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HARTMAN-WINTNER-TYPE INEQUALITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH PRABHAKAR DERIVATIVE

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ABSTRACT

In this paper, we consider a nonlocal fractional boundary value problem with Prabhakar derivative and obtained a Hartman-Wintner type inequality for it.

Mathematics Subject Classification: *26A33; 33E12; 34A08; 34A40.*

Keywords: *Hartman-Wintner-type inequality; Fractional boundary value problem, nonlocal boundary conditions; Prabhakar derivative.*

1. INTRODUCTION

Hartman and Wintner [11] considered the following boundary value problem

$$\begin{cases} x''(t)+q(t)x(t)=0\ , & a<t<b, \\ x(a)=x(b)=0\ , \end{cases} \tag{1.1}$$

and proved the inequality if (1.1) have a nontrivial solution

$$\int_a^b (b-s)(s-a)q^+(s)ds > b-a, \tag{1.2}$$

provided (1.1) has a nontrivial solution, where $q^+(s)=max\{q(s),0\}$.

Lyapunov [14] proved that if (1.1) have a nontrivial solution, then

$$\int_a^b |q(s)|\,ds > \frac{4}{b-a}. \tag{1.3}$$

This inequality (1.3) is useful in various branches of mathematics such as oscillation theory, disconjugacy and eigenvalue problems. The Lyapunov inequality (1.3) can be deduced from (1.2) using the fact that

$$\max_{s\in[a,b]}(b-s)(s-a)=\frac{(b-a)^2}{4}. \tag{1.4}$$

The generalizations and extensions of the Lyapunov inequality (1.3) exist in the literature [3, 2, 5, 4, 17, 15, 19, 22]. Recently, some Lyapunov type inequalities were obtained for different fractional boundary value problem using various differential operators [8, 9, 12, 13, 20, 21, 16, 1].

Cabrera and *et al.* [6] have considered the following nonlocal fractional boundary value problem

$$\begin{cases} D_a^{\alpha}x(t)+q(t)x(t)=0\ , & a<t<b, \\ x(a)=x'(a)=0\ , & x'(b)=\beta x(\xi), \end{cases} \tag{1.5}$$

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where D_a^α denotes the standard Riemann-Liouville fractional derivative of order α, $a<\xi<b$, $0 \le \beta(\xi-a)^{\alpha-1} < (\alpha-1)(b-a)^{\alpha-2}$, $q(t)$ is continuous real valued function on $[a,b]$. The authors obtained the Hartman-Wintner-type inequality

$$\int_a^b (b-s)^{\alpha-2}(s-a)\,|q(s)|\,ds \ge \left(1+\frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2}-\beta(\xi-a)^{\alpha-1}}\right)^{-1}\Gamma(\alpha). \tag{1.6}$$

Motivated by above work, in this paper we consider the following nonlocal fractional boundary problem

$$\begin{cases} D^\gamma_{\rho,\mu,\omega,a^+}x(t)+q(t)x(t)=0\ ,\ a<t<b,\quad 2<\mu\le 3 \\ x(a)=x'(a)=0\ , x'(b)=\beta x(\xi), \end{cases} \tag{1.7}$$

where $D^\gamma_{\rho,\mu,\omega,a^+}$ denotes the Prabhakar derivative of order μ.

$a<\xi<b, 0\le \beta(\xi-a)^{\mu-1}<(\mu-1)(b-a)^{\mu-2}$, $q:[a,b]\to\mathbb{R}$ is real valued continuous function and obtained the Hartman-Wintner-type inequality for it.

2. PRELIMINARIES

In this section, we give some basic definitions and lemmas that will be important to us in the sequel.

Definition 2.1: *[18] . The generalized Mittag-Leffler function with three parameters is defined as,*

$$E^\gamma_{\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k z^k}{\Gamma(\rho k+\mu)k!},\qquad \gamma,\rho,\mu\in\mathbb{C},\Re(\rho)>0\ , \tag{2.1}$$

where $(\gamma)_k$ is Pochhammer symbol defined by,

$(\gamma)_0=1,\quad (\gamma)_k=\gamma(\gamma+1)...(\gamma+k-1)$, for $k=1,2,...$.

For $\gamma=1$, the generalized Mittag-Leffler function (2.1) reduces to the two-parameter Mittag-Leffler function given by

$$E_{\rho,\mu}(z):=E^1_{\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+\mu)},\quad \rho,\mu\in\mathbb{C},\quad \Re(\rho)>0\ , \tag{2.2}$$

and for $\mu=\gamma=1$, this function coincides with the classical Mittag-Leffler function $E_\rho(z)$

$$E_\rho(z):=E^1_{\rho,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+1)},\qquad \rho\in\mathbb{C},\quad \Re(\rho)>0\ . \tag{2.3}$$

Also, for $\gamma=0$ we have $E_{\rho,\mu}(z)=\dfrac{1}{\Gamma(\mu)}$.

Definition 2.2: *[10]. Let* $f\in L^1[0,b]$, $0<x<b\le\infty$, *the prabhakar integral operator including generalized Mittag-Leffler function (2.1) is defined as follows*

$$E^\gamma_{\rho,\mu,\omega,0+}f(x)dx=\int_0^x (x-u)^{\mu-1}E^\gamma_{\rho,\mu}(\omega(x-u)^\rho)f(u)du,\quad x>0 \tag{2.4}$$

where $\rho,\mu,\omega,\gamma\in\mathbb{C}$, with $\Re(\rho),\Re(\mu)>0.$

If for $\gamma=0$, the prabhakar integral operator coincides with the Riemann-Liouville fractional integral of order μ;

$$E^0_{\rho,\mu,\omega,0+}f(x)=I^\mu_{0+}f(x),$$

where the Riemann-Lioville fractional integral is defined as

$$I^\mu_{0+}f(x)=\frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}f(t)dt,\quad \mu\in\mathbb{C},\Re(\mu)>0\ . \tag{2.4}$$

Definition 2.3: [10]. *Let* $f \in L^1[0,b]$, $0 < x < b \le \infty$, *the Prabhakar derivative is defined as*

$$D^{\gamma}_{\rho,\mu,\omega,0+} f(x) = \frac{d^m}{dx^m} E^{-\gamma}_{\rho,m-\mu,\omega,0+} f(x), \tag{2.6}$$

where $\rho,\mu,\omega,\gamma \in \mathbb{C}$, with $\Re(\rho) > 0$, $\Re(\mu) > o$, $m-1 < \Re(\mu) < m$.

We note that the Prabhakar derivative genralizes the Riemann-Liouville fractional derivative

$$D^{\mu}_{0+} f(x) = \frac{d^m}{dx^m}(I^{m-\mu}_{0+} f)(x), \quad \mu \in \mathbb{C}, \Re(\mu) > 0, m-1 < \Re(\mu) < m. \tag{2.7}$$

Lemma 2.1: [7]. *If* $f(x) \in C(a,b) \bigcap L(a,b)$, *then*

$$D^{\gamma}_{\rho,\mu,\omega,a+} E^{\gamma}_{\rho,\mu,\omega,a+} f(x) = f(x), \tag{2.8}$$

and if $f(x), D^{\gamma}_{\rho,\mu,\omega,a+} f(x) \in C(a,b) \bigcap L(a,b)$ then for $m-1 < \mu \le m$,
we have

$$\begin{aligned} E^{\gamma}_{\rho,\mu,\omega,a+} D^{\gamma}_{\rho,\mu,\omega,a+} f(x) = {} & f(x) + c_1 (x-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(x-a)^{\rho}) \\ & + c_2 (x-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(x-a)^{\rho}) + \ldots \\ & + c_m (x-a)^{\mu-m} E^{\gamma}_{\rho,\mu-m+1}(\omega(x-a)^{\rho}). \end{aligned} \tag{2.9}$$

3. MAIN RESULTS

Theorem 3.1: *Assume that* $2 < \mu \le 3$ *and* $x \in C[a,b]$. *If the nonlocal fractional boundary value problem (1.7) has unique nontrivial solution, then it satisfies*

$$x(t) = \int_a^b G(t,s)x(s)ds + \frac{\beta(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho}) - \beta(\xi-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})} \times \int_a^b G(\xi,s)x(s)ds,$$

where the Green's function is defined as

$$G(t,s) = \begin{cases} \dfrac{(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})(b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})} \\ -(t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}), \qquad a \le s \le t \le b \\ \dfrac{(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})(b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}, a \le t \le s \le b. \end{cases} \tag{3.1}$$

Proof: From lemma 2.1, the general solution to (1.7) in $C[a,b]$ can be written as follows

$$\begin{aligned} x(t) = {} & c_1(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho}) + c_2(t-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(t-a)^{\rho}) \\ & + c_3(t-a)^{\mu-3} E^{\gamma}_{\rho,\mu-2}(\omega(t-a)^{\rho}) - \int_a^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) q(s)x(s)ds. \end{aligned}$$

Employing the first boundary condition $x(a) = x'(a) = 0$ we obtain $c_2 = c_3 = 0$. Therefore

$$x(t) = c_1(t-a)^{\mu-2} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho}) - \int_a^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}) q(s)x(s)ds.$$

For second boundary condition we obtain,

$$x'(t) = c_1(t-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(t-a)^{\rho}) - \int_a^t (t-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(t-s)^{\rho}) q(s)x(s)ds.$$

Employing the second boundary condition $x'(b) = \beta x(\xi)$ we get

$$\begin{aligned} & c_1(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho}) - \int_a^b (b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho}) q(s)x(s)\, ds \\ & \qquad = \beta c_1(\xi-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho}) - \beta \int_a^{\xi} (\xi-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-s)^{\rho}) q(s)x(s)ds, \end{aligned}$$

$$\Rightarrow c_1(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta c_1(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})$$

$$=\int_a^b(b-s)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})q(s)x(s)ds-\beta\int_a^{\xi}(\xi-s)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-s)^{\rho})q(s)x(s)ds,$$

$$\Rightarrow c_1[(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})]$$
$$=\int_a^b(b-s)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})q(s)x(s)ds-\beta\int_a^{\xi}(\xi-s)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-s)^{\rho})q(s)x(s)ds,$$

$$\Rightarrow c_1=\frac{1}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}$$
$$\times\int_a^b(b-s)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})q(s)x(s)d$$
$$-\frac{\beta}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}$$
$$\times\int_a^{\xi}(\xi-s)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-s)^{\rho})q(s)x(s)ds.$$

Thus the solution $x(t)$ becomes

$$x(t)=\frac{(t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}$$
$$\times\int_a^b(b-s)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})q(s)x(s)\ ds$$
$$-\frac{\beta(t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}$$
$$\times\int_a^{\xi}(\xi-s)^{\mu-1}E^{\gamma}_{\rho,\mu-1}(\omega(\xi-s)^{\rho})q(s)x(s)ds$$
$$-\int_a^t(t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho})q(s)x(s)ds.$$

Taking into account that

$$\frac{E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}$$
$$=\frac{E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}\left(\frac{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right)$$
$$=\frac{E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}$$
$$\times\left(1+\frac{\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right),$$

we have

$$x(t)=\frac{(t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}$$
$$\left(1+\frac{\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right)$$

$$\times\int_a^b (b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})q(s)x(s)ds$$
$$-\frac{\beta(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}$$
$$\times\int_a^{\xi}(\xi-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-s)^{\rho})q(s)x(s)ds-\int_a^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho})q(s)x(s)ds.$$

On simplifying,

$$x(t)=\frac{(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}\int_a^t (b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})q(s)x(s)ds$$
$$+\frac{(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}\int_t^b (b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})q(s)x(s)ds$$
$$+\left(\frac{(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}\right)\left(\frac{\beta(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right)$$
$$\times\int_a^{\xi}(b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})q(s)x(s)ds+\frac{(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}$$
$$\times\left(\frac{\beta(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right)$$
$$\times\int_{\xi}^b (b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})q(s)x(s)ds$$
$$-\frac{\beta(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}$$
$$\times\int_a^{\xi}(\xi-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-s)^{\rho})q(s)x(s)ds-\int_a^t (t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho})q(s)x(s)ds.$$

Further, on rearranging the terms, we have

$$x(t)=\int_a^t\left(\frac{(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})(b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}\right.$$
$$\left.-(t-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho})\right)q(s)x(s)\,ds$$
$$+\int_t^b\left(\frac{(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})(b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}\right)q(s)x(s)ds$$
$$+\frac{\beta(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}$$
$$\times\int_a^{\xi}\left(\frac{(\xi-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})(b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}\right.$$
$$\left.-(\xi-s)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-s)^{\rho})\right)q(s)x(s)ds+\frac{(t-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}$$
$$\times\left(\frac{\beta(\xi-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}{(b-a)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1} E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right)$$
$$\times\int_{\xi}^b (b-s)^{\mu-2} E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})q(s)x(s)d\ ,$$

therefore,

$$x(t)=\int_a^b G(t,s)q(s)x(s)ds$$
$$+\frac{\beta(t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\int_a^b G(\xi,s)q(s)x(s)ds,$$

where the Green's function $G(t,s)$ is as in (3.1).

Theorem 3.2: *The Green's function (3.1) satisfies the following properties:*

(a) $G(t,s)\geq 0,$ for all $(t,s)\in[a,b]\times[a,b]$;

(b) $G(t,s)$ is nondecreasing function with respect to the first variable;

(c) $0\leq G(a,s)\leq G(t,s)\leq G(b,s), \quad (t,s)\in[a,b]\times[a,b].$

proof (a): We set two function as

$$g_1(t,u)=\frac{(t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})(b-s)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}$$
$$-(t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}), \qquad a\leq s\leq t\leq b,$$

and

$$g_2(t,u)=\frac{(t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})(b-s)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}, \qquad a\leq t\leq s\leq b.$$

It is clear that $g_2(t,u)\geq 0$. So to prove (a), we should show that $g_1(t,u)\geq 0$, or equivalently

$$\frac{(t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})(b-s)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}\geq(t-s)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho}),$$

therefore it is sufficient to prove that

$$(i)\quad \frac{(t-a)^{\mu-1}(b-s)^{\mu-2}}{(b-a)^{\mu-1}}\geq(t-s)^{\mu-1}$$

$$(ii)\quad \frac{E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}\geq E^{\gamma}_{\rho,\mu}(\omega(t-s)^{\rho})$$

Consider,

$$\begin{aligned}\frac{(t-a)^{\mu-1}(b-s)^{\mu-2}}{(b-a)^{\mu-1}}-(t-s)^{\mu-1}&=(t-a)^{\mu-1}\left(\frac{b-s}{b-a}\right)^{\mu-2}-((t-a)-(s-a))^{\mu-1}\\&=(t-a)^{\mu-1}\left(\frac{(b-a)-(s-a)}{b-a}\right)^{\mu-2}-(t-a)^{\mu-1}\left(1-\frac{s-a}{t-a}\right)^{\mu-1}\\&=(t-a)^{\mu-1}\left(1-\frac{s-a}{b-a}\right)^{\mu-2}-(t-a)^{\mu-1}\left(1-\frac{s-a}{t-a}\right)L^{\mu-1}\\&\geq(t-a)^{\mu-1}\left(1-\frac{s-a}{b-a}\right)^{\mu-2}-(t-a)^{\mu-1}\left(1-\frac{s-a}{b-a}\right)^{\mu-1}\quad(\because t\leq b)\\&=(t-a)^{\mu-1}\left(1-\frac{s-a}{b-a}\right)^{\mu-2}\left(\frac{s-a}{b-a}\right)\geq 0.\end{aligned}$$

Hence (i) is proved. For the proof of (ii) refer (Theorem 2 , in [7]).

Proof (b): Proof of this is similar to (Theorem 2 , in [7]) for $\mu=\mu-1$.

Proof (c): Proof of this follows from **(**b).

Theorem 3.3: *Suppose that problem (1.7) has a nontrivial continuous solution, then*

$$\int_a^b \Big[\frac{(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})(b-s)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})} - (b-s)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-s)^{\rho})\Big]|q(s)|\,ds$$

$$\geq \left(1+\frac{\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right)^{-1}.$$

Proof: Consider the Banach space

$C[a,b]=\{u:[a,b]\to\mathbb{R}\,|\,u \ is \ continuous\}$ with norm

$\|u\|_{\infty} = max\{|u(t)|: a\leq t\leq b\}, u\in C[a,b].$

By theorem 3.1, a solution $x\in C[a,b]$ of (1.7) has the expression

$$x(t)=\int_a^b G(t,s)q(s)x(s)ds$$
$$+\frac{\beta(t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\int_a^b G(\xi,s)q(s)x(s)\,ds,\quad a\leq t\leq b.$$

From this, for any $t\in[a,b]$, we have

$$|x(t)|\leq \|x\|_{\infty}\int_a^b |G(t,s)||q(s)|\,ds$$
$$+\frac{\beta(t-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-a)^{\rho})\|x\|_{\infty}}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\int_a^b |G(\xi,s)||q(s)|\,ds,$$

therefore,

$$|x(t)|\leq \|x\|_{\infty}\int_a^b |G(b,s)||q(s)|\,ds$$
$$+\frac{\beta(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})\|x\|_{\infty}}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\int_a^b |G(b,s)||q(s)|\,ds,$$

which yields

$$\|x\|_{\infty}\leq \|x\|_{\infty}\left(1+\frac{\beta(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right)\int_a^b G(b,s)|q(s)|\,ds,$$ As x is a nontrivial, we have

$$1\leq\left(1+\frac{\beta(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right)\int_a^b |G(b,s)||q(s)\,ds.$$

$$\int_a^b |G(b,s)||q(s)|\,ds\geq\left(1+\frac{\beta(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right)^{-1},$$

therefore

$$\int_a^b\left(\frac{(b-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-a)^{\rho})(b-s)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-s)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})}-(b-s)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(b-s)^{\rho})\right)|q(s)|\,ds$$

$$\geq\left(1+\frac{\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}{(b-a)^{\mu-2}E^{\gamma}_{\rho,\mu-1}(\omega(b-a)^{\rho})-\beta(\xi-a)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\xi-a)^{\rho})}\right)^{-1}$$

Hence the result.

4. REFERENCES

1. N. Abuj and D. Pachpatte, *Lyapunov type inequality for fractional differential equation with k-Prabhakar derivative*, Classical Analysis and ODEs (math.CA) (2017), arXiv:1702.01562v1 [math.CA], 1-11.
2. R. Brown, D. Hinton, *Opial's inequality and oscillation of 2nd order equations*, *Proc. Amer. Math. Soc.* 125 (1997), 1123-1129.
3. S. Cheng, *A discrete analouge of the inequality of Lyapunov*, *Hokkaido Math. J.* 12 (1983), 105-112.
4. S. Cheng, *Lyapunov type inequalities for differential and difference equations*, *Fasc. Math.* 23 (1991), 25-41.
5. D. Cakmak, *Lyapunov-type integral inequalities for certain higher order differential equations*, *Appl. Math. Comput.* 216 (2010), 368-373.
6. I. Cabrera, K. Sadarangani and B. Samet, *Hartman-Wintner-type inequalities for a class of nonlocal fractional boundary value problems*, *Math. Methods Appl. Sci.* 40 (2017), 129-136.
7. S. Eshaghi and A. Ansari, *Lyapunov inequality for fractional differential equations with Prabhakar derivative*, *Math. Inequal. Appl.* 19 (2016), 349-358.
8. R. Ferreira, *A Lyapunov-type inequality for a fractional boundary value problem*, *Fract. Calc. Appl. Anal.* 16(2013), 978-984.
9. R. Ferriera, *On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function*, *J. Math. Anal. Appl.* 412 (2014), 1058-1063.
10. R. Garra, R. Gorenflo, F. Polito and Z. Tomovski, *Hilfer Prabhakar derivative and some applications*, *Appl. Math. Comput.* 242 (2014), 576-589.
11. P. Hartman, A. Wintner, *On an oscillation criterion of Lyapunov*, *Amer. J. Math.* 73 (1951), 885-890.
12. M. Jleli and B. Samet, *Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions*, *Math. Inequal. Appl.* 18 (2015), 443-451.
13. M. Jleli and B. Samet, *Lyapunov-type inequalities for fractional boundary value problems*, *Electr. j. differ. equ.* 88 (2015), 1-11.
14. A. Lyapunov, *Probleme generel de la stabilite du mouvement*, *Ann. Fac. Sci. Toulouse Math.* 2 (1907), 203-407.
15. C. Lee, C. Yeh, C. Hong, R. Agarwal *Lyapunov and Wirtinger inequalities*, *Appl. Math. Lett.* 17 (2004), 847-853.
16. D. Pachpatte, N. Abuj and A. Khandagale, *Lyapunov type inequality for hybrid fractional differential equation with Prabhakar derivative*, *Int. J. Pure Appl. Math.* 113 (2017), 563-574.
17. B. Pachpatte, *On Lyapunov-type inequalities for certain higher order differential equations*, *J. Math. Anal. Appl.* 195 (1995), 527-536.
18. T. R. Prabhakar, *A Singular integral equation with a generalized Mittag-Leffler function in the kernel*, *Yokohama Math. J.* 19(1971), 7-15.
19. N. Parhi, S. Panigrahi, *On Lyapunov-type inequality for third order differential equations*, *J. Math. Anal. Appl.* 233 (1999), 445-460.
20. J. Rong, C. Bai, *Lyapunov-type inequality for a fractional differential equation with fractional boundary condition*, *Adv. Difference Equ.* 82 (2015), 1-10.
21. O. Regan, B. Samet, *Lyapunov-type inequality for a class of fractional differential equations*, *J. Inequal. Appl.* 247 (2015), 1-10.
22. A. Tiryaki, M. Unal, D. Cakmak, *Lyapunov-type inequalities for nonlinear systems*, *Math. Anal. Appl.* 332 (2007) 497-511.
23. X. Yang, K. Lo, *Lyapunov-type inequality for a class of even-order differential equations*, *Appl. Math. Comput.* 215 (2010), 3884-3890.

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