde{y}, \tilde{z}) + a_2 \max\{\tilde{G}(\tilde{x}, T\tilde{x}, T\tilde{x}), \tilde{G}(\tilde{y}, T\tilde{y}, T\tilde{y})\} \\ & + a_3 \max\{\tilde{G}(\tilde{x}, T\tilde{z}, T\tilde{z}), \tilde{G}(\tilde{x}, T\tilde{y}, T\tilde{y})\} \\ & + a_4 \max\left\{\tilde{G}(\tilde{x}, \tilde{y}, \tilde{z}), \tilde{G}(\tilde{x}, T\tilde{x}, T\tilde{x}), \tilde{G}(\tilde{y}, T\tilde{y}, T\tilde{y}), \frac{\tilde{G}(\tilde{x},T\tilde{y},T\tilde{y})+\tilde{G}(\tilde{x},T\tilde{z},T\tilde{z})}{2}\right\}\end{aligned} \tag{3.2.1}$$

Where $a_1, a_2, a_3, a_4 \geq 0$ and $0 \leq a_1 + a_2 + 2a_3 + 2a_4 < 1$. Then T has a unique fixed point $\tilde{u}$ and T is G-continuous at $\tilde{u}$.

Proof: Let $x_0 \in SE(\tilde{X})$ be an arbitrary soft element and define the sequence $\{\tilde{x}_n\}$ by

$$T\tilde{x}_0 = \tilde{x}_1, T\tilde{x}_1 = \tilde{x}_2, T\tilde{x}_2 = \tilde{x}_3, \dots \dots \dots \dots T\tilde{x}_n = \tilde{x}_{n+1}$$

Here we may assume that $\tilde{x}_n \neq \tilde{x}_{n+1}$ for each $n \in N \cup \{0\}$.

Consider

$$\begin{aligned}\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) &= \tilde{G}(T\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n) \\ &\leq a_1\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) + a_2 \max\{\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_{n-1}, T\tilde{x}_{n-1}), \tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n)\} \\ &\quad + a_3 \max\{\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n), \tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n)\} \\ &\quad + a_4 \max\left\{\begin{matrix}\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n), \tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_{n-1}, T\tilde{x}_{n-1}), \tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n), \\ \dfrac{\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n) + \tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n)}{2}\end{matrix}\right\} \\ &\leq a_1\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) + a_2 \max\{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n), \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})\} + a_3\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \\ &\quad + a_4 \max\{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n), \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}), \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})\} \\ &\leq a_1\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) + a_2 \max\{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n), \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})\} \\ &\quad + a_3\{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) + \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})\} \\ &\quad + a_4 \max\{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n), \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}), \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) + \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})\}\end{aligned}$$

If $\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) > \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)$, then

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) < (a_1 + a_2 + 2a_3 + 2a_4)\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})$$

Which is a contradiction and therefore

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq \left(\frac{a_1 + a_2 + a_3 + a_4}{1 - a_3 - a_4}\right) \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)$$

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq k\, \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)$$

Let $k = \frac{a_1+a_2+a_3+a_4}{1-a_3-a_4} < 1$

Repeated n times, we get

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq k^n\, \tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)$$

Then, for all $n, m \in N, n < m$ we have

$$\begin{aligned}\tilde{G}(\tilde{x}_n, \tilde{x}_m, \tilde{x}_m) &\leq \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) + \tilde{G}(\tilde{x}_{n+1}, \tilde{x}_{n+2}, \tilde{x}_{n+2}) + \cdots + \tilde{G}(\tilde{x}_{m-1}, \tilde{x}_m, \tilde{x}_m)\\ &\leq (k^n + k^{n+1} + \cdots + k^{m-1})\tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)\\ &\leq \frac{k^n}{1-k}\tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)\end{aligned}$$

Therefore $\{\tilde{x}_n\}$ is soft G-Cauchy sequence. Since $(\tilde{X}, \tilde{G}, E)$ is soft G-complete, there exists $\tilde{u} \in SE(\tilde{X})$ such that $\{\tilde{x}_n\}$ soft G-converges to $\tilde{u}$.

Form (3.2.1) we have

$$\begin{aligned}\tilde{G}(\tilde{x}_{n+1}, T\tilde{u}, T\tilde{u}) &= \tilde{G}(T\tilde{x}_n, T\tilde{u}, T\tilde{u})\\ &\leq a_1\tilde{G}(\tilde{x}_n, \tilde{u}, \tilde{u}) + a_2 \max\{\tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n), \tilde{G}(\tilde{u}, T\tilde{\tilde{u}}, T\tilde{u})\}\\ &\quad + a_3 \max\{\tilde{G}(\tilde{x}_n, T\tilde{u}, T\tilde{u}), \tilde{G}(\tilde{x}_n, T\tilde{u}, T\tilde{u})\}\\ &\quad + a_4 \max\left\{\tilde{G}(\tilde{x}_n, \tilde{u}, \tilde{u}), \tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n), \tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u}), \frac{\tilde{G}(\tilde{x}_n, T\tilde{u}, T\tilde{u}) + \tilde{G}(\tilde{x}_n, T\tilde{u}, T\tilde{u})}{2}\right\}\end{aligned}$$

Taking the limit as $n \to \infty$, and using the fact that the function G is continuous on its variable then we have

$$\tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u}) \leq (a_2 + a_3 + a_4)\tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u})$$

This contradiction implies that $\tilde{u} = T\tilde{u}$

To prove uniqueness, suppose that $\tilde{u}$ and $\tilde{v}$ are two fixed points of T. Then by inequality (3.2.1) we have

$$\begin{aligned}G(\tilde{u}, \tilde{v}, \tilde{v}) &= G(T\tilde{u}, T\tilde{v}, T\tilde{v})\\ &\leq a_1\tilde{G}(\tilde{u}, \tilde{v}, \tilde{v}) + a_2 \max\{\tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u}), \tilde{G}(\tilde{v}, T\tilde{v}, T\tilde{v})\}\\ &\quad + a_3 \max\{\tilde{G}(\tilde{u}, T\tilde{v}, T\tilde{v}), \tilde{G}(\tilde{u}, T\tilde{v}, T\tilde{v})\}\\ &\quad + a_4 \max\left\{\tilde{G}(\tilde{u}, \tilde{v}, \tilde{v}), \tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u}), \tilde{G}(\tilde{v}, T\tilde{v}, T\tilde{v}), \frac{\tilde{G}(\tilde{u}, T\tilde{v}, T\tilde{v}) + \tilde{G}(\tilde{u}, T\tilde{v}, T\tilde{v})}{2}\right\}\end{aligned}$$

$$\Rightarrow \tilde{G}(\tilde{u}, \tilde{v}, \tilde{v}) \leq (a_1 + a_3 + a_4)\tilde{G}(\tilde{u}, \tilde{v}, \tilde{v})$$

$$\Rightarrow \tilde{G}(\tilde{u}, \tilde{v}, \tilde{v}) = 0$$

Which implies that $\tilde{u} = \tilde{v}$.

To show that T is soft G-continuous at $\tilde{u}$. Let $\{\tilde{y}_n\}$ be a sequence of soft elements in $\tilde{X}$ such that $\{\tilde{y}_n\} \to \tilde{u}$ then we can deduce that

$$\begin{aligned}G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) &= G(T\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)\\ &\leq a_1\tilde{G}(\tilde{u}, \tilde{y}_n, \tilde{y}_n) + a_2 \max\{\tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u}), \tilde{G}(\tilde{y}_n, T\tilde{y}_n, T\tilde{y}_n)\}\\ &\quad + a_3 \max\{\tilde{G}(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n), \tilde{G}(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)\}\\ &\quad + a_4 \max\left\{\tilde{G}(\tilde{u}, \tilde{y}_n, \tilde{y}_n), \tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u}), \tilde{G}(\tilde{y}_n, T\tilde{y}_n, T\tilde{y}_n), \frac{\tilde{G}(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) + \tilde{G}(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)}{2}\right\}\\ &\leq a_1\tilde{G}(\tilde{u}, \tilde{y}_n, \tilde{y}_n) + a_2\, \tilde{G}(\tilde{y}_n, T\tilde{y}_n, T\tilde{y}_n) + a_3\tilde{G}(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)\\ &\quad + a_4 \max\{\tilde{G}(\tilde{u}, \tilde{y}_n, \tilde{y}_n), \tilde{G}(\tilde{y}_n, T\tilde{y}_n, T\tilde{y}_n), \tilde{G}(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)\} \qquad (3.2.2)\end{aligned}$$

Now following three cases are arise:

Case-I: If $\max\{\tilde{G}(\tilde{u},\tilde{y}_n,\tilde{y}_n),\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n),\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\}=\tilde{G}(\tilde{u},\tilde{y}_n,\tilde{y}_n)$ then condition (3.2.2) reduces to

$$G(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\leq\frac{(a_1+a_4)\tilde{G}(\tilde{u},\tilde{y}_n,\tilde{y}_n)+a_2\tilde{G}(\tilde{y}_n,\tilde{u},\tilde{u})}{1-(a_2+a_3)}$$

Taking the limit as $n\longrightarrow\infty$ from which we see that

$$G(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\longrightarrow 0$$

Case - II: If $\max\{\tilde{G}(\tilde{u},\tilde{y}_n,\tilde{y}_n),\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n),\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\}=\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n)$ then condition (3.2.2) reduces to

$$G(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\leq\frac{a_1\tilde{G}(\tilde{u},\tilde{y}_n,\tilde{y}_n)+(a_2+a_4)\tilde{G}(\tilde{y}_n,\tilde{u},\tilde{u})}{1-(a_2+a_3+a_4)}$$

Taking the limit as $n\longrightarrow\infty$ from which we see that

$$G(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\longrightarrow 0$$

Case - III: If $\max\{\tilde{G}(\tilde{u},\tilde{y}_n,\tilde{y}_n),\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n),\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\}=\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)$ then condition (3.2.2) reduces to

$$G(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\leq\frac{a_1\tilde{G}(\tilde{u},\tilde{y}_n,\tilde{y}_n)+a_2\tilde{G}(\tilde{y}_n,\tilde{u},\tilde{u})}{1-(a_2+a_3+a_4)}$$

Taking the limit as $n\longrightarrow\infty$ from which we see that

$$G(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\longrightarrow 0$$

Taking the limit as $n\to\infty$ from which we see that $\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\to 0$ and so, by proposition (2.17) we have that the sequence $T\tilde{y}_n$ is G – convergent to $T\tilde{u}=\tilde{u}$ therefore proposition (2.20) implies that T is G-continuous at $\tilde{u}$.

Theorem 3.3: Let $(\tilde{X},\tilde{G},E)$ be a soft G-metric space and $T:(\tilde{X},\tilde{G},E)\to(\tilde{X},\tilde{G},E)$ be a mapping that satisfies the following condition for all $\tilde{x},\tilde{y},\tilde{z}\in SE(\tilde{X})$

$$\tilde{G}(T\tilde{x},T\tilde{y},T\tilde{z})\leq\alpha\frac{\tilde{G}(\tilde{x},T\tilde{y},T\tilde{y})+\tilde{G}(\tilde{x},T\tilde{z},T\tilde{z})}{2}$$
$$+\beta\frac{\tilde{G}(\tilde{x},T\tilde{y},T\tilde{y})[\tilde{G}(\tilde{x},T\tilde{y},T\tilde{y})+\tilde{G}(\tilde{x},T\tilde{z},T\tilde{z})+\tilde{G}(\tilde{y},T\tilde{x},T\tilde{x})+\tilde{G}(\tilde{z},T\tilde{x},T\tilde{x})]}{2[\tilde{G}(\tilde{x},T\tilde{y},T\tilde{y})+\tilde{G}(\tilde{y},T\tilde{x},T\tilde{x})]}\tag{3.3.1}$$

Where $0\leq(\alpha+\beta)<\frac{1}{2}$. Then T has a unique fixed point $\tilde{u}$ and T is G-continuous at $\tilde{u}$.

Proof: Let $x_0\in SE(\tilde{X})$ be an arbitrary soft element and define the sequence $\{\tilde{x}_n\}$ by

$$T\tilde{x}_0=\tilde{x}_1,T\tilde{x}_1=\tilde{x}_2,T\tilde{x}_2=\tilde{x}_3,\dots\dots\dots\dots T\tilde{x}_n=\tilde{x}_{n+1}$$

Here we may assume that $\tilde{x}_n\neq\tilde{x}_{n+1}$ for each $n\in N\cup\{0\}$.

Consider

$$\begin{aligned}\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1})&=\tilde{G}(T\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)\\&\leq\alpha\frac{\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)+\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)}{2}\\&\quad+\beta\frac{\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)[\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)+\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)+\tilde{G}(\tilde{x}_n,T\tilde{x}_{n-1},T\tilde{x}_{n-1})+\tilde{G}(\tilde{x}_n,T\tilde{x}_{n-1},T\tilde{x}_{n-1})]}{2[\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)+\tilde{G}(\tilde{x}_n,T\tilde{x}_{n-1},T\tilde{x}_{n-1})]}\\&\leq\alpha\frac{\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})+\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})}{2}\\&\quad+\beta\frac{\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})[\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})+\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})+\tilde{G}(\tilde{x}_n,\tilde{x}_n,\tilde{x}_n)+\tilde{G}(\tilde{x}_n,\tilde{x}_n,\tilde{x}_n)]}{2[\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})+\tilde{G}(\tilde{x}_n,\tilde{x}_n,\tilde{x}_n)]}\\&\leq\alpha\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})+\beta\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})\end{aligned}$$

$$(1-\alpha-\beta)\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1})\leq(\alpha+\beta)\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)$$

$$\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1})\leq\frac{(\alpha+\beta)}{(1-\alpha-\beta)}\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)$$

$$\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1})\leq K\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)$$

$$Let\ K=\frac{(\alpha+\beta)}{(1-\alpha-\beta)}\tag{3.3.2}$$

On further decomposing we can write

$$\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)\leq K\tilde{G}(\tilde{x}_{n-2},\tilde{x}_{n-1},\tilde{x}_{n-1})\tag{3.3.3}$$

By combination of (3.3.2) and (3.3.3) we have

$$\tilde{G}(x_n,x_{n+1},x_{n+1})\leq K^2\tilde{G}(\tilde{x}_{n-2},\tilde{x}_{n-1},\tilde{x}_{n-1})$$

On continuing this process n times

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq K^n \tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)$$

Then for all $n, m \in N, n < m$ we have

$$\begin{aligned}\tilde{G}(\tilde{x}_n, \tilde{x}_m, \tilde{x}_m) &\leq \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) + \tilde{G}(\tilde{x}_{n+1}, \tilde{x}_{n+2}, \tilde{x}_{n+2}) + \ldots + \tilde{G}(\tilde{x}_{m-1}, \tilde{x}_m, \tilde{x}_m)\\ &\leq (K^n + K^{n+1} + \cdots \ldots + K^{m-1}) \tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)\\ &\leq \frac{K^n}{1-K} \tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)\end{aligned}$$

Therefore $\{\tilde{x}_n\}$ is soft G-Cauchy sequence. Since $(\tilde{X}, \tilde{G}, E)$ is soft G-complete, there exists $\tilde{u} \in SE(\tilde{X})$ such that $\{\tilde{x}_n\}$ soft G-converges to $\tilde{u}$.

Form (3.3.1) we have

$$\begin{aligned}\widetilde{G}(\tilde{x}_n, T\tilde{u}, T\tilde{u}) &= \tilde{G}(T\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u})\\ &\leq \alpha \frac{\tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u}) + \tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u})}{2}\\ &\quad + \beta \frac{\tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u})[\tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u}) + \tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u}) + \tilde{G}(\tilde{u}, T\tilde{x}_{n-1}, T\tilde{x}_{n-1}) + \tilde{G}(\tilde{u}, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})]}{2[\tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u}) + \tilde{G}(\tilde{u}, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})]}\\ &\leq \alpha \frac{\tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u}) + \tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u})}{2}\\ &\quad + \beta \frac{\tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u})[\tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u}) + \tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u}) + \tilde{G}(u, \tilde{x}_n, \tilde{x}_n) + \tilde{G}(\tilde{u}, \tilde{x}_n, \tilde{x}_n)]}{2[\tilde{G}(\tilde{x}_{n-1}, T\tilde{u}, T\tilde{u}) + \tilde{G}(\tilde{u}, \tilde{x}_n, \tilde{x}_n)]}\end{aligned}$$

Taking the limit of both sides of above as $n \to \infty$ yields

$$\tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u}) \leq (\alpha + \beta) \tilde{G}(\tilde{u}, T\tilde{u}, T\tilde{u})$$

This contradiction implies that $\tilde{u} = T\tilde{u}$.

To prove uniqueness, suppose that $\tilde{u}$ and $\tilde{v}$ are two fixed point for T. Then

$$\begin{aligned}G(\tilde{u}, \tilde{v}, \tilde{v}) &= G(T\tilde{u}, T\tilde{v}, T\tilde{v})\\ &\leq \alpha \frac{G(\tilde{u}, T\tilde{v}, T\tilde{v}) + G(\tilde{u}, T\tilde{v}, T\tilde{v})}{2} + \beta \frac{G(\tilde{u}, T\tilde{v}, T\tilde{v})[G(\tilde{u}, T\tilde{v}, T\tilde{v}) + G(\tilde{u}, T\tilde{v}, T\tilde{v}) + G(\tilde{v}, T\tilde{u}, T\tilde{u}) + G(\tilde{v}, T\tilde{u}, T\tilde{u})]}{2[G(\tilde{u}, T\tilde{v}, T\tilde{v}) + G(\tilde{v}, T\tilde{u}, T\tilde{u})]}\end{aligned}$$

$$G(\tilde{u}, \tilde{v}, \tilde{v}) \leq \alpha G(\tilde{u}, \tilde{v}, \tilde{v}) + \beta G(\tilde{u}, \tilde{v}, \tilde{v})$$

$$G(\tilde{u}, \tilde{v}, \tilde{v}) \leq (\alpha + \beta) G(\tilde{u}, \tilde{v}, \tilde{v})$$

$$\Rightarrow G(\tilde{u}, \tilde{v}, \tilde{v}) = 0$$

Since $(\alpha + \beta) < 1$

$$\Rightarrow \tilde{u} = \tilde{v}$$

To show that T is soft G-continuous at $\tilde{u}$. Let $\{\tilde{y}_n\}$ be a sequence of soft elements in $\tilde{X}$ such that $\{\tilde{y}_n\} \to \tilde{u}$ then we can deduce that

$$\begin{aligned}G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) &= G(T\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)\\ &\leq \alpha \frac{G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) + G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)}{2}\\ &\quad + \beta \frac{G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)[G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) + G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) + G(\tilde{y}_n, T\tilde{u}, T\tilde{u}) + G(\tilde{y}_n, T\tilde{u}, T\tilde{u})]}{2[G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) + G(\tilde{y}_n, T\tilde{u}, T\tilde{u})]}\end{aligned}$$

$$G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) \leq (\alpha + \beta) G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)$$

$$[1 - (\alpha + \beta)] G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) \leq 0$$

$$G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) \leq 0$$

Taking the limit as $n \to \infty$ from which we see that $\tilde{G}(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) \to 0$ and so, by proposition (2.17) we have that the sequence $T\tilde{y}_n$ is G – convergent to $T\tilde{u} = \tilde{u}$ therefore proposition (2.20) implies that T is G-continuous at $\tilde{u}$.

Theorem 3.4: Let $(\tilde{X}, \tilde{G}, E)$ be a soft G-metric space and $T: (\tilde{X}, \tilde{G}, E) \to (\tilde{X}, \tilde{G}, E)$ be a mapping that satisfies the following condition for all $\tilde{x}, \tilde{y}, \tilde{z} \in SE(\tilde{X})$

$$\begin{aligned}\tilde{G}(T\tilde{x}, T\tilde{y}, T\tilde{z}) &\leq \alpha\, min\{\tilde{G}(\tilde{x}, T\tilde{x}, T\tilde{x}), \tilde{G}(\tilde{y}, T\tilde{y}, T\tilde{y}), \tilde{G}(\tilde{z}, T\tilde{z}, T\tilde{z}), \tilde{G}(\tilde{x}, \tilde{y}, \tilde{z})\}\\ &\quad + \beta \left[\frac{\tilde{G}(\tilde{x}, T\tilde{x}, T\tilde{x}) + \tilde{G}(\tilde{y}, T\tilde{x}, T\tilde{x}) + \tilde{G}(\tilde{x}, T\tilde{y}, T\tilde{y})}{1 + \tilde{G}(\tilde{x}, T\tilde{x}, T\tilde{x})\ \tilde{G}(\tilde{y}, T\tilde{x}, T\tilde{x})\ \tilde{G}(\tilde{x}, T\tilde{y}, T\tilde{y})}\right] \qquad (3.4.1)\end{aligned}$$

Where $\alpha, \beta \geq 0$ and $\alpha + 3\beta < 1$ Then T has a unique fixed point $\tilde{u}$ and T is G-continuous at $\tilde{u}$.

Proof: Let $x_0 \in SE(\tilde{X})$ be an arbitrary soft element and define the sequence $\{\tilde{x}_n\}$ by

$$T\tilde{x}_0 = \tilde{x}_1, T\tilde{x}_1 = \tilde{x}_2, T\tilde{x}_2 = \tilde{x}_3, \dots\dots\dots\dots T\tilde{x}_n = \tilde{x}_{n+1}$$

$$\begin{aligned}
\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) &= \tilde{G}(T\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n) \\
&\leq \alpha\, min\{\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_{n-1}, T\tilde{x}_{n-1}), \tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n), \tilde{G}(\tilde{x}_n, T\tilde{x}_n, T\tilde{x}_n), \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)\} \\
&\quad +\beta \left[\frac{\tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_{n-1}, T\tilde{x}_{n-1}) + \tilde{G}(\tilde{x}_n, T\tilde{x}_{n-1}, T\tilde{x}_{n-1}) + \tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n)}{1 + \tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})\ \tilde{G}(\tilde{x}_n, T\tilde{x}_{n-1}, T\tilde{x}_{n-1})\ \tilde{G}(\tilde{x}_{n-1}, T\tilde{x}_n, T\tilde{x}_n)}\right] \\
&\leq \alpha\, min\{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n), \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}), \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}), \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)\} \\
&\quad +\beta \left[\frac{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) + \tilde{G}(\tilde{x}_n, \tilde{x}_n, \tilde{x}_n) + \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})}{1 + \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)\ \tilde{G}(\tilde{x}_n, \tilde{x}_n, \tilde{x}_n)\ \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})}\right] \\
&\leq \alpha\, min\{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n), \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})\} \\
&\quad +\beta[\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) + \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})] \qquad (3.4.2)
\end{aligned}$$

Here two cases are arise

Case – I: If $min\{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n), \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})\} = \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)$

Then condition (3.4.2) reduces to

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq \alpha\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) + \beta[\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) + \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})]$$

$$(1-\beta)\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq (\alpha + 2\beta)\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)$$

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq \frac{(\alpha + 2\beta)}{(1-\beta)}\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)$$

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq K\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)$$

Let $K = \frac{(\alpha+2\beta)}{(1-\beta)}$

On continuing this process *n* times

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq K^n\tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)$$

Case – II: If $min\{\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n), \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})\} = \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1})$

Then condition (3.4.2) reduces to

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq \alpha\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) + \beta[\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n) + \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_{n+1}, \tilde{x}_{n+1})]$$

$$(1-\alpha-\beta)\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq 2\beta\ \tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)$$

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq \frac{2\beta}{(1-\alpha-\beta)}\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)$$

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq K\tilde{G}(\tilde{x}_{n-1}, \tilde{x}_n, \tilde{x}_n)$$

Let $K = \frac{2\beta}{(1-\alpha-\beta)}$

On continuing this process *n* times

$$\tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) \leq K^n\tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)$$

Then for all $n, m \in N, \quad n < m$ we have

$$\begin{aligned}
\tilde{G}(\tilde{x}_n, \tilde{x}_m, \tilde{x}_m) &\leq \tilde{G}(\tilde{x}_n, \tilde{x}_{n+1}, \tilde{x}_{n+1}) + \tilde{G}(\tilde{x}_{n+1}, \tilde{x}_{n+2}, \tilde{x}_{n+2}) + \ldots + \tilde{G}(\tilde{x}_{m-1}, \tilde{x}_m, \tilde{x}_m) \\
&\leq (K^n + K^{n+1} + \cdots \ldots + K^{m-1})\tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1) \\
&\leq \frac{K^n}{1-K}\tilde{G}(\tilde{x}_0, \tilde{x}_1, \tilde{x}_1)
\end{aligned}$$

Therefore $\{\tilde{x}_n\}$ is soft G-Cauchy sequence. Since $(\tilde{X}, \tilde{G}, E)$ is soft G-complete, there exists $\tilde{u} \in SE(\tilde{X})$ such that $\{\tilde{x}_n\}$ soft G-converges to $\tilde{u}$.

Form (3.4.1) we have

$$\begin{aligned}\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}) &= \tilde{G}(\tilde{x}_n,T\tilde{u},T\tilde{u}) = \tilde{G}(T\tilde{x}_{n-1},T\tilde{u},T\tilde{u})\\ &\le \alpha\, min\{\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_{n-1},T\tilde{x}_{n-1}),\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}),\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}),\tilde{G}(\tilde{x}_{n-1},\tilde{u},\tilde{u})\}\\ &\quad +\beta\left[\frac{\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_{n-1},T\tilde{x}_{n-1})+\tilde{G}(\tilde{u},T\tilde{x}_{n-1},T\tilde{x}_{n-1})+\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})}{1+\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_{n-1},T\tilde{x}_{n-1})\,\tilde{G}(\tilde{u},T\tilde{x}_{n-1},T\tilde{x}_{n-1})\,\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})}\right]\\ &\le \alpha\, min\{\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n),\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}),\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}),\tilde{G}(\tilde{x}_{n-1},\tilde{u},\tilde{u})\}\\ &\quad +\beta\left[\frac{\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)+\tilde{G}(\tilde{u},\tilde{x}_n,\tilde{x}_n)+\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})}{1+\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)\,\tilde{G}(\tilde{u},\tilde{x}_n,\tilde{x}_n)\,\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})}\right]\end{aligned}$$

Taking the limit as taking the limit as $n \to \infty$

$$\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}) \le \beta\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})$$

Since $\beta < 1$.

Which implies that

$$\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}) = 0$$

And hence $\tilde{u} = T\tilde{u}$.

To prove uniqueness suppose that $\tilde{u}$ and $\tilde{v}$ are two fixed point for T. Then

$$\begin{aligned}\tilde{G}(\tilde{u},\tilde{v},\tilde{v}) &= \tilde{G}(T\tilde{u},T\tilde{v},T\tilde{v})\\ &\le \alpha\, min\{\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}),\tilde{G}(\tilde{v},T\tilde{v},T\tilde{v}),\tilde{G}(\tilde{v},T\tilde{v},T\tilde{v}),\tilde{G}(\tilde{u},\tilde{v},\tilde{v})\}\\ &\quad +\beta\left[\frac{\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})+\tilde{G}(\tilde{v},T\tilde{u},T\tilde{u})+\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})}{1+\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})\,\tilde{G}(\tilde{v},T\tilde{u},T\tilde{u})\,\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})}\right]\\ &\le \beta\tilde{G}(\tilde{v},\tilde{u},\tilde{u})\end{aligned}$$

$$\tilde{G}(\tilde{u},\tilde{v},\tilde{v}) \le 2\beta\tilde{G}(u,\tilde{v},v)$$

a contradiction. Therefore, $\tilde{G}(\tilde{u},\tilde{v},\tilde{v}) = 0$

Hence $\tilde{u} = \tilde{v}$

To show that T is soft G-continuous at $\tilde{u}$. Let $\{\tilde{y}_n\}$ be a sequence of soft elements in $\tilde{X}$ such that $\{\tilde{y}_n\} \to \tilde{u}$ then we can deduce that

Using (3.4.1)

$$\begin{aligned}\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n) &= \tilde{G}(T\tilde{u},T\tilde{y}_n,T\tilde{y}_n)\\ &\le \alpha\, min\{\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}),\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n),\tilde{G}(\tilde{y}_n,T\tilde{y}_n,T\tilde{y}_n),\tilde{G}(\tilde{u},\tilde{y}_n,\tilde{y}_n)\}\\ &\quad +\beta\left[\frac{\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})+\tilde{G}(\tilde{y}_n,T\tilde{u},T\tilde{u})+\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)}{1+\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})\tilde{G}(\tilde{y}_n,T\tilde{u},T\tilde{u})\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)}\right]\\ &\le \beta[\tilde{G}(\tilde{y}_n,\tilde{u},\tilde{u})+\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n)]\end{aligned}$$

$$\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n) \le \frac{\beta}{1-\beta}\tilde{G}(\tilde{y}_n,\tilde{u},\tilde{u})$$

Taking the limit as $n \to \infty$ from which we see that $\tilde{G}(\tilde{u},T\tilde{y}_n,T\tilde{y}_n) \to 0$ and so, by proposition (2.17) we have that the sequence $T\tilde{y}_n$ is G – convergent to $T\tilde{u} = \tilde{u}$ therefore proposition (2.20) implies that T is G-continuous at $\tilde{u}$.

Theorem 3.5: Let $(\tilde{X},\tilde{G},E)$ be a soft G-metric space and $T:(\tilde{X},\tilde{G},E) \to (\tilde{X},\tilde{G},E)$ be a mapping that satisfies the following condition for all $\tilde{x},\tilde{y},\tilde{z} \in SE(\tilde{X})$

$$\begin{aligned}\tilde{G}(T\tilde{x},T\tilde{y},T\tilde{z}) &\le \alpha\tilde{G}(\tilde{x},\tilde{y},\tilde{z})+\beta[\tilde{G}(\tilde{x},T\tilde{y},T\tilde{y})+\tilde{G}(\tilde{x},T\tilde{z},T\tilde{z})]\\ &\quad +\gamma\frac{\tilde{G}(\tilde{x},\tilde{y},\tilde{z})[1+\tilde{G}(\tilde{x},T\tilde{y},T\tilde{y})]}{1+\tilde{G}(\tilde{x},T\tilde{z},T\tilde{z})}+\delta\left[\frac{\tilde{G}(\tilde{x},T\tilde{y},T\tilde{y}).\tilde{G}(\tilde{y},T\tilde{y},T\tilde{y})}{\tilde{G}(\tilde{z},T\tilde{y},T\tilde{y})}\right] \qquad (3.5.1)\end{aligned}$$

Where $\alpha,\beta,\gamma,\delta \ge 0$ and $\alpha+4\beta+\gamma+2\delta < 1$

Then T has a unique fixed point $\tilde{u}$ and T is G-continuous at $\tilde{u}$.

Proof: Let $x_0 \in SE(\tilde{X})$ be an arbitrary soft element and define the sequence $\{\tilde{x}_n\}$ by

$$T\tilde{x}_0 = \tilde{x}_1, T\tilde{x}_1 = \tilde{x}_2, T\tilde{x}_2 = \tilde{x}_3, \dots\dots\dots\dots T\tilde{x}_n = \tilde{x}_{n+1}$$

Here we may assume that $\tilde{x}_n \ne \tilde{x}_{n+1}$

Consider,

$$\begin{aligned}\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1}) &= \tilde{G}(T\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)\\
&\le \alpha\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)+\beta\left[\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)+\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)\right]\\
&\quad+\gamma\frac{\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)[1+\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)]}{1+\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n)}+\delta\left[\frac{\tilde{G}(\tilde{x}_{n-1},T\tilde{x}_n,T\tilde{x}_n).\tilde{G}(\tilde{x}_n,T\tilde{x}_n,T\tilde{x}_n)}{\tilde{G}(\tilde{x}_n,T\tilde{x}_n,T\tilde{x}_n)}\right]\\
&\le \alpha\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)+\beta\left[\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})+\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})\right]\\
&\quad+\gamma\frac{\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)[1+\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})]}{1+\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})}+\delta\left[\frac{\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1}).\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1})}{\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1})}\right]\\
&\le \alpha\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)+2\beta\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})\\
&\quad+\gamma\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)+\delta\tilde{G}(\tilde{x}_{n-1},\tilde{x}_{n+1},\tilde{x}_{n+1})\\
&\le \alpha\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)+2\beta\left[\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)+\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1})\right]\\
&\quad+\gamma\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)+\delta\left[\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)+\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1})\right]\end{aligned}$$

$$\begin{aligned}(1-2\beta-\delta)\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1}) &\le (\alpha+2\beta+\gamma+\delta)\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)\\
&\le \frac{\alpha+2\beta+\gamma+\delta}{1-2\beta-\delta}\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)\end{aligned}$$

$$\Rightarrow \tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1}) \le K\tilde{G}(\tilde{x}_{n-1},\tilde{x}_n,\tilde{x}_n)$$

Let $\frac{\alpha+2\beta+\gamma+\delta}{1-2\beta-\delta}=K$

On continuing this process *n* times

$$\tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1}) \le K^n\tilde{G}(\tilde{x}_0,\tilde{x}_1,\tilde{x}_1)$$

Then for all $m,n\in N, n<m$ we have

$$\begin{aligned}\tilde{G}(\tilde{x}_n,\tilde{x}_m,\tilde{x}_m) &\le \tilde{G}(\tilde{x}_n,\tilde{x}_{n+1},\tilde{x}_{n+1})+\tilde{G}(\tilde{x}_{n+1},\tilde{x}_{n+2},\tilde{x}_{n+2})+\ldots+\tilde{G}(\tilde{x}_{m-1},\tilde{x}_m,\tilde{x}_m)\\
&\le (K^n+K^{n+1}+\cdots\ldots+K^{m-1})\tilde{G}(\tilde{x}_0,\tilde{x}_1,\tilde{x}_1)\\
&\le \frac{K^n}{1-K}\tilde{G}(\tilde{x}_0,\tilde{x}_1,\tilde{x}_1)\end{aligned}$$

Therefore $\{\tilde{x}_n\}$ is soft G-Cauchy sequence. Since $(\tilde{X},\tilde{G},E)$ is soft G-complete, there exists $\tilde{u}\in SE(\tilde{X})$ such that $\{\tilde{x}_n\}$ soft G-converges to $\tilde{u}$.

Form (3.5.1) we have

$$\begin{aligned}\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}) &= \tilde{G}(\tilde{x}_n,T\tilde{u},T\tilde{u}) = \tilde{G}(T\tilde{x}_{n-1},T\tilde{u},T\tilde{u})\\
&\le \alpha\tilde{G}(\tilde{x}_{n-1},\tilde{u},\tilde{u})+\beta\left[\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})+\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})\right]\\
&\quad+\gamma\frac{\tilde{G}(\tilde{x}_{n-1},\tilde{u},\tilde{u})[1+\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})]}{1+\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u})}+\delta\left[\frac{\tilde{G}(\tilde{x}_{n-1},T\tilde{u},T\tilde{u}).\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})}{\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})}\right]\end{aligned}$$

Taking the limit as taking the limit as $n\to\infty$

$$\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u}) \le (2\beta+\delta)\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})$$

Since$(2\beta+\delta)<1$

$$\tilde{G}(\tilde{u},T\tilde{u},T\tilde{u})=0$$

And hence $\tilde{u}=T\tilde{u}$

To prove uniqueness suppose that $\tilde{u}$ and $\tilde{v}$ are two fixed point for T. Then

$$\begin{aligned}\tilde{G}(\tilde{u},\tilde{v},\tilde{v}) &= \tilde{G}(T\tilde{u},T\tilde{v},T\tilde{v})\\
&\le \alpha\tilde{G}(\tilde{u},\tilde{v},\tilde{v})+\beta\left[\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})+\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})\right]\\
&\quad+\gamma\frac{\tilde{G}(\tilde{u},\tilde{v},\tilde{v})\left[1+\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})\right]}{1+\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v})}+\delta\left[\frac{\tilde{G}(\tilde{u},T\tilde{v},T\tilde{v}).\tilde{G}(\tilde{v},T\tilde{v},T\tilde{v})}{\tilde{G}(\tilde{v},T\tilde{v},T\tilde{v})}\right]\end{aligned}$$

$$\Rightarrow \tilde{G}(\tilde{u},\tilde{v},\tilde{v}) \le (\alpha+2\beta+\gamma+\delta)\tilde{G}(\tilde{u},\tilde{v},\tilde{v})$$

Since $(\alpha+2\beta+\gamma+\delta)<1$

$$\Rightarrow \tilde{G}(\tilde{u},\tilde{v},\tilde{v})=0$$

To show that T is soft G-continuous at $\tilde{u}$. Let $\{\tilde{y}_n\}$ be a sequence of soft elements in $\tilde{X}$ such that $\{\tilde{y}_n\}\to\tilde{u}$ then we can deduce that

Using (3.5.1)

$$\begin{aligned} G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) &= G(T\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) \\ &\leq \alpha G(\tilde{u}, \tilde{y}_n, \tilde{y}_n) + \beta[G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) + G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)] \\ &\quad +\gamma \frac{G(\tilde{u}, \tilde{y}_n, \tilde{y}_n)[1+G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)]}{1+G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n)} + \delta\left[\frac{G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n).G(\tilde{y}_n, T\tilde{y}_n, T\tilde{y}_n)}{G(\tilde{y}_n, T\tilde{y}_n, T\tilde{y}_n)}\right] \end{aligned}$$

$$\Longrightarrow G(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) \leq \frac{\alpha+\gamma}{[1-(2\beta+\delta)]} G(\tilde{u}, \tilde{y}_n, \tilde{y}_n)$$

Taking the limit as $n \to \infty$ from which we see that $\tilde{G}(\tilde{u}, T\tilde{y}_n, T\tilde{y}_n) \to 0$ and so, by proposition (2.17) we have that the sequence $T\tilde{y}_n$ is G – convergent to $T\tilde{u} = \tilde{u}$ therefore proposition (2.20) implies that T is G-continuous at $\tilde{u}$.

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