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DIVISOR DEGREE ENERGY OF GRAPHS

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ABSTRACT

In this paper, we introduce the concepts of divisor degree matrix DD(G) of a simple graph G and obtain eigenvalues of DD(G). We also introduce divisor degree energy (DDE) of graphs denoted by E_{DD} (G) and find DDE of some standard graphs and also we establish the relation between energy and DDE of graphs.

Keywords: *Energy, Divisor degree matrix, Divisor degree energy, Eigenvalues.*

Mathematics Subject Classification: *05C50.*

1. INTRODUCTION

In the study of spectral graph theory, we use the spectra of certain matrix associated with the graph, such as the adjacency matrix, the Laplacian matrix and other related matrices. Some useful information about the graph can be obtained from the spectra of these various matrices.

An adjacency matrix of *G* denoted by $A = A(G) = [a_{ij}]$ is a square matrix of order of *n* where

$$a_{ij} = \begin{cases} 1, & if\ v_i\ is\ adjacent\ to\ v_j \\ 0, & otherwise \end{cases}$$

The Characteristic polynomial $|\lambda I - A|$ of *A* is called the characteristic polynomial of *G* and is denoted by

$$P_G(\lambda) \text{ or } P(G) = \sum_{i=1}^{n} a_i \lambda^{n-i} \quad .$$

The eigenvalues of *A*, which are the zeros of $|\lambda I - A|$ are the eigenvalues of *G* and form spectrum denoted by *spec* (*G*). If the distinct eigenvalues of *G* are $\lambda_1, \lambda_2, \dots, \lambda_n$ with multiplicities $t_1, t_2, \dots, t_n$ respectively, then *spec* (G) is written as $\left[{\lambda_1}^{t_1}, {\lambda_2}^{t_2}, \dots, {\lambda_n}^{t_n}\right]$. Since the adjacency matrix is a real symmetric matrix, all its eigenvalues are real and hence the eigenvalues can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Let *G* be a graph with $spec(G) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Then, the energy of *G*, denoted by *E(G)* is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

Leverrier's Algorithm: This method allows us to find the characteristic polynomial of any $n \times n$ matrix *A* using the trace of the matrix A^K, where $k = 1,2, \dots, n$. Let $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be the set of all eigenvalues of *A* which is also called the spectrum of *A*. Note that $s_k = trace\ (A^k) = \sum_{i=1}^{n} \lambda_i^k$, for all $k = 1,2, \dots, n$.

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Let $K_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$ be the characteristic polynomial of the matrix *A*, then for $k \le n$, the Newton's identities hold true:

$$p_k = -\frac{1}{k}[s_k + p_1 s_{k-1} + \cdots + p_{k-1}s_1] \quad (k = 1,2,\dots,n)$$

Various types of energies are studied in the mathematical literature [5]. Motivated by recent works on energy of a graph, in this paper we introduce divisor degree matrix associated with a graph and study its spectrum.

Let *G* be a simple graph with *n* vertices $v_1, v_2, v_3, \dots, v_n$ and *m* edges. Let d_i be the degree of v_i, $i = 1,2,\dots,n.$ Define

$$a_{ij} = \begin{cases} \left[\dfrac{d_i}{d_j}\right] + \left[\dfrac{d_j}{d_i}\right], & if\ v_i and\ v_j are\ adjacent \\ 1, & if\ v_i and\ v_j are\ adjacent\ and\ d_i = d_j \\ 0, & otherwise \end{cases}$$

where [*x*] denote integral part of real number *x*. Then the $n \times n$ matrix of *G* has its eigenvalues as $\lambda_1, \lambda_2, \dots, \lambda_n$.

The divisor degree energy (*DDE*)of a graph is define as $E_{DD}(G) = \sum_{i=1}^{n} |\lambda_i|$.

Since DD(G) is a symmetric matrix with zero trace, these divisor degree eigenvalues are real with sum equal to zero.

Example 1.1: Consider the graph *G*

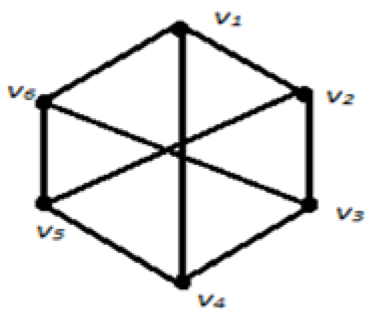


Figure-1.1

The divisor degree matrix of the graph *G* in *Fig* 1.1 is

$$DD(G) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of the divisor degree matrix DD(G) is

$$|\lambda I - DD(G)| = \begin{bmatrix} \lambda & -1 & 0 & -1 & 0 & -1 \\ -1 & \lambda & -1 & 0 & -1 & 0 \\ 0 & -1 & \lambda & -1 & 0 & -1 \\ -1 & 0 & -1 & \lambda & -1 & 0 \\ 0 & -1 & 0 & -1 & \lambda & -1 \\ -1 & 0 & -1 & 0 & -1 & \lambda \end{bmatrix}$$

From the Leverrier's algorithm [4], it follows that $P(G) = \lambda^6 - 9\lambda^4$ and the divisor degree eigenvalues of *G* are $\lambda_1 = 3, \lambda_2 = -3, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, \lambda_6 = 0$. Thus, $E_{DD}(G) = 6$.

Example 1.2: Consider the graph *G*

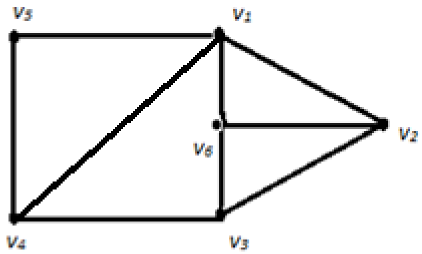


Figure-1.2

The divisor degree matrix of the graph *G* in *Fig* 1.2 is

$$DD(G) = \begin{bmatrix} 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of the divisor degree matrix *DD(G)* is

$$|\lambda I - DD(G)| = \begin{bmatrix} \lambda & -1 & 0 & -1 & -2 & -1 \\ -1 & \lambda & -1 & 0 & 0 & -1 \\ 0 & -1 & \lambda & -1 & 0 & -1 \\ -1 & 0 & -1 & \lambda & -1 & 0 \\ -2 & 0 & 0 & -1 & \lambda & 0 \\ -1 & -1 & -1 & 0 & 0 & \lambda \end{bmatrix}$$

From the Leverrier's algorithm [4], it follows that $P(G) = \lambda^6 - 12\lambda^4 - 8\lambda^3 + 23\lambda^2 + 16\lambda - 4$ and the divisor degree eigenvalues of G are $\lambda_1 \cong 3.470, \lambda_2 \cong 1.442, \lambda_3 \cong 0.198, \lambda_4 \cong -2.498, \lambda_5 \cong -1.612, \lambda_6 \cong -1$. Thus, $E_{DD}(G) \cong 10.22$.

We now give the explicit expression for the coefficient a_i of λ^{n-i} $(i = 0,1,2,3)$ in the characteristic polynomial of the divisor degree matrix *DD(G)*. It is clear that $a_0 = 1$ and $a_1 = trace\ DD(G) = 0$.

(i) We have

$$a_2 = \sum_{1 \le j < k \le n} \begin{vmatrix} 0 & a_{jk} \\ a_{kj} & 0 \end{vmatrix}$$

But $$\left\{ \begin{vmatrix} 0 & a_{jk} \\ a_{kj} & 0 \end{vmatrix} = \begin{cases} -\left(\left[\frac{d_j}{d_k}\right] + \left[\frac{d_k}{d_j}\right] \right)^2, & v_j \text{ and } v_k \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases} \right.$$

Thus

$$a_2 = -\frac{1}{2} \sum_{j=1}^{n} \left(\sum_{j \sim k} \left(\left[\frac{d_j}{d_k}\right] + \left[\frac{d_k}{d_j}\right] \right)^2 \right)$$

where $\sum_{j \sim k}$ indicates summation over all pair of adjacent vertices v_j, v_k .

Example 1.3: For the graph G in Fig 1.2 the coefficient a_2 of λ^2 in the characteristic polynomial of the divisor degree matrix DD(G) is equal to

$$-\frac{1}{2}\sum_{j=1}^{n} \left(\sum_{j \sim k} \left(\left[\frac{d_j}{d_k}\right] + \left[\frac{d_k}{d_j}\right] \right)^2 \right) = -\frac{1}{2} \begin{pmatrix} (1^2 + 2^2 + 1^2 + 1^2) + (1^2 + 1^2 + 1^2) + (1^2 + 1^2 + 1^2) \\ + \ (1^2 + 1^2 + 1^2) + (2^2 + 1^2) + (1^2 + 1^2 + 1^2) \end{pmatrix}$$
$$= -12$$

(ii) We have

$$a_3 = (-1)^3 \sum_{1\le i<j<k\le n} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}$$

$$a_3 = -2 \sum_{\Delta v_i v_j v_k} a_{ij} a_{jk} a_{ki}$$

where $\sum_{\Delta v_i v_j v_k}$ indicates summation over all pair of adjacent vertices v_i, v_j and $v_k$of triangles in a graph.

Remark 1.4:
a) Number of terms in the above sum is equal to the number of triangles in the graph.
b) If there is no triangle in the graph *G*, then $a_3 = 0$.

Example 1.5: For the graph *G* in Fig 1.2 the coefficient a_3 of λ^3 in the characteristic polynomial of the divisor degree matrix DD(G) is equal to

$$-2 \sum_{\Delta v_i v_j v_k} a_{ij} a_{jk} a_{ki} = -2\big((1\times 1\times 1) + (1\times 1\times 1) + (1\times 1\times 2)\big) = -8\ .$$

Theorem 1.6: If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the divisor degree eigenvalues of a graph *G*, then
$\sum_{i=1}^{n} \lambda_i^2 = -2a_2$

Proof: We have

$$\sum_{i=1}^{n} \lambda_i^2 = trace\big(DD(G)\big)^2 = \sum_{i=1}^{n}\big(\sum_{j=1}^{n} a_{ij} a_{ji}\big) = \sum_{i=1}^{n}\left(\sum_{i\sim j}\left(\left[\frac{d_i}{d_j}\right] + \left[\frac{d_j}{d_i}\right]\right)^2\right)$$

$$= -2a_2.$$

2. THE DIVISOR DEGREE ENERGY OF SOME STANDARD GRAPHS

We observe that the adjacent matrix of energy and divisor degree matrix of a regular graphs are same. So, *DDE* is equal to the energy of the regular graphs [8]. Hence, we have the following results.

Result 2.1:
i) $E_{DD}(K_{\boldsymbol{n}}) = 2(n-1)$
ii) $E_{DD}(K_{n,n}) = 2n$
iii) $spec\ (C_n) = 2\cos\left(\frac{2\pi j}{n}\right)$ $(j = 0,1,\dots,n-1)$.

Theorem 2.2: If *G* is a path P_n of order *n*, $(n \ge k+2)$, then

$$P(P_n) = \lambda^{k+2} - 8\lambda^k + 8k\sum_{r=0}^{\left[\frac{k}{2}\right]-1} \frac{(-1)^r}{r+1}\binom{k-r-2}{r}\lambda^{k-2(r+1)} - \sum_{r=0}^{\left[\frac{k}{2}\right]-1}(-1)^r\binom{k-r-1}{r+1}\lambda^{k-2r},$$
$$n \ge k+2, k \in N\ .$$

Proof: For $n = 3$,

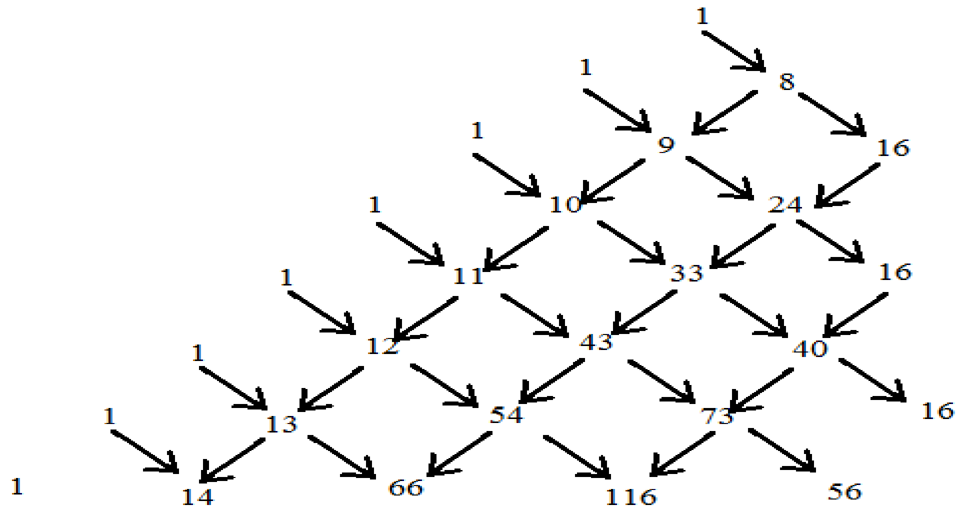
$$DD(P_3) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$$|\lambda I - DD(P_3)| = \begin{vmatrix} \lambda & -2 & 0 \\ -2 & \lambda & -2 \\ 0 & -2 & \lambda \end{vmatrix}$$

From the Leverrier's algorithm [4], it follows that $P(P_3) = \lambda^3 - 8\lambda$

Similarly, for $n = 4, P(P_4) = \lambda^4 - 9\lambda^2 + 16$
$n = 5, P(P_5) = \lambda^5 - 10\lambda^3 + 24\lambda$
$n = 6, P(P_6) = \lambda^6 - 11\lambda^4 + 33\lambda^2 - 16$
$n = 7, P(P_7) = \lambda^7 - 12\lambda^5 + 43\lambda^3 - 40\lambda$ and so on.

Diagrammatic representation of the characteristic polynomial of path P_n of order *n*, $n \ge 3$



The numbers in the row can be obtained by the following rule. The first number is 1 each and 16 is the last number in each odd rows except first row. The second number in each row is obtained by adding the previous row's first number. The third number in each row is obtained by adding the previous second row's second number and previous row's third number. Likewise, the fourth number in each row is obtained by adding the previous second row's third number and previous row's fourth number and so on. This helps us to generalize the characteristic polynomial of a path P_n of order $n, n \geq 3$,

$$\begin{aligned} P(\lambda) &= \lambda^n - (8+n-3)\lambda^{n-2} + \left(8(n-2) + \tfrac{(n-4)(n-5)}{2}\right)\lambda^{n-4} - \left(\tfrac{8(n-2)(n-5)}{2} + \tfrac{(n-5)(n-6)(n-7)}{3\times2\times1}\right)\lambda^{n-6} \\ &\quad + \left(\tfrac{8(n-2)(n-6)(n-7)}{3\times2} + \tfrac{(n-6)(n-7)(n-8)(n-9)}{4\times3\times2}\right)\lambda^{n-8} - \cdots \\ &= \lambda^{k+2} - 8\lambda^k + \left(8k\lambda^{k-2} - \tfrac{8k(k-3)}{2}\lambda^{k-4} + \tfrac{8k(k-4)(k-5)}{3\times2}\lambda^{k-6} - \cdots\right) \\ &\quad - \left(\tfrac{(k-1)}{1}\lambda^k - \tfrac{(k-2)(k-3)}{2}\lambda^{k-2} + \tfrac{(k-3)(k-4)(k-5)}{3\times2}\lambda^{k-4} - \cdots\right) \end{aligned}$$

where $n \geq k+2, k \in N$.

$$P(P_n) = \lambda^{k+2} - 8\lambda^k + 8k\sum_{r=0}^{\left[\frac{k}{2}\right]-1} \frac{(-1)^r}{r+1}\binom{k-r-2}{r}\lambda^{k-2(r+1)} - \sum_{r=0}^{\left[\frac{k}{2}\right]-1}(-1)^r\binom{k-r-1}{r+1}\lambda^{k-2r}, \quad n \geq k+2, k \in N.$$

Lemma 2.3 [2]: If M is a non singular square matrix then we have $\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M||Q - PM^{-1}N|$.

Theorem 2.4: If G is a complete bipartite graph K_{n_1,n_2}, then $P(K_{n_1,n_2}) = \lambda^{n_1+n_2-2}(\lambda^2 - n_1n_2k^2)$ where $n_2 = n_1k + r$, $(0 \leq r \leq n_1 - 1)$ and $k = \left[\frac{n_2}{n_1}\right]$ is the integral part of real number $\frac{n_2}{n_1}$, $(n_1 \leq n_2)$ and $E_{DD}(K_{n_1,n_2}) = 2k\sqrt{n_1n_2}$.

Proof: Without loss of generality we partition the vertex set of the complete bipartite graph K_{n_1,n_2} into disjoint sets $A = \{u_1, u_2, \ldots, u_{n_1}\}$ and $B = \{v_1, v_2, \ldots, v_{n_2}\}$ such that no two vertices in either sets are adjacent to each other. Then the characteristic polynomial of K_{n_1,n_2} is

$$|\lambda I - DD(K_{n_1,n_2})| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & -k & -k & \cdots & -k & -k \\ 0 & \lambda & \cdots & 0 & -k & -k & \cdots & -k & -k \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda & -k & -k & \cdots & -k & -k \\ -k & -k & \cdots & -k & \lambda & 0 & \cdots & 0 & 0 \\ -k & -k & \cdots & -k & 0 & \lambda & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -k & -k & \cdots & -k & 0 & 0 & \cdots & \lambda & 0 \\ -k & -k & \cdots & -k & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}$$

$$= \begin{vmatrix} \lambda I_{n_1} & N^T \\ N & \lambda I_{n_2} \end{vmatrix} \quad (1)$$

where $N = \begin{bmatrix} -k & -k & \cdots & -k & -k \\ -k & -k & \cdots & -k & -k \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -k & -k & \cdots & -k & -k \\ -k & -k & \cdots & -k & -k \end{bmatrix}$ is a matrix of order $n_2 \times n_1$ and N^T is the transpose of N of order $n_1 \times n_2$.

Applying lemma 2.3 to the expression (1), we get

$$\left|\lambda I - DD\left(K_{n_1,n_2}\right)\right| = \lambda^{n_1}\left|\lambda I_{n_2} - N\frac{I_{n_1}}{\lambda}N^T\right| = \lambda^{n_1-n_2}\left|\lambda^2 I_{n_2} - NN^T\right| \qquad (2)$$

Now, $NN^T = \begin{bmatrix} n_1k^2 & n_1k^2 & \cdots & n_1k^2 & n_1k^2 \\ n_1k^2 & n_1k^2 & \cdots & n_1k^2 & n_1k^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n_1k^2 & n_1k^2 & \cdots & n_1k^2 & n_1k^2 \\ n_1k^2 & n_1k^2 & \cdots & n_1k^2 & n_1k^2 \end{bmatrix}$ is a square matrix of order n_2.

Substituting NN^T in (2), we get

$$\left|\lambda I - DD\left(K_{n_1,n_2}\right)\right| = \lambda^{n_1-n_2}\begin{vmatrix} \lambda^2 - n_1k^2 & -n_1k^2 & \cdots & -n_1k^2 & -n_1k^2 \\ -n_1k^2 & \lambda^2 - n_1k^2 & \cdots & -n_1k^2 & -n_1k^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -n_1k^2 & -n_1k^2 & \cdots & \lambda^2 - n_1k^2 & -n_1k^2 \\ -n_1k^2 & -n_1k^2 & \cdots & -n_1k^2 & \lambda^2 - n_1k^2 \end{vmatrix} \qquad (3)$$

Subtract row n_2 from the rows $1,2,\dots,n_2-1$ of (3), we get

$$= \lambda^{n_1-n_2}\begin{vmatrix} \lambda^2 & 0 & \cdots & 0 & -\lambda^2 \\ 0 & \lambda^2 & \cdots & 0 & -\lambda^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda^2 & -\lambda^2 \\ -n_1k^2 & -n_1k^2 & \cdots & -n_1k^2 & \lambda^2 - n_1k^2 \end{vmatrix}$$

$$= \lambda^{n_1-n_2}(\lambda^2)^{n_2-1}\begin{vmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -1 \\ -n_1k^2 & -n_1k^2 & \cdots & -n_1k^2 & \lambda^2 - n_1k^2 \end{vmatrix}$$

Thus, $P\left(K_{n_1,n_2}\right) = \lambda^{n_1+n_2-2}(\lambda^2 - n_1n_2k^2)$, where $n_2 = n_1k + r, (0 \le r \le n_1 - 1)$ and $E_{DD}\left(K_{n_1,n_2}\right) = 2k\sqrt{n_1n_2}$.

Corollary 2.5: For a star K_{1,n_2}, $E_{DD}\left(K_{1,n_2}\right) = 2n_2\sqrt{n_2}$.

Theorem 2.6: If G is a r-regular graph with triangle free and $n = 2r$ then $E_{DD}(G) = n$.

Proof: We have

$$|\lambda I - DD(G)| = \begin{vmatrix} \lambda & -1 & 0 & -1 & \cdots & 0 & -1 \\ -1 & \lambda & -1 & 0 & \cdots & -1 & 0 \\ 0 & -1 & \lambda & -1 & \cdots & 0 & -1 \\ -1 & 0 & -1 & \lambda & \cdots & -1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 0 & -1 & \cdots & \lambda & -1 \\ -1 & 0 & -1 & 0 & \cdots & -1 & \lambda \end{vmatrix}$$

Add row 1 to rows $2,3,\dots,n$, we get

$$|\lambda I - DD(G)| = \left(\lambda - \frac{n}{2}\right)\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & \lambda & -1 & 0 & \cdots & -1 & 0 \\ 0 & -1 & \lambda & -1 & \cdots & 0 & -1 \\ -1 & 0 & -1 & \lambda & \cdots & -1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 0 & -1 & \cdots & \lambda & -1 \\ -1 & 0 & -1 & 0 & \cdots & -1 & \lambda \end{vmatrix}$$

Subtract column n from the column $2, 4, 6, \ldots, n-2$, and column 1 from the column $3, 5, 7, \ldots, n-1$ respectively, we get

$$|\lambda I - DD(G)| = \lambda^{n-2}\left(\lambda - \frac{n}{2}\right)\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & -1 & 0 & -1 & \cdots & 0 & \lambda \end{vmatrix}$$

$$= \lambda^{n-2}\left(\lambda - \frac{n}{2}\right)\left(\lambda + \frac{n}{2}\right)$$

Hence $E_{DD}(G) = n$.

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