A Pál Type \((0, 1; 0)\) Interpolation Process on Laguerre Polynomial

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ABSTRACT

In the present paper, we have considered the problem in which \( \{x_i\}_{i=1}^{n} \) and \( \{\xi_i\}_{i=1}^{n} \) be the two sets of interscaled nodal points on the interval \([0, \infty)\). Here we deal with the problem in which one set consists of the nodes of \( L_n^k(x) \) and other consists of the nodes of \( L_n^{k-1}(x) \). We investigate the existence, uniqueness explicit representation of interpolatory polynomial. Estimation of the fundamental polynomials have also been obtained.

Keywords: lacunary Interpolation, Pál - Type Interpolation, Laguerre Polynomial.

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1. INTRODUCTION

J. Balázs [2] was the first to give the solution of the problem with the nodes as the zeros of ultra spherical polynomial \( p_n^{(\alpha)}(x) \) \((\alpha > -1)\) and the weight function \((x) = (1 - x^2)^{\frac{1+\alpha}{2}}, \ x \in [-1, 1] \). He proved that generally there do exist any polynomial \( R_n(x) \) of degree \( \leq 2n-1 \) satisfying the conditions:

\[
R_n(x) = g_i^*, \quad (\omega R_n)’(\xi_i) = g_i^{**} \quad \text{for} \ i = 1(1)n
\]

where \( g_i^* \) and \( g_i^{**} \) are arbitrary real numbers. However taking an additional condition

\[
R_n(0) = \sum_{i=1}^{n} a_i x_i^l \tag{1.3}
\]

where 0 is not a nodal point. In 1984, L. Szili [13] studied analogous problem with the nodes as the roots of \( H_n(x) \), the Hermite polynomial and weight function \( \omega(x) = e^{(-x^2)} \). Pál [10] proved that for a given arbitrary numbers \( \{a_i^\prime\}_{i=1}^{n} \) and \( \{\beta_i^\prime\}_{i=1}^{n} \) there exists a unique polynomial of degree \( \leq 2n-1 \) satisfying the conditions:

\[
R_n(x) = a_i^*, \quad \text{for} \ i = 1(1)n \quad (\omega R_n)’(\xi_i) = \beta_i^* \quad \text{for} \ i = 1(1)n - 1, \quad \text{with an initial condition} \quad R_n(0) = 0 \quad \text{where} \ a \ \text{is a given point, different from the nodal points} \ {x_i}_{i=1}^{n} \ \text{and} \ {y_j}_{j=1}^{n} \ \text{. In this paper we study Pál – type interpolational polynomial with} \ \omega_{n+k}(x) = x^k L_n^{(k)}(x) \ \text{. We have determined the existence, uniqueness, explicit representation and estimation of fundamental polynomials of such special kind of mixed type of interpolation on interval} \ [0, \infty) \text{. Let} \ \{x_i\}_{i=1}^{n} \ \text{and} \ \{\xi_i\}_{i=1}^{n} \ \text{be the two sets of interscaled nodal points on the interval} \ [0, \infty) \text{. We seek to determine a polynomial} \ R_n(x) \ \text{of minimal possible degree} \ \leq 3n+k \text{ satisfying the interpolatory conditions:}
\]

\[
R_n(x) = g_i^*, \quad R_n’(x_k) = g_i^{*}\quad \text{, } \quad R_n(0) = g_0^j \quad \text{, } \quad j = 0, 1, \ldots, k \tag{1.3}
\]

where \( g_i^* \), \( g_i^{**} \) and \( g_0^j \) are arbitrary real numbers. Here Laguerre polynomials \( L_n^{(k)}(x) \) and \( L_n^{(k-1)}(x) \) have zeroes \( \{x_i\}_{i=1}^{n} \ \text{and} \ \{\xi_i\}_{i=1}^{n} \ \text{respectively and} \ x_0 = 0 \). We prove existence, uniqueness, explicit representation and estimation of fundamental polynomials.

2. PRELIMINARIES

In this section we shall give some well-known results which are as follows:

As we know that the Laguerre polynomial is a constant multiple of a confluent hypergeometric function so the differential equation is given by

\[
xD^2 L_n^{(k)}(x) + (1 + k - x)DL_n^{(k)}(x) + nL_n^{(k)}(x) = 0 \tag{2.1}
\]

\[
L_n^{(k-1)'}(x) = -L_n^{(k)}(x) \tag{2.2}
\]
Similarly using the identities
\begin{align}
L_n^{(k)}(x) &= l_n^{(k+1)}(x) - L_{n-1}^{(k+1)}(x) \\
xL_n^{(k)}(x) &= nL_n^{(k)}(x) - (n+k)L_{n-1}^{(k)}(x)
\end{align}

We can easily find a relation
\begin{align}
\frac{d}{dx}[x^kL_n^k(x)] &= (n+k)x^{k-1}L_n^{(k-1)}(x)
\end{align}

By the following conditions of orthogonality and normalization we define Laguerre polynomial $L_n^{(k)}(x)$, for $k > -1$
\begin{align}
\int_0^\infty e^{-x}x^kL_n^{(k)}(x)L_m^{(k)}(x)dx &= \Gamma(k+1)\binom{n+k}{n} \delta_{nm}, m = 0, 1, 2, \ldots.
\end{align}

\begin{align}
L_n^{(k)}(x) &= \sum_{\mu=0}^{n} \binom{n+k}{n-\mu} \frac{(-x)^\mu}{\mu!}
\end{align}

The fundamental polynomials of Lagrange interpolation are given by
\begin{align}
l_j(x) &= \frac{l_n^{(k)}(x)}{l_n^{(k)}(x_j)(x-x_j)} = \delta_{ij} \\
l_j'(x) &= \frac{l_n^{(k-1)}(x)}{l_n^{(k-1)}(x_j)(x-x_j)} = \delta_{ij} \\
l_j''(y_j) &= \begin{cases} \frac{l_n^{(k-1)}(y_j)}{l_n^{(k-1)}(y_j)(y_j-y_j)} - \frac{(k-y_j)}{y_j} & i \neq j \\ i = j = 1(1)n \end{cases} \\
l_j'(y_j) &= \frac{1}{(x_j-y_j)} \frac{l_n^{(k)}(y_j)}{l_n^{(k)}(x_j)(y_j-x_j)} - \frac{l_n^{(k)}(y_j)}{l_n^{(k)}(x_j)(x_j-y_j)}, \quad j = 1(1)n
\end{align}

For the roots of $L_n^{(k)}(x)$ we have
\begin{align}
x_k^2 &\sim \frac{k^2}{n} \\
\eta(x)|S_n^{(j)}(x)| &= O(1) \quad \text{where} \ \eta(x) \ \text{is the weight function} \\
|L_n^{(k)}(x_j)| &\sim j^{-k-\frac{3}{2}n+\frac{1}{2}}, \quad 0 < x_j \leq \Omega, \quad n = 1, 2, 3, \ldots.
\end{align}

\begin{align}
|L_n^k(x_j)| &= \begin{cases} x^{-\frac{k-1}{2}} \binom{k-1}{n} \frac{1}{(n+k)^{\frac{4}{2}}}, & cn^{-1} \leq x \leq \Omega \\ 0(n^k), & 0 \leq x \leq cn^{-1} \end{cases}
\end{align}

3. NEW RESULTS

**Theorem 1:** For $n > 1$ fixed integer let $\{g_i\}_{i=1}^n$, $\{g_i^*\}_{i=1}^n$, $\{g_i^{**}\}_{i=1}^n$ and $\{g_0^{(j)}\}_{j=0}^k$ are arbitrary real numbers then there exists a unique polynomial $R_n(x)$ of minimal possible degree $\leq 3n+k$ on the nodal points (1.1) satisfying the condition (1.2) and (1.3). The polynomial $R_n(x)$ can be written in the form
\begin{align}
R_n(x) &= \sum_{j=1}^n U_j(x)g_j + \sum_{j=1}^n V_j(x)g_j^* + \sum_{j=1}^n W_j(x)g_j^{**} + \sum_{j=0}^k C_j(x)g_0^{(j)}
\end{align}

where $U_j(x)$, $V_j(x)$, $W_j(x)$ and $C_j(x)$ are fundamental polynomials of degree $\leq 3n+k$ given by
\begin{align}
U_j(x) &= \frac{x^{k+1} l_n^{(k)}(x_j) [l_n^{(k)}(x)]^2 [1 - z(x-x_j)]}{x_j^{k+1} l_n^{(k)}(x_j)} \\
V_j(x) &= \frac{x^{k+1} l_n^{(k)}(x_j) [l_n^{(k)}(x)]^2 [1 - z(x-x_j)]}{x_j^{k+1} l_n^{(k)}(x_j) l_n^{(k-1)}(x_j)}
\end{align}
(3.4) \[ W_j(x) = \frac{x^{k+1}l_j^r(x)l_n^k(x)^2}{y_j^{k+1}l_n^k(y_j)^2}, \]

(3.5) \[ C_j(x) = p_j(x)x^j \left[ L_n^{(k-1)}(x) \right]^2 L_n^k(x) x^{k}l_n^k(x) l_n^{(k-1)}(x) \left[ \frac{C_j - \frac{i_{n-1}(x)p_j(x)q_j(x)}{x^k-j}}{x^{k-j}} \right], \]

(3.6) \[ C_k(x) = \frac{1}{k! l_n^k(0)^2} x^k l_n^{(k-1)}(x) l_n^k(x)^2 \]

where \( p_j(x) \) and \( q_j(x) \) are polynomials of degree at most \( k-j-1 \).

**Theorem 2:** Let the interpolatory function \( f: \mathbb{R} \rightarrow \mathbb{R} \) be continuously differentiable such that, \( C(m) = \{ f(x) : f \) is continuous in \([0, \infty), f(x) = O(x^m) \) as \( x \rightarrow \infty \); where \( m \geq 0 \) is an integer, then for every \( f \in C(m) \) and \( k \geq 0 \)

\[ R_n(x) = \sum_{j=1}^{n} a_j^{**} U_j(x) + \sum_{j=1}^{n} b_j^{**} V_j(x) + \sum_{j=1}^{n} y_j^{**} W_j(x) + \sum_{j=0}^{kb} q_{j=0} q_{j=0}^{(j)} C_j(x) \]

satisfies the relations:

(3.8) \[ |R_n(x) - f(x)| = O(1) \omega \left( f, \frac{\log n}{\sqrt{n}} \right), \quad \text{for } 0 \leq x \leq cn^{-1} \]

(3.9) \[ |R_n(x) - f(x)| = O(1) \omega \left( f, \frac{\log n}{\sqrt{n}} \right), \quad \text{for } cn^{-1} \leq x \leq \Omega \]

where \( \omega \) is the modulus of continuity.

4. **PROOF OF THEOREM 1**

Let \( U_j(x) , V_j(x) , W_j(x) \) and \( C_j(x) \) are polynomials of degree \( \leq 3n+k \) satisfying conditions (4.1), (4.2), (4.3) and (4.4) respectively.

(4.1) \[
\begin{align*}
&U_j(x) = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases} , \\
&U_j'(x) = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases} , \\
&U_j(0) = 0 , \\
&i = 1(1)n , \\
&l = 0,1,...,k
\end{align*}
\]

For \( j = 1,2, ..., n \)

(4.2) \[
\begin{align*}
&V_j(x) = 0 , \\
&V_j'(x) = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases} , \\
&V_j(0) = 0 , \\
&i = 1(1)n , \\
&l = 0,1,...,k
\end{align*}
\]

For \( j = 1,2, ..., n \)

(4.3) \[
\begin{align*}
&W_j(x) = 0 , \\
&W_j'(x) = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases} , \\
&W_j(0) = 0 , \\
&i = 1(1)n , \\
&l = 0,1,...,k
\end{align*}
\]

and for \( l = 0,1,...,k \)

(4.4) \[
\begin{align*}
&C_k(x) = 0 , \\
&C_k'(x) = 0 , \\
&C_k(0) = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases} , \\
&i = 1(1)n
\end{align*}
\]

To determine \( W_j(x) \) let

(4.5) \[ W_j(x) = C_j x^{k+1} l_j^r(x) [L_n^k(x)]^2 \]

where \( C_j \) is a constant. \( l_j^r(x) \) is defined in (2.8). \( W_j(x) \) is a polynomial of degree \( \leq 3n+k \)
By using (2.9) and (4.3) we determine
\[(4.6) \quad C_1 = \frac{1}{y_{j}^{(k+1)}[y_{j}^{(k)}]^2}\]
Hence we find the third fundamental polynomial \(W_j(x)\) of degree \(\leq 3n+k\)

To find second fundamental polynomial let
\[(4.7) \quad V_j(x) = C_2 x^{k+1} [l_j(x)]^2 L_n^{(k-1)}(x) + C_4 x^{k+1} (x-x_j) [l_j(x)]^2 L_n^{(k-1)}(x)\]
where \(C_2\) is arbitrary constant. By using (2.8) and (4.2) we determine
\[(4.8) \quad C_2 = \frac{1}{x_{j}^{(k+1)}[y_{j}^{(k-1)}(x_j)]}\]
\[(4.9) \quad C_4 = \frac{2}{x_{j}^{(k+1)}[y_{j}^{(k-1)}(x_j)]}\]
Hence we find the second fundamental polynomial \(V_j(x)\) of degree \(\leq 3n+k\)

Again let
\[(4.10) \quad U_j(x) = C_3 x^{k+1} [l_j(x)]^2 L_n^{(k-1)}(x) + C_4 x^{k+1} (x-x_j) [l_j(x)]^2 L_n^{(k-1)}(x)\]
where \(C_3\) and \(C_4\) are arbitrary constant, \(l_j(x)\) is defined in (2.8). \(U_j(x)\) is polynomial of degree \(\leq 3n+k\) satisfying the conditions (4.1) by which we obtain
\[(4.11) \quad C_3 = \frac{1}{x_{j}^{(k+1)}[y_{j}^{(k-1)}(x_j)]}\]
\[(4.12) \quad C_4 = \frac{2}{x_{j}^{(k+1)}[y_{j}^{(k-1)}(x_j)]}\]
Hence we find the first fundamental polynomial \(U_j(x)\) of degree \(\leq 3n+k\)

To find \(C_j(x)\), we assume \(C_j(x)\) for fixed \(j \in \{0,1,\ldots,k-1\}\) in the form
\[(4.13) \quad C_j(x) = p_j(x)x^j[L_n^{(k-1)}(x)]^2 L_n^k(x) + x^j p_j(x) L_n^{(k-1)}(x) g_n(x)\]
Where \(p_j(x)\) and \(g_n(x)\) are polynomials of degree \(k-j+1\) and \(n\) respectively. Now it is clear that \(C_j^{(l)}(0) = 0\) for \(l = 0, \ldots, j-1\) and since \(L_n^{(k)}(x_j) = 0\) and \(L_n^{(k-1)}(y_i) = 0\) we get \(C_j(x_j) = 0\) and \(C_j(y_i) = 0\) for \(i = 1(1)n\).

The coefficient of the polynomial \(p_j(x)\) are calculated by the system
\[(4.14) \quad C_j^{(l)}(0) = \frac{d^k}{dx^l} [p_j(x)x^j[L_n^{(k-1)}(x)]^2 L_n^k(x)]_{x=0} = \delta_{ijj} \quad (l = j, \ldots, k-1)\]

Now from the equation \(C_j^{(k)}(0) = 0\), we get
\[(4.15) \quad c_j = g_n(0) = \frac{-1}{(n+k)! L_n^{(k-1)}(0)} \frac{d^k}{dx^k} [p_j(x)x^j[L_n^{(k-1)}(x)]^2 L_n^k(x)]_{x=0}\]
Now using the condition \(C_j'(x_j) = 0\) of (4.7), we get
\[(4.16) \quad g_n(x_j) = - \frac{x_j^{k-1} p_j(x_j) L_n^k(x_j)}{x_j^{k-j}}\]
where \(q_j(x)\) is a polynomial of degree \(k-j\)

Using (4.12) and (4.14) we obtain \(C_j(x)\) of degree \(\leq 3n+k\) satisfying the conditions (4.4)

### 5. ESTIMATION OF THE FUNDAMENTAL POLYNOMIALS

**Lemma 5.1:** Let the fundamental polynomial \(U_j(x)\), for \(j = 1,2,\ldots,n\) be given by (3.2) then we have
\[(5.1) \quad \sum_{j=1}^{n} |U_j(x)| = O(1), \quad \text{for } 0 \leq x \leq cn^{-1}\]
\[(5.2) \quad \sum_{j=1}^{n} |U_j(x)| = O(1), \quad \text{for } cn^{-1} \leq x \leq \Omega\]
where \(U_j(x)\) is given in equation (3.2)

**Proof:** From (3.2) we have
\[(5.3) \quad |U_j(x)| \leq \frac{|x_j^{k+1} [U_j(x)]^2 L_n^{(k-1)}(x_j)|}{|p_j^{(k+1)}(x_j)| |L_n^{(k-1)}(x_j)|} + \frac{2|x_j^{k+1} |p_j^{(k+1)}(x_j)| |L_n^{(k-1)}(x_j)|}{|p_j^{(k+1)}(x)| |L_n^{(k-1)}(x)|}\]
Proof.

(5.4) \[ \sum_{j=1}^{n}|U_j(x)| \leq \sum_{j=1}^{n} \left[ \frac{k^{(k+1)}}{x_j} \right] \left[ n^{(k-1)}(x_j) \right] + \sum_{j=1}^{n} \left[ \frac{k^{(k+1)}}{x_j} \right] \left[ n^{(k-1)}(x_j) \right] \]

where \[ \zeta_1 = \sum_{j=1}^{n} \left[ \frac{k^{(k+1)}}{x_j} \right] \left[ n^{(k-1)}(x_j) \right] \]

Thus (6.2) and Lemmas 5.1, 5.2, 5.3 completes the proof of the theorem.

Lemma 3.3.2: Let the fundamental polynomial \( V_j(x) \), for \( j = 1, \ldots, n \) be given by (3.3) then we have

(5.4) \[ \sum_{j=1}^{n}|V_j(x)| = O(n^{-1}), \quad \text{for } 0 \leq x \leq cn^{-1} \]

(5.5) \[ \sum_{j=1}^{n}|V_j(x)| = O(1), \quad \text{for } cn^{-1} \leq x \leq \Omega \]

where \( V_j(x) \) is given in equation (3.3)

Proof: From (3.3) we have

(5.6) \[ \sum_{j=1}^{n}|V_j(x)| \leq \sum_{j=1}^{n} \left[ \frac{k^{(k+1)}}{x_j} \right] \left[ n^{(k-1)}(x_j) \right] \]

Using (2.16), we get the result.

Lemma 5.3: Let the fundamental polynomial \( W_j(x) \), for \( j = 1, \ldots, n \) be given by (3.4) then we have

(5.7) \[ \sum_{j=1}^{n}|W_j(x)| = O(n^{-1}), \quad \text{for } 0 \leq x \leq cn^{-1} \]

(5.8) \[ \sum_{j=1}^{n}|W_j(x)| = O(1), \quad \text{for } cn^{-1} \leq x \leq \Omega \]

where \( W_j(x) \) is given in equation (3.4).

Proof: From (3.4) we have

(5.9) \[ \sum_{j=1}^{n}|W_j(x)| \leq \sum_{j=1}^{n} \left[ \frac{k^{(k+1)}}{x_j} \right] \left[ n^{(k)}(x_j) \right] \]

By equations (5.9) and (2.16), we yield the result.

Now we prove our main theorem in § 6.

6. PROOF OF MAIN THEOREM 3.2

Since \( R_n(x) \) given by equation (3.1) is exact for all polynomial \( S_n(x) \) of degree \( \leq 3n+k \), we have

(6.1) \[ Q_n(x) = \sum_{j=1}^{n} Q_n(x_j)U_j(x) + \sum_{j=1}^{n} Q_n(x_j)V_j(x) + \sum_{j=1}^{n} Q_n(x_j)W_j(x) + \sum_{j=0}^{n} Q_n(x_0)C_j(x) \]

From equation (3.2.1) and (3.4.1) we get

(6.2) \[ |f(x) - R_n(x)| \leq |f(x) - Q_n(x)| + |Q_n(x) - R_n(x)| \]

Thus (6.2) and Lemmas 5.1, 5.2, 5.3 completes the proof of the theorem.
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