A NOTE ON INCLINE ALGEBRAS

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ABSTRACT

In this paper, we prove that -
(1) In the definition of an incline algebra $K$ with zero element 0, the conditions
   (i) $a + 0 = a$ for all $a \in K$ and (ii) $a * 0 = 0 * a = 0$ for all $a \in K$ are equivalent and hence any one of them
   can be deleted.
(2) "Every irreducible ideal of an incline algebra is not prime" is shown by giving an example.

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0. INTRODUCTION

Sun Shin Ahn, young bae Jun and Hee Sik Kim [1] introduced and studied the concepts - Sub incline, ideal, quotients
of an incline algebras, prime ideal, irreducible ideal and maximal ideal of an incline algebra and their properties.

1. PRELIMINARIES

Definition 1.1: [1]. Incline algebra: A system $(K, +, *)$, where $K$ is a non empty set "$+$" and "*" are binary
operations on $K$ satisfying the following axioms is called an incline algebra.
(i) $x + y = y + x$ (+ is commutative)
(ii) $x + (y + z) = (x + y) + z$ (+ is associative)
(iii) $x * (y * z) = (x * y) * z$ (* is associative)
(iv) $x * (y + z) = (x * y) + (x * z)$ (* is left distributive)
(v) $(y + z) * x = (y * x) + (z * x)$ (* is right distributive)
(vi) $x + x = x$ (+ is idempotent)
(vii) $x + (x * y) = x$
(viii) $y + (x * y) = y$ for all $x, y, z \in K$

Definition 1.2: [1]. Let $(K, +, *)$ be an incline algebra.
(i) $K$ is called commutative if $x * y = y * x$ for all $x, y \in K$.
(ii) An element $0 \in K$ is called a zero element if $x + 0 = x$ and $x * 0 = 0 * x = 0$ for all $x \in K$
(iii) An element $1 \in K$ is called a multiplicative identity if $x * 1 = 1 * x = x$ for all $x \in K$
Clearly, every distributive lattice \((K, \lor, \land)\) is an incline algebra \((K, +, \ast)\) with \(+ = \lor\) and \(\ast = \land\). Every incline algebra is not a distributive as the following example shows.

**Example 1.3:** Consider the system \((K, +, \ast)\) where \(K = \{0, 1\}\) and the binary operations \(+\) and \(\ast\) are given by

\[
\begin{array}{c|c|c}
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{c|c|c}
\ast & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

This system is an incline algebra but not a distributive lattice since \(0 = 1 \land 1\) (by the definition of \(\land\)) \(\neq 1\).

**Note 1.3.1:** Let \((K, +, \ast)\) be an incline algebra. From axioms (i), (ii), (vi), \((K, +)\) is a semi lattice and hence the binary relation \(\leq\) on \(K\), defined by "\(x \leq y \Leftrightarrow x + y = y\)" is a partial ordering on \(K\), such that for any \(x, y \in K\), \(x \lor y = 1u.b\{x, y\}\) exists and \(x \lor y = x + y\).

**Definition 1.4:** [1]. A sub incline of an incline algebra \((K, +, \ast)\) is a non empty subset \(M\) of \(K\) which is closed under the operations \(+\) and \(\ast\).

**Definition 1.5:** [1]. A sub incline \(M\) of an incline algebra \((K, +, \ast)\) is called an ideal if \(\forall a, b \in K\), \(a \ast b \in M, x \ast y \in M\).

**Note 1.5.1:** An ideal \(M\) of an incline algebra \(K\) is called proper if \(K \neq M\). By the definition, every ideal of an incline algebra is a sub incline. Converse is not true as the following example shows.

**Example 1.6:** Consider the incline algebra \((K, +, \ast)\) where \(K = \{0, 1, a\}\) the binary operations \(+\) and \(\ast\) are given by

\[
\begin{array}{c|c|c|c}
+ & 0 & 1 & a \\
\hline
0 & 0 & 1 & a \\
1 & 1 & 1 & a \\
a & a & a & a \\
\end{array}
\quad
\begin{array}{c|c|c|c}
\ast & 0 & 1 & a \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
\end{array}
\]

Here \(M = \{0, a\}\) is clearly, a sub incline of \(K\). Clearly \(1 \leq a\) (since \(1 + a = a\)), \(a \in M\), but \(1 \notin M\). So, \(M\) is not an ideal of \(K\).

**Definition 1.7:** [1]. A proper ideal \(I\) of an incline algebra \((K, +, \ast)\)

(i) prime if

"\(a, b \in K, a \ast b \in I \Rightarrow a \in I \text{ or } b \in I\)"

(ii) maximal ideal if

"\(N\) is an ideal of \(K\), \(I \subseteq N\), \(\Rightarrow I = N \text{ or } N = K\)"

(iii) an irreducible ideal if

"\(A \cap B = I \Rightarrow A = I \text{ or } B = I\)" for any ideals \(A\) and \(B\) of \(K\).

**Theorem 1.8:** [1]. Let \(I\) be a proper ideal of an incline algebra \(K\). The following statements are equivalent.

(a) \(I\) is an irreducible ideal.

(b) \(I\) is prime.

(c) \(A \cap B \subseteq I \Rightarrow A \subseteq I \text{ or } B \subseteq I\) for any ideals \(A\) and \(B\) of \(K\).

**2. MAIN RESULTS OF THE PAPER**

We begin with the following

**Theorem 2.1:** Let \((K, +, \ast)\) be an incline algebra. For any \(x, y \in K\), \(x \ast y\) is a lower bound of \(\{x, y\}\)

i.e., \(x \ast y \leq x, \ x \ast y \leq y\).
Proof: Let \(x, y \in K\). Now, \(x + x \cdot y = x\) (by (vii) of def 1.1)
\[\Rightarrow x \cdot y + x = x\] (by (i) of def 1.1)
\[\Rightarrow x \cdot y \leq x\]
\[\Rightarrow y + x \cdot y = y\] (by (viii) of def 1.1)
\[\Rightarrow x \cdot y + y = y\] (by (i) of def 1.1)
\[\Rightarrow x \cdot y \leq y\]
Hence, \(x \cdot y\) is a lower bound of \(\{x, y\}\).

Note 2.1.1: Interchanging \(x\) and \(y\) in theorem 2.1, we have that for any \(x, y \in K\), \(y \cdot x\) is also a lower bound of \(\{x, y\}\).

Theorem 2.2: Let \(K\) be an incline algebra. Let \(0 \in K\). Then, the following statements are equivalent.

(i) \(a + 0 = a\) for all \(a \in K\):
(ii) \(0\) is the least element of \(K\); i.e., \(0 \leq a\) for all \(a \in K\)
(iii) \(a \cdot 0 = 0 = 0 \cdot a\) for all \(a \in K\)

Proof:
(i) \(\Rightarrow\) (ii): Trivial by the definition of \(\leq\).

(ii) \(\Rightarrow\) (iii): Assume (ii). Let \(a \in K\). By theorem 2.1, \(a \cdot 0 \leq 0\), \(0 \cdot a \leq 0\). Since \(0\) is the least element of \(K\),
we have \(0 \leq a \cdot 0\) and \(0 \leq 0 \cdot a\).
Hence \(a \cdot 0 = 0 = 0 \cdot a\).

(iii) \(\Rightarrow\) (i): Assume (iii). For any \(a \in K\),
\[a = a + a \cdot 0\] (by (vii) of definition 1.1)
\[a = a + 0\] (by our assumption).
Hence the theorem.

Note 2.2.1: Since (i) and (iii) are equivalent in theorem 2.2, we can retain any one of "\(a + 0 = a\) for all \(a \in K\)" and "\(a \cdot 0 = 0 = 0 \cdot a\)" for all \(a \in K\)" in the definition 1.2 (ii) of zero element in the preliminaries.

Theorem 2.3: Let \(I\) be a proper ideal of an incline algebra \(K\). Consider the following statements.

(a) \(I\) is an irreducible ideal.
(b) \(I\) is prime.
(c) \(A \cap B \subseteq I \Rightarrow A \subseteq I\) or \(B \subseteq I\) for any ideals \(A\) and \(B\) of \(K\).

Then, (b) \(\Rightarrow\) (c) \(\Rightarrow\) (a) holds.

Proof:
(b) \(\Rightarrow\) (c): Assume (b). Suppose (c) fails i.e., there exist ideals \(A, B\) of \(K\) such that \(A \subseteq I\), \(B \subseteq I\) and \(A \cap B \subseteq I\).
So, there exist elements \(x, y\) in \(K\) such that \(x \in A - I\), \(y \in B - I\). By theorem 2.1, \(x \cdot y \leq x\) and \(x \cdot y \leq y\).
Since \(A\) and \(B\) are ideals, \(x \cdot y \in A \cap B\). Since \(A \cap B \subseteq I\), we have that \(x \cdot y \in I\).
Since \(I\) is prime (by our assumption), either \(x \in I\) or \(y \in I\), a contradiction. Hence (c) holds.

(c) \(\Rightarrow\) (a): Trivial.

Note 2.3.1: In [1], it is prove that the statements (a),(b) and (c) of theorem 2.3 are equivalent(see theorem 1.8. in the preliminaries). But this is not true as the following example shows.
Example 2.4: Consider the incline algebra $(K, +, *)$ of the example 1.6. Clearly, $I = \{0\}$ and $J = \{0, 1\}$ are the only proper ideals of $K$. Clearly, $I$ and $J$ are irreducible ideals of $K$. $I$ is not a prime ideal since $a \in I$ and $a * a = 0 \in I$. Similarly, $J$ is not a prime ideal.

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